Frequency Response

Lecture #11

Chapter 10
What Is this Course All About?

• To Gain an Appreciation of the Various Types of Signals and Systems
• To Analyze The Various Types of Systems
• To Learn the Skills and Tools needed to Perform These Analyses.
• To Understand How Computers Process Signals and Systems
Frequency Response of LTI Systems

• Let’s review the Frequency Response for continuous-time systems

• First, some definitions:
  – The unit impulse function
  – The unit Step function
A Special Function – Unit Impulse Function

• The unit impulse function, $\delta(t)$, also known as the Dirac delta function, is defined as:

$$\delta(t) = 0 \text{ for } t \neq 0;$$

$$= \text{ undefined for } t = 0$$

and has the following special property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$
Uses of Delta Function

- Modeling of electrical, mechanical, physical phenomenon:
  - point charge,
  - impulsive force,
  - point mass
  - point light
Unit Impulse Function Continued

- A consequence of the delta function is that it can be approximated by a narrow pulse as the width of the pulse approaches zero while the area under the curve = 1

\[
\lim_{\varepsilon \to 0} \delta(t) \approx \frac{1}{\varepsilon} \quad \text{for} \quad -\varepsilon/2 < t < \varepsilon/2; = 0 \quad \text{otherwise.}
\]
Unit Impulse Function
Continued

\[ \int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt \]

Let's approximate \( \delta(t-\tau) \) with a pulse of height \( \frac{1}{\varepsilon} \) and width \( \varepsilon \)

where \( \varepsilon = t-\tau \), we have

\[ \approx \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t) \frac{1}{\varepsilon} dt \]

If we take the limit of this integral as \( \varepsilon \to 0 \),
the approximate approaches the original integral

\[ = \lim_{\varepsilon \to 0} \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t) \frac{1}{\varepsilon} dt \to \lim_{\varepsilon \to 0} f(\tau) \frac{1}{\varepsilon} \to f(\tau), \]

since as \( \varepsilon \to 0 \), \( t \to \tau \)
Another Special Function – Unit Step Function

• The unit step function, $u(t)$ is defined as:
  
  $u(t) = 1$ for $t \geq 0$;
  
  $= 0$ for $t < 0$.

and is related to the delta function as follows:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$
Integration of the Delta Function

- $\delta(t) \rightarrow u(t)$
- $u(t) \rightarrow tu(t)$ \hspace{1cm} 1\textsuperscript{st} order
- $tu(t) \rightarrow \frac{t^2}{2!}u(t)$ \hspace{1cm} 2\textsuperscript{nd} order
- $\ldots$
- $\ldots$
- $\ldots$
- $\rightarrow \frac{t^n}{n!}u(t)$ \hspace{1cm} n\textsuperscript{th} order
Signal Representations using the Unit Step Function

- $x(t) = e^{-\sigma t} \cos(\omega t)u(t)$

- $x(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$
Convolution of the Unit-Impulse Response

- As with Discrete-time system, we find that the Unit-Impulse Response of the a Continuous-time system, $h(t)$, is key to determining the output of the system to any input:

\[ y(t) = h(t) \otimes x(t) = \int h(\tau)x(t - \tau)d\tau \]

- Let’s apply the complex exponential (sinusoidal) signal, $x(t) = Ae^{j\phi}e^{j\omega t}$, for all $t$

\[ y(t) = h(t) \otimes x(t) = \int h(\tau)x(t - \tau)d\tau \]

\[ = \int h(\tau) Ae^{j\phi}e^{j\omega(t-\tau)}d\tau = \left\{ \int h(\tau)e^{-j\omega\tau}d\tau \right\} Ae^{j\phi}e^{j\omega t} \]
Frequency Response

- As in Discrete-time systems, the Frequency Response of the LTI system, $H(j\omega)$, is:

\[
y(t) = \left\{ \int_{\tau} h(\tau)e^{-j\omega \tau} d\tau \right\} Ae^{j\phi} e^{j\omega t}
\]

\[
H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt
\]

\[
y(t) = H(j\omega) Ae^{j\phi} e^{j\omega t}
\]

- Again this is true only for sinusoidal signals!!!!
Examples

• Assume we have a system, \( H(j3\pi) = 2 - j2 \) with input \( x(t) = 10e^{j3\pi t} \), then

\[
y(t) = H(j3\pi)10e^{j3\pi t} = (2 - j2)10e^{j3\pi t}
\]

\[
= (2\sqrt{2}e^{-j\frac{\pi}{4}})(10e^{j3\pi t})
\]

\[
= 20\sqrt{2}e^{j(3\pi t-j\frac{\pi}{4})}
\]

