Computing

Lecture #13
Chapter 13
What Is this Course All About?

• To Gain an Appreciation of the Various Types of Signals and Systems
• To Analyze The Various Types of Systems
• To Learn the Skills and Tools needed to Perform These Analyses.
• To Understand How Computers Process Signals and Systems
What did we learn so far

• Learned about Signals and Systems
  – Continuous-time vs Discrete-time
  – Sinusoids
  – Complex Exponentials
  – Periodic Signals
• How to analyze them
  – Sampling
  – Time Domain
  – Frequency Domain
• How to Process Them
  – Filters
What do we still have left to learn

• How does a Computer handle signals?
  – Sinusoids
  – Bio Med Signals

• The Computer can’t handle Continuous-time signals

• The Computer must first sample the signal
Some more Background

• We saw that the Fourier Series can be used to handle any periodic signal since it can be decomposed into frequency components.

• But most signals are not periodic
  – ECG, EEG, EMG, etc.
  – Voice Signals
  – Video Signals
Fourier Transform

- We can handle non-periodic signals in a similar fashion as periodic signals.
- That is, we can decompose them into frequency components.
- However, the way we get there is different than the way we use for periodic signals.
- Periodic signal frequency decomposition use the Fourier Series which generates a frequency spectrum.
- Non-periodic signal frequency decomposition use the Fourier Transform which generates a frequency DENSITY spectrum.
Fourier Analysis and Fourier Transform

• Recall this is Fourier Series
\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t} = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \phi_k) \] where \( A_0 = a_0 \); \( a_k = \frac{1}{2} A_k e^{j\phi_k} \); \( a_{-k} = a_k^* \); \( f_o = \frac{1}{T_o} \)

\[ a_k = \frac{1}{T_o} \int_{0}^{T_o} x(t) e^{-j(\frac{2\pi}{T_0})kt} \, dt \]

• Here’s what the Fourier Transform looks like for continuous signals, CTFT:
\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \]
\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega \]
Fourier Transforms vs. Fourier Series

• Note that $X(j\omega)$ is a Spectral Density function; that is, if $x(t)$ is voltage, then $X(j\omega)$ is volts/rad.
  – Note in Fourier series analysis, $a_k$ would also be volts is $x(t)$ is voltage.
• Note that $X(j\omega)$ is a continuous function of $\omega$ and the limits of integration are over all values of $t$.
  – Note in Fourier series analysis, $a_k$ is a discrete function of $kf_o$ (and are $f_o$ Hz apart) and the limits of summation are over one period of $x(t)$
• A proof of how $X(j\omega)$ is formulated is beyond our scope but, briefly, $X(j\omega)$ can be obtained by starting with the Fourier Series of $x(t)$ (as if it were periodic) and letting $f_o$ go to zero (i.e., $T_o$ goes to infinity which make the second repetition of $x(t)$ move to infinity and makes $x(t)$ non-periodic.
  – This would make the spectral components move closer to each other (infinitely closer – make $2\pi kf_o$ a continuous variable $\omega$)
  – This will also make $a_k$ approach zero but the ratio of $a_k/f_o$, which is a spectral density, remains finite.
Discrete-time Fourier Transform

• If this is the continuous-time Fourier Transform
  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]

• Then replacing \( t \) with \( nT_s \) and the integral with a summation, then the Discrete-Time Fourier Transform, DTFT, can be shown to be:
  \[ X(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s} \]
  \[ X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \]
  Recalling that \( \hat{\omega} = \omega T_s \)
Discrete-time Fourier Transform

• Note that this looks very similar to the Frequency Response of a system

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]

\[ H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} \]

• As a matter of fact, the Fourier Transform of the Impulse response is the Frequency Response

\[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = \sum_{k=0}^{M} h[k]e^{-j\omega k} = H(e^{j\omega}) \]
Discrete Fourier Transform

- The DTFT yields a spectrum which is a continuous function of $\hat{\omega}$
  \[ X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \]
- How do we get around this? Sample the spectrum.

When we sampled in the time domain, we replaced $t$ by $nT_s$ where $T_s$ is the distance (in time) between samples.
Therefore to sample in the frequency domain we replace $\omega = 2\pi f$ by $2\pi kf_\Delta$ where $f_\Delta$ is the distance (in frequency) between spectrum samples.

Note that since $\hat{\omega} = \omega T_s = \frac{\omega}{f_s} \Rightarrow \hat{\omega} = \frac{2\pi kf_\Delta}{f_s}$

\[ X(e^{j\hat{\omega}}) = X[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi kf_\Delta}{f_s}n} \]

- Let us assume that there are only $L$ samples for time domain and $N$ samples for the spectrum.

\[ X[k] = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi kf_\Delta}{f_s}n} \]

- Since $f_s$ is the maximum frequency in the spectrum, then $f_\Delta = \frac{f_s}{N}$. This is just the resolution of the displaced spectrum.

\[ X[k] = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi kf_\Delta}{f_s}n} = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k}{N}n} = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi kn}{N}} \]

- This is called the Discrete Fourier Transform
Discrete Fourier Transform

- Since the computer can only process discrete functions of finite time, we have to define a new Fourier Transform called the Discrete Fourier Transform, DFT.
  - Do not confuse this with the Discrete-time Fourier Transform, DTFT.
- It is defined as

  \[ X(k) = \sum_{n=0}^{L-1} x[n] e^{-j \frac{2\pi kn}{N}} \]

  where there are the \( L \) samples of \( x[n] \),
  we evaluate the Spectrum over \( N \) frequencies, i.e., \( 0 \leq k \leq N - 1 \),
  and each frequency is \( f_\Delta \) apart and chose \( f_\Delta = \frac{f_s}{N} \)
  since \( f_s \) is the maximum frequency of the spectrum.

