Frequency Response

Lecture #11

Chapter 10

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.
- To Understand How Computers Process Signals and Systems

Frequency Response of LTI Systems

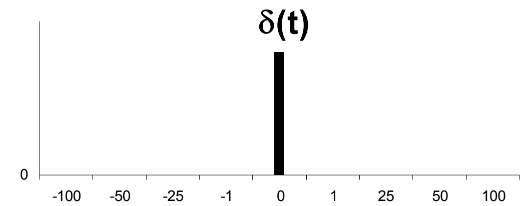
- Let's review the Frequency Response for continuous-time systems
- First, some definitions:
  - The unit impulse function
  - The unit Step function

A Special Function – Unit Impulse Function

- The unit impulse function, $\delta(t)$, also known as the Dirac delta function, is defined as:

$\delta(t) = 0$ for $t \neq 0$;

$= \text{undefined}$ for $t = 0$



and has the following special property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

$$\therefore \int_{-\infty}^{\infty} \delta(t)dt = 1$$

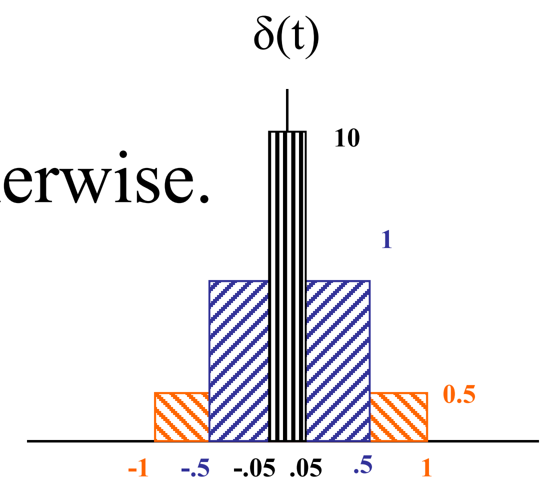
Uses of Delta Function

- Modeling of electrical, mechanical, physical phenomenon:
  - point charge,
  - impulsive force,
  - point mass
  - point light

Unit Impulse Function Continued

- A consequence of the delta function is that it can be approximated by a narrow pulse as the width of the pulse approaches zero while the area under the curve = 1

$$\lim_{\varepsilon \to 0} \delta(t) \approx 1/\varepsilon \text{ for } -\varepsilon/2 < t < \varepsilon/2; = 0 \text{ otherwise.}$$



Unit Impulse Function Continued

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt$$

Let's approximate $\delta(t-\tau)$ with a pulse of height $\frac{1}{\varepsilon}$ and width ε

where $\varepsilon = t\text{-}\tau$, we have

$$\approx \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t)\frac{1}{\varepsilon}dt$$

If we take the limit of this integral as $\varepsilon \to 0$,

the approximate approaches the original integral

$$= \lim_{\varepsilon\to 0} \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t)\frac{1}{\varepsilon}dt \to \lim_{\varepsilon\to 0} f(\tau)\frac{1}{\varepsilon}\varepsilon \to f(\tau),$$

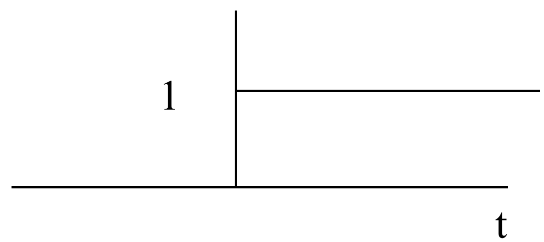
since as $\varepsilon \to 0$, $\mathrm{t} \to \tau$

Another Special Function – Unit Step Function

- The unit step function, $u(t)$ is defined as:

$u(t) = 1$ for $t \geq 0$;

$= 0$ for $t < 0$.



and is related to the delta function as follows:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Integration of the Delta Function

- $\delta(t) \longrightarrow u(t)$
- $u(t) \longrightarrow tu(t)$ *1st order*
- $tu(t) \longrightarrow \frac{t^2}{2!}u(t)$ *2nd order*