• Assume \( h(t) = 2e^{-2t}u(t) \), calculate the Frequency Response:

\[
H(j\omega) = \int_{-\infty}^{\infty} 2e^{-2t}u(t)e^{-j\omega t} dt = \int_{0}^{\infty} 2e^{-2t-j\omega t} dt
\]

\[
= \frac{2}{-(2+j\omega)} e^{-2\omega} \bigg|_{0}^{\infty} = \frac{2}{-(2+j\omega)} [e^{-(2+j\omega)\infty} - e^{-(2-j\omega)0}]
\]

\[
= \frac{2}{-(2+j\omega)} [e^{-2\omega} e^{-j\omega \infty} - 1] = \frac{2[0e^{-j\omega \infty} - 1]}{-(2+j\omega)} = \frac{2}{2+j\omega}
\]
Plotting the Frequency Response

- Let’s plot the magnitude and phase of the Frequency Response vs. Frequency
- What kind of filter is this?

\[ H(j\omega) = \frac{2}{2 + j\omega} \]

\[ |H(j\omega)| = \frac{2}{\sqrt{2^2 + \omega^2}} \]

\[ \angle H(j\omega) = 0 - \angle\{2 + j\omega\} = -\tan^{-1}\left(\frac{\omega}{2}\right) \]
Frequency Response to a Cosine Input

• If the input to an LTI system is
  \[ x(t) = A \cos(\omega t + \phi), \]
• and if the impulse response is real-valued, then the output will be
  \[ y(t) = AM \cos(\omega t + \phi + \phi) \]
• where the frequency response is
  \[ H(j\omega) = M e^{j\phi} \]
• To show this:
  \[ x(t) = A \cos(\omega t + \phi) = \frac{1}{2}A \{ e^{j\phi}e^{j\omega t} + e^{-j\phi}e^{-j\omega t} \} \]
• Using superposition
  \[ y(t) = \frac{1}{2}A \{ H(j\omega) e^{j\phi}e^{j\omega t} + H(-j\omega) e^{-j\phi}e^{-j\omega t} \} \]
Frequency Response to a Cosine Input

- If the impulse response is real-valued then

\[
y(t) = \frac{1}{2}A \left\{ H(j\omega) e^{j\phi} e^{j\omega t} + H^*(j\omega) e^{-j\phi} e^{-j\omega t} \right\}
\]

\[
y(t) = \frac{1}{2}A \left\{ M e^{j\psi} e^{j\phi} e^{j\omega t} + (M e^{j\psi})^* e^{-j\phi} e^{-j\omega t} \right\}
\]

\[
y(t) = \frac{1}{2}A \left\{ M e^{j\psi} e^{j\phi} e^{j\omega t} + M e^{-j\psi} e^{-j\phi} e^{-j\omega t} \right\}
\]

\[
y(t) = \frac{1}{2}A \left\{ M e^{j(\omega t + \phi + \psi)} + M e^{-j(\omega t + \phi + \psi)} \right\}
\]

\[
y(t) = AM \cos(\omega t + \phi + \psi)
\]
Proof of Conjugate Symmetry

\[ H^*(j\omega) = \left( \int_{-\infty}^{\infty} h(t)e^{-j\omega t} \, dt \right)^* = \left( \int_{-\infty}^{\infty} h(t)^* e^{+j\omega t} \, dt \right) \]

\[ = \int_{-\infty}^{\infty} h(t)e^{-j(\omega)t} \, dt = H(-j\omega) \]

\[ h(t)^* = h(t) \text{ Only is } h(t) \text{ is real - valued} \]

If \( H(j\omega) = Me^{j\psi} \), then \( H(-j\omega) = H^*(j\omega) = (Me^{j\psi})^* = Me^{-j\psi} \)
Examples

• Assume we have a system, given by the following $H(j\omega)$ and the input $x(t) = 3 \cos(40\pi t - \pi)$ is applied. Determine the output signal.

$$H(j\omega) = \frac{40\pi}{40\pi + j\omega}$$

$$|H(j40\pi)| = \frac{40\pi}{\sqrt{(40\pi)^2 + 40^2\pi^2}} = \frac{1}{\sqrt{2}}$$

$$\angle H(j40\pi) = -\tan^{-1}\left(\frac{40\pi}{40\pi}\right) = -\tan^{-1}(1) = -\frac{\pi}{4}$$

$$y(t) = 3\left(\frac{1}{\sqrt{2}}\right)\cos(40\pi t - \pi - \frac{\pi}{4})$$

$$= 2.1213\cos(40\pi t - \frac{5\pi}{4})$$
An Example

• An LTI system has an impulse response of
  \[ h(t) = \delta(t) - 200\pi e^{-200\pi t}u(t) \]

