Frequency Response

Lecture #12
Chapter 10
Ideal Filters

• We want to study $H(j\omega)$ functions which provide frequency selectivity such as:
  – Low Pass
  – High Pass
  – Band Pass

• However, we will look at ideal filtering, that is, filter which have ideal performance but are very difficult to construct.
A simple Filter – Ideal Delay

- Ideal Delay Filter => $y(t) = x(t - t_d)$: the output is same as the input except shifted in time by an amount $t_d$ seconds.
- The impulse response is just $h(t) = \delta(t - t_d)$
- The frequency response is then

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t - t_d)e^{-j\omega t} dt$$

$$= e^{-j\omega t_d}$$

Alternatively, if $x(t) = e^{j(\omega t + \phi)}$, then

$$y(t) = x(t - t_d) = e^{j[\omega(t-t_d) + \phi]} = e^{-j\omega t_d} e^{j(\omega t + \phi)}$$

But $y(t) = H(j\omega)e^{j(\omega t + \phi)}$

And therefore,

$$H(j\omega) = e^{-j\omega t_d}$$

- The Frequency Response of an Ideal Delay filter has a constant magnitude with a phase that is linear with frequency
- Therefore, it does not affect the magnitude of the input. It only effects the phase by an amount of $-\omega t_d$
Example

A signal of the form \( x(t) = 10e^{j\pi/4}e^{j200\pi t} \) is input to an ideal delay filter with delay of 0.001 sec.

The frequency response is : \( H(j\omega) = e^{-j\omega0.001} \)

\( H(j200\pi) = e^{-j200\pi(0.001)} \)

Then the output signal becomes :

\[
y(t) = H(j\omega)x(t) = H(j200\pi)10e^{j\pi/4}e^{j200\pi t}
= e^{-j200\pi(0.001)}10e^{j\pi/4}e^{j200\pi t} = e^{-j0.2\pi}10e^{j\pi/4}e^{j200\pi t}
= 10e^{j(200\pi + \frac{\pi}{4} - 0.2\pi)} = 10e^{j(200\pi + 0.05\pi)}
\]

Or rewritten as : \( y(t) = 10e^{j[200\pi(t-0.001) + \frac{\pi}{4}]} \)
Ideal Low Pass Filter

• This filter only passes frequencies below a value $\omega_{co}$ and attenuates all frequencies above $\omega_{co}$.
• We call $\omega_{co}$ the cutoff frequency.
• Therefore, the frequency response of a low pass filter is:

$$H_{lp}(j\omega) = \begin{cases} 
1 & |\omega| \leq \omega_{co} \\
0 & |\omega| > \omega_{co}
\end{cases}$$
Ideal High Pass Filter

- This filter only passes frequencies above a value $\omega_{co}$ and attenuates all frequencies below $\omega_{co}$.
- We call $\omega_{co}$ the cutoff frequency.
- Therefore, the frequency response of a high pass filter is:

$$H_{hp}(j\omega) = \begin{cases} 
1 & |\omega| \geq \omega_{co} \\
0 & |\omega| < \omega_{co}
\end{cases}$$
Ideal Band Pass Filter

- This filter only passes frequencies above a value $\omega_{co1}$ and below a value $\omega_{co2}$ and attenuates all other frequencies outside this range.

- We call $\omega_{co1}$ the lower (or low) cutoff frequency and $\omega_{co2}$ the upper (or high) cutoff frequency.

- Therefore, the frequency response of a bandpass filter is:

$$H_{bp}(j\omega) = \begin{cases} 
0 & |\omega| < \omega_{co1} \\
1 & \omega_{co1} \leq |\omega| \leq \omega_{co2} \\
0 & |\omega| > \omega_{co2}
\end{cases}$$
Application of Ideal Filters

- We will apply a band pass filter to a periodic square wave filter out its fundamental frequency.
- Let our input signal have a period of $T_o = 500\mu s$ or $f_o = 2\text{kHz} \Rightarrow \omega_o = 2\pi(2000)$ rad/sec and its form over one period is:

$$x(t) = \begin{cases} 
2 & 0 \leq t < \frac{T_o}{2} \\
0 & \frac{T_o}{2} \leq t < T_o 
\end{cases}$$
Application of Ideal Filters

- Since $x(t)$ is a period, let’s calculate the Fourier series for to decompose the input into its frequency components.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_o} \int_0^{T_o/2} 2 e^{-jk\omega_0 t} dt$$

$$= 2 \frac{1}{T_o (-j k \omega_0)} e^{-jk\omega_0 T_o/2} \mid_{0}^{T_o/2}$$

$$= \frac{1}{jk\pi} [1 - e^{-j\pi}]$$

Recall that $e^{-jk\pi} = \cos k\pi - j \sin k\pi = 1$ for even values of $k$

or $e^{-jk\pi} = (-1)^k$ for odd values of $k$

$$a_0 = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) dt = \frac{1}{T_o} \int_0^{T_o/2} 2 dt = 1$$

$$a_k = \frac{1}{jk\pi} [1 - e^{-jk\pi}] = \frac{1}{jk\pi} [1 - (-1)^k] = \frac{2}{jk\pi} = \frac{2}{k\pi} e^{-j\pi/2} \quad \text{for odd values of } k$$

$$= 0 \quad \text{for even values of } k$$

Diagram: A line graph with values at $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$.
Application of Ideal Filters

- Now let’s apply an ideal band pass filter with low frequency cutoff of 1,250 Hz and high frequency cutoff of 2,750 Hz which has a bandwidth of 1500 Hz and is centered around 2000 Hz which is the fundamental frequency of this square wave.
Application of Ideal Filters

If the filter is LTI, then the output signal is also periodic with same fundamental frequency. Therefore, $y(t)$ can be written as a Fourier Series.