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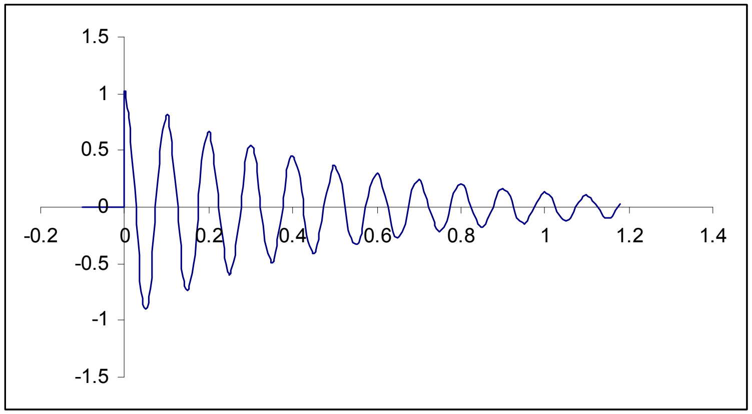
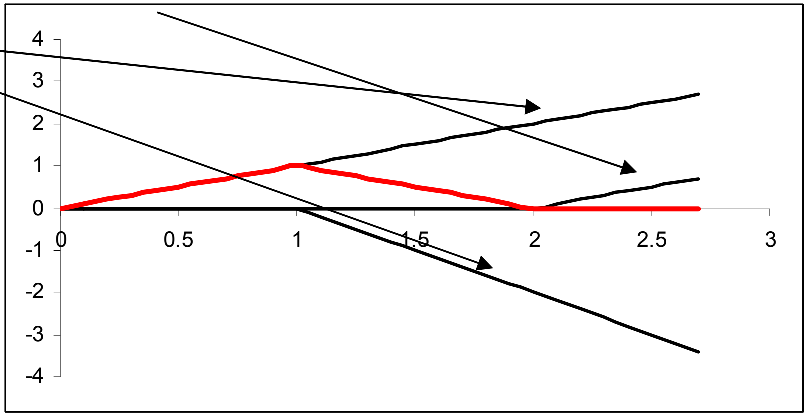
- $\longrightarrow \frac{t^n}{n!}u(t)$ n^{th} order

Signal Representations using the Unit Step Function

- $x(t) = e^{-\sigma t} \cos(\omega t) u(t)$



- $x(t) = \underbrace{t\,u(t)} - 2\underbrace{(t\text{-}1)u(t\text{-}1)} + \underbrace{(t\text{-}2)\,u(t\text{-}2)}$



Convolution of the Unit-Impulse Response

- As with Discrete-time system, we find that the Unit-Impulse Response of the a Continuous-time system, $h(t)$, is key to determining the output of the system to any input:

$$y(t) = h(t) \otimes x(t) = \int_{\tau} h(\tau)x(t-\tau)d\tau$$

- Let's apply the complex exponential (sinusoidal) signal, $x(t)=Ae^{j\phi}e^{j\omega t}$, for all t

$$y(t) = h(t) \otimes x(t) = \int_{\tau} h(\tau)x(t-\tau)d\tau$$

$$= \int_{\tau} h(\tau)Ae^{j\phi}e^{j\omega(t-\tau)}d\tau = \{\int_{\tau} h(\tau)e^{-j\omega\tau}d\tau\}Ae^{j\phi}e^{j\omega t}$$

Frequency Response

- As in Discrete-time systems, the Frequency Response of the LTI system, $H(j\omega)$, is:

$$y(t) = \{\int_{\tau} h(\tau)e^{-j\omega\tau}d\tau\}Ae^{j\phi}e^{j\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$$y(t) = H(j\omega)Ae^{j\phi}e^{j\omega t}$$

- Again this is true only for sinusoidal signals!!!!

Examples

- Assume we have a system, $H(j3\pi) = 2 - j2$ with input $x(t) = 10e^{j3\pi t}$, then

$$y(t) = H(j3\pi)10e^{j3\pi t} = (2 - j2)10e^{j3\pi t}$$

$$= (2\sqrt{2}e^{-j\frac{\pi}{4}})(10e^{j3\pi t})$$

$$= 20\sqrt{2}e^{j(3\pi t - j\frac{\pi}{4})}$$

- Assume $h(t) = 2e^{-2t}u(t)$, calculate the Frequency Response:

$$H(j\omega) = \int_{-\infty}^{\infty} 2e^{-2t}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} 2e^{-2t-j\omega t}dt$$

$$= \frac{2}{-(2+j\omega)}e^{-(2+j\omega)t}\Big|_0^{\infty} = \frac{2}{-(2+j\omega)}[e^{-(2+j\omega)\infty} - e^{-(2-j\omega)0}]$$

$$= \frac{2}{-(2+j\omega)}[e^{-2\infty}e^{-j\omega\infty} - 1] = \frac{2[0e^{-j\omega\infty} - 1}{-(2+j\omega)} = \frac{2}{2+j\omega}$$