• The following signal is applied:
  \[ x(t)=10+20\delta(t-0.1)+40\cos(200\pi t+0.3\pi) \text{ for all } t \]

• The input has 3 parts: a constant, an impulse and a cosine wave. We will take each part separately and use the easiest method to find the solution.
An Example

- Let’s first find the frequency response of the system from the impulse response:

\[
H(j\omega) = \int_{-\infty}^{\infty} [\delta(t) - 200\pi e^{-200\pi t} u(t)] e^{-j\omega t} dt
\]

\[
= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt - 200\pi \int_{-\infty}^{\infty} e^{-200\pi t} u(t) e^{-j\omega t} dt
\]

\[
= 1 - 200\pi \int_{0}^{\infty} e^{-(200\pi + j\omega)t} dt = 1 + \left. \frac{200\pi}{200\pi + j\omega} e^{-(200\pi + j\omega)t} \right|_{0}^{\infty}
\]

\[
= 1 - \frac{200\pi}{200\pi + j\omega} = \frac{j\omega}{200\pi + j\omega}
\]
An Example

- Now let’s take the first (constant, \( \omega = 0 \)) part and the third (cosine) part and evaluate the solution using the frequency response:

The first part of the input: \( 10 \)

\[
H(j \omega) = \frac{j \omega}{200\pi + j \omega}
\]

\( 10 \mapsto H(j0)10 = \frac{j0}{200\pi + j0}10 = 0 \)

The third part of the input: \( 40\cos(200\pi t + 0.3\pi) \)

\[
H(j \omega) = \frac{j \omega}{200\pi + j \omega}
\]

\( 40\cos(200\pi t + 0.3\pi) \mapsto 40|H(j200\pi)|\cos[200\pi t + 0.3\pi + \angle H(j200\pi)] \)

\[
H(j200\pi) = \frac{j200\pi}{200\pi + j200\pi} = \frac{j}{1+j} = \frac{1\angle \pi/2}{\sqrt{2}\angle \pi/4} = \frac{1}{\sqrt{2}}\angle \pi/4
\]

\( 40\cos(200\pi t + 0.3\pi) \mapsto 40\frac{1}{\sqrt{2}}\cos[200\pi t + 0.3\pi + .25\pi] = \frac{40}{\sqrt{2}}\cos[200\pi t + 0.55\pi] \)
An Example

• Now for the second part of the input (the impulse function), we will apply the impulse response:

The second part of the input: \( 20\delta(t - 0.1) \)

\[
20\delta(t - 0.1) \leftrightarrow 20h(t - 0.1) = 20[\delta(t - 0.1) - 200\pi e^{-200\pi(t-0.1)}u(t - 0.1)]
\]

\[
20\delta(t - 0.1) \leftrightarrow 20\delta(t - 0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t - 0.1)
\]

• The Complete solution by superposition is:

\[
y(t) = 0 + 20\delta(t - 0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t - 0.1) + \frac{40}{\sqrt{2}} \cos(200\pi t + 0.55\pi)
\]
Frequency Response to a Sum of Cosine Inputs

- We can extend this to the case when
  \[ x(t) = \sum A_k \cos(\omega_k t + \phi_k) \]
- By superposition
  \[ y_k(t) = AM_k \cos(\omega_k t + \phi_k + \phi_k) \]
- where the frequency response is
  \[ H_k(j\omega) = M_k e^{j\phi_k} \]
- Then:
  \[ y(t) = \sum AM_k \cos(\omega_k t + \phi_k + \phi_k) \]
- This model can also be used when analyzing periodic input signals since a Fourier Series can be generated which has the similar form: \[ x(t) = \sum A_k \cos(k\omega_o t + \phi_k) \] where \( \omega_o \) is the fundamental frequency.
Homework

• Exercises:
  – 10.1-10.3

• Problems:
  – 10.1, 10.2,
  – 10.4 Use Matlab to plot the frequency response and submit your code
  – Using unit step functions, construct a single pulse of magnitude 10 starting at $t=5$ and ending at $t=10$.
  – Repeat with 2 pulses where the second is of magnitude 5 starting at $t=15$ and ending at $t=25$. 