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_o t}$$

By superposition of these complex exponential signals at $k\omega_o$

$$b_k = H(jk\omega_o) a_k$$

But since $H(j\omega)$ is only defined for $1250 \leq \omega \leq 2750$, then only the terms for which $|k| = 1$ will be left upon this multiplication

$$b_k = H(jk\omega_o) a_k = \frac{2}{\pi k} e^{-j\pi/2} \quad \text{for } |k| = 1$$

$$y(t) = b_1 e^{j2\pi(2000)t} + b_{-1} e^{-j2\pi(2000)t} = \frac{2}{\pi(1)} e^{-j\pi/2} e^{j2\pi(2000)t} + \frac{2}{\pi(-1)} e^{-j\pi/2} e^{-j2\pi(2000)t}$$

$$= \frac{2}{\pi} (e^{j(2\pi(2000)t - \pi/2)} - e^{j(2\pi(2000)t + \pi/2)})$$

$$= \frac{2}{\pi} (e^{j(2\pi(2000)t - \pi/2)} + e^{j\pi} e^{-j(2\pi(2000)t + \pi/2)})$$

$$= \frac{2}{\pi} (e^{j(2\pi(2000)t - \pi/2)} + e^{-j(2\pi(2000)t - \pi/2)})$$

$$= \frac{4}{\pi} \cos(2\pi(2000)t - \pi/2)$$
Time Domain or Frequency Domain

• We have seen that a LTI can be represented by its impulse response in the time domain and by its frequency response in the frequency domain.

• In general when working with sinusoids (or complex exponentials) either single or summed signals, it is easier to work in the Frequency Domain.

• If the signal consists of impulses, step functions, or other non-sinusoidal signals (e.g., signals which are progressive integrations of the impulse function), convolution of the impulse response (Time Domain) is usually easiest.
An Example

• An LTI system has an impulse response of
  \[ h(t) = \delta(t) - 200\pi e^{-200\pi t}u(t) \]

• The following signal is applied:
  \[ x(t)=10+20\delta(t - 0.1)+40\cos(200\pi t+0.3\pi) \text{ for all } t \]

• The input has 3 parts: a constant, an impulse and a cosine wave. We will take each part separately and use the easiest method to find the solution.
An Example

• Let’s first find the frequency response of the system from the impulse response:

\[
H(j\omega) = \int_{-\infty}^{\infty} [\delta(t) - 200\pi e^{-200\pi t} u(t)] e^{-j\omega t} dt
\]

\[
= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt - 200\pi \int_{-\infty}^{\infty} e^{-200\pi t} u(t) e^{-j\omega t} dt
\]

\[
= 1 - 200\pi \int_{0}^{\infty} e^{-(200\pi + j\omega)t} dt = 1 + \frac{200\pi}{200\pi + j\omega} e^{-(200\pi + j\omega)t} \bigg|_{0}^{\infty}
\]

\[
= 1 - \frac{200\pi}{200\pi + j\omega} = \frac{j\omega}{200\pi + j\omega}
\]
An Example

• Now let’s take the first (constant, $\omega = 0$) part and the third (cosine) part and evaluate the solution using the frequency response:

The first part of the input: $10$

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$$10 \mapsto H(j0)10 = \frac{j0}{200\pi + j0}10 = 0$$

The third part of the input: $40\cos(200\pi t + 0.3\pi)$

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$$40\cos(200\pi t + 0.3\pi) \mapsto 40|H(j200\pi)|\cos[200\pi t + 0.3\pi + \angle H(j200\pi)]$$

$$H(j200\pi) = \frac{j200\pi}{200\pi + j200\pi} = \frac{j}{1+j} = \frac{\frac{\pi}{4}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \angle \frac{\pi}{4}$$

$$40\cos(200\pi t + 0.3\pi) \mapsto 40\frac{1}{\sqrt{2}} \cos[200\pi t + 0.3\pi + 0.25\pi] = \frac{40}{\sqrt{2}} \cos[200\pi t + 0.55\pi]$$
An Example

• Now for the second part of the input (the impulse function), we will apply the impulse response:

The second part of the input: \(20\delta(t - 0.1)\)

\[20\delta(t - 0.1) \leftrightarrow 20h(t - 0.1) = 20[\delta(t - 0.1) - 200\pi e^{-200\pi(t-0.1)}u(t - 0.1)]\]

\[20\delta(t - 0.1) \leftrightarrow 20\delta(t - 0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t - 0.1)\]

• The Complete solution by superposition is:

\[y(t) = 0 + 20\delta(t - 0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t - 0.1)\]

\[+ \frac{40}{\sqrt{2}} \cos(200\pi t + 0.55\pi)\]
Homework

• Exercises:
  – 10.4-10.7

• Problems:
  – 10.5, 10.6,
  – 10.7 Use Matlab to plot $x(t)$; show your code