Plotting the Frequency Response

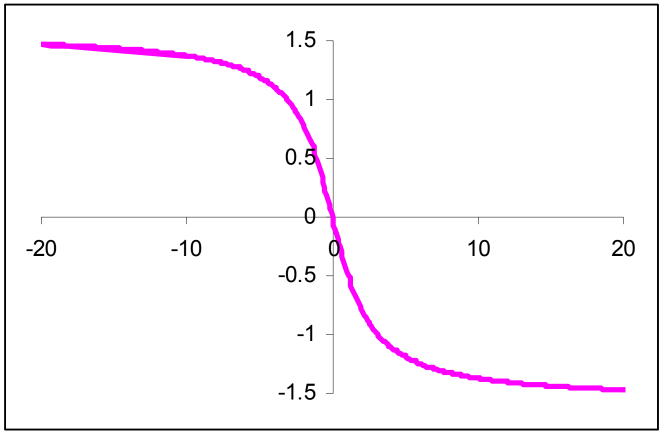
- Let's plot the magnitude and phase of the Frequency Response vs. Frequency
- What kind of filter is this?

$$H(j\omega) = \frac{2}{2 + j\omega}$$

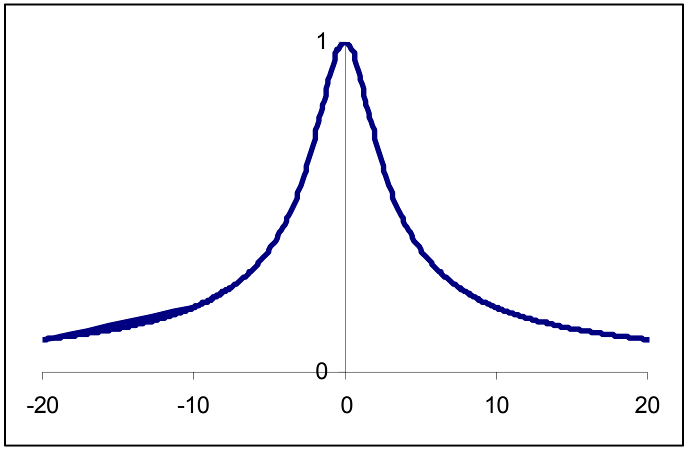
$$\left|H(j\omega)\right| = \frac{2}{\sqrt{2^2 + \omega^2}}$$

$$\angle H(j\omega) = 0 - \angle\{2 + j\omega\} = -\tan^{-1}(\frac{\omega}{2})$$

Phase



Magnitude



Frequency Response to a Cosine Input

- If the input to an LTI system is

$$x(t)=A\cos(\omega t+\phi),$$

- and if the impulse response is real-valued, then the output will be

$$y(t)=AM\cos(\omega t+\phi+\varphi)$$

- where the frequency response is

$$H(j\omega)=M\,e^{j\varphi}$$

- To show this:

$$x(t)=\mathrm{A}\cos(\omega t+\phi)=½\mathrm{A}\{e^{j\phi}e^{j\omega t}+e^{-j\phi}e^{-j\omega t}\}$$

- Using superposition

$$y(t)=½\mathrm{A}\{H(j\omega)\,e^{j\phi}e^{j\omega t}+H(-j\omega)\,e^{-j\phi}e^{-j\omega t}\}$$

Frequency Response to a Cosine Input

- If the impulse response is real-valued then

$$y(t)= ½A\{H(j\omega)\, e^{j\phi}e^{j\omega t} + H^*(j\omega)\, e^{-j\phi}e^{-j\omega t}\}$$

$$y(t)= ½A\{Me^{j\psi}e^{j\phi}e^{j\omega t} + (Me^{j\psi})^*\, e^{-j\phi}e^{-j\omega t}\}$$

$$y(t)= ½A\{Me^{j\psi}e^{j\phi}e^{j\omega t} + Me^{-j\psi}e^{-j\phi}e^{-j\omega t}\}$$

$$y(t)= ½A\{Me^{j(\omega t+\phi+\psi)} + Me^{-j(\omega t+\phi+\psi)}\}$$

$$y(t)=AM\cos(\omega t + \phi + \psi)$$

Proof of Conjugate Symmetry

$$H^*(j\omega) = \left(\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right)^* = \int_{-\infty}^{\infty} h(t)^* e^{+j\omega t} dt$$

$$= \int_{-\infty}^{\infty} h(t) e^{-j(-\omega)t} dt = H(-j\omega)$$

$$h(t)^* = h(t) \text{ Only is } h(t) \text{ is real-valued}$$

$$\text{If } H(j\omega) = Me^{j\psi}$$

$$\text{Then } H(-j\omega) = H^*(j\omega) = (Me^{j\psi})^* = Me^{-j\psi}$$

Examples

- Assume we have a system, given by the following $H(j\omega)$ and the input $x(t) = 3\cos(40\pi t - \pi)$ is applied. Determine the output signal.

$$H(j\omega) = \frac{40\pi}{40\pi + j\omega}$$

$$\left|H(j40\pi)\right| = \frac{40\pi}{\sqrt{(40\pi)^2 + 40\pi^2}} = \frac{1}{\sqrt{2}}$$

$$\angle H(j40\pi) = -\tan^{-1}\left(\frac{40\pi}{40\pi}\right) = -\tan^{-1}(1) = -\frac{\pi}{4}$$

$$y(t) = 3\left(\frac{1}{\sqrt{2}}\right)\cos\left(40\pi t - \pi - \frac{\pi}{4}\right)$$

$$= 2.1213\cos\left(40\pi t - \frac{5\pi}{4}\right)$$

An Example

- An LTI system has an impulse response of

$$h(t) = \delta(t) - 200\pi e^{-200\pi t}u(t)$$

- The following signal is applied:

$$x(t)=10+20\delta(t - 0.1)+40\cos(200\pi t+0.3\pi) \text{ for all } t$$

- The input has 3 parts: a constant, an impulse and a cosine wave. We will take each part separately and use the easiest method to find the solution.

An Example

- Let's first find the frequency response of the system from the impulse response:

$$H(j\omega) = \int_{-\infty}^{\infty} [\delta(t) - 200\pi e^{-200\pi t} u(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt - 200\pi \int_{-\infty}^{\infty} e^{-200\pi t} u(t) e^{-j\omega t} dt$$

$$= 1 - 200\pi \int_{0}^{\infty} e^{-(200\pi + j\omega)t} dt = 1 + \frac{200\pi}{200\pi + j\omega} e^{-(200\pi + j\omega)t} \Big|_{0}^{\infty}$$

$$= 1 - \frac{200\pi}{200\pi + j\omega} = \frac{j\omega}{200\pi + j\omega}$$

An Example

- Now let's take the first (constant, $\omega = 0$) part and the third (cosine) part and evaluate the solution using the frequency response:

$$\text{The first part of the input}: 10$$

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$$10 \mapsto H(j0)10 = \frac{j0}{200\pi + j0}10 = 0$$

$$\text{The third part of the input}: 40\cos(200\pi t + 0.3\pi)$$

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$$40\cos(200\pi t + 0.3\pi) \mapsto 40\left|H(j200\pi)\right|\cos[200\pi t + 0.3\pi + \angle H(j200\pi)]$$

$$H(j200\pi) = \frac{j200\pi}{200\pi + j200\pi} = \frac{j}{1+j} = \frac{1\angle\frac{\pi}{2}}{\sqrt{2}\angle\frac{\pi}{4}} = \frac{1}{\sqrt{2}}\angle\frac{\pi}{4}$$

$$40\cos(200\pi t + 0.3\pi) \mapsto 40\frac{1}{\sqrt{2}}\cos[200\pi t + 0.3\pi + .25\pi] = \frac{40}{\sqrt{2}}\cos[200\pi t + 0.55\pi]$$

An Example

- Now for the second part of the input (the impulse function), we will apply the impulse response:

$$\text{The second part of the input}: 20\delta(t-0.1)$$

$$20\delta(t-0.1) \mapsto 20h(t-0.1) = 20[\delta(t-0.1) - 200\pi e^{-200\pi(t-0.1)}u(t-0.1)]$$

$$20\delta(t-0.1) \mapsto 20\delta(t-0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t-0.1)$$

- The Complete solution by superposition is:

$$\begin{aligned} y(t) &= 0 \\ &+ 20\delta(t-0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t-0.1) \\ &+ \frac{40}{\sqrt{2}}\cos(200\pi t + 0.55\pi) \end{aligned}$$

Frequency Response to a Sum of Cosine Inputs

- We can extend this to the case when

$$x(t)=\Sigma\, A_k\cos(\omega_k t+\phi_k)$$

- By superposition

$$y_k(t)=AM_k \cos(\omega_k t + \phi_k +\varphi_k)$$

- where the frequency response is

$$H_k(j\omega_k) = M_k\, e^{\,j\varphi_k}$$

- Then:

$$y(t) =\Sigma\, AM_k \cos(\omega_k t + \phi_k +\varphi_k)$$

- This model can also be used when analyzing periodic input signals since a Fourier Series can be generated which has the similar form: $x(t)=\Sigma\, A_k\cos(k\omega_o t+\phi_k)$ where ω_o is the fundamental frequency.

Homework

- Exercises:
  - 10.1-10.3
- Problems:
  - 10.1, 10.2,
  - 10.4 Use Matlab to plot the frequency response and submit your code
  - Using unit step functions, construct a single pulse of magnitude 10 starting at t=5 and ending at t=10.
  - Repeat with 2 pulses where the second is of magnitude 5 starting at t=15 and ending at t=25.