Therefore, \( f_\Delta = \frac{f_s}{N} = \frac{1}{NT_s} \). We call this the resolution of the spectrum.
Discrete Fourier Transform

Let's start with the DTFT: \( X(e^{j\hat{\omega}}) = \sum_{-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}; \hat{\omega} = \omega T_s \)

Let's divide the spectrum is into \( N \) frequencies equally spaced \( f_\Delta \) Hz apart (i.e., we are sampling the spectrum).

![Diagram of frequency samples](image)

Therefore, let's define the \( k \)th sample in the frequency domain as \( \omega_k = 2\pi f_x = 2\pi kf_\Delta \) where \( k \) goes from 1 to \( N \).

When \( k = N \), the highest frequency in the spectrum is \( \omega_N = \frac{2\pi N}{T_o} = 2\pi Nf_\Delta = 2\pi f_s \).
If \( f_s \) meets the Nyquist rate, then the one-sided spectrum of \( x[n] = X(e^{j\hat{\omega}}) \) must end at or below \( \frac{f_s}{2} \).

Therefore, \( \hat{\omega}_k = \omega_k T_s = 2\pi k f_s T_s = \frac{2\pi k f_s}{N} T_s = \frac{2\pi k}{N} \).

Let's substitute \( \hat{\omega}_k \) for \( \hat{\omega} \) in the DTFT: 

\[
X(e^{j\hat{\omega}_k}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}_k n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi k}{N} n}
\]

This sum will only be a function of \( k \). In addition, let's assume that there are \( L \) samples of \( x[n] \).

Then, we have the Discrete Fourier Transform, DFT as 

\[
X[k] = X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k}{N} n}
\]
Computer Processing

• Computers use the DFT to determine the spectrum of a signal $x(t)$.
• There are different computer algorithms for processing the DFT
  – The most widely used algorithm is called the Fast Fourier Transform: FFT
• Note that the DFT is just like a discrete Fourier Series in the Frequency Domain

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{j \frac{2\pi}{T_o} kt} \quad \Leftrightarrow \quad X[k] = \sum_{n=0}^{L-1} x[n] e^{-j \frac{2\pi}{N} kn}$$
How to Evaluate the DFT
Method 1: Expand $n$ first, then $k$.

- What is the DFT for the $x[n]=\{1, 1, 0, 0\}$ assuming $N=4$

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4} kn} = \sum_{n=0}^{3} x[n] e^{-j\frac{\pi}{2} kn}$$

$$X[0] = 1 + 1e^{-j\frac{\pi0}{2}} = 2$$

$$X[1] = 1 + 1e^{-j\frac{\pi1}{2}} = 1 - j = \sqrt{2}e^{-j\frac{\pi}{4}}$$

$$X[2] = 1 + 1e^{-j\frac{\pi2}{2}} = 1 + 1e^{-j\pi} = 1 - 1 = 0$$

$$X[3] = 1 + 1e^{-j\frac{\pi3}{2}} = 1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$X[k] = \{2, \sqrt{2}e^{-j\frac{\pi}{4}}, 0, \sqrt{2}e^{j\frac{\pi}{4}}\}$$
Method 2: Expand \( k \) first, then \( n \).

- What is the DFT for the \( x[n]=\{1, 1, 0, 0\} \) assuming \( N=4 \)

\[
X[k] = \sum_{n=0}^{3} x[n]e^{-j2\pi kn/N}
\]

\[
X[0] = \sum_{n=0}^{3} x[n]e^{-j2\pi 0n/N}
\]

\[
= x[0]e^{-j0} + x[1]e^{-j0} + x[2]e^{-j0} + x[3]e^{-j0}
\]

\[
= 1 + 1 + 0 + 0 = 2
\]

\[
X[1] = \sum_{n=0}^{3} x[n]e^{-j2\pi 1n/N}
\]

\[
= x[0]e^{-j0} + x[1]e^{-j\pi} + x[2]e^{-j\pi} + x[3]e^{-j\pi}
\]

\[
= 1 + (-j) + 0 + 0 = \sqrt{2}e^{-j\pi/4}
\]

\[
X[2] = \sum_{n=0}^{3} x[n]e^{-j2\pi 2n/N}
\]

\[
= x[0]e^{-j\pi} + x[1]e^{-j\pi} + x[2]e^{-j\pi} + x[3]e^{-j3\pi}
\]

\[
= 1 + (-1) + 0 + 0 = 0
\]

\[
X[3] = \sum_{n=0}^{3} x[n]e^{-j2\pi 3n/N}
\]

\[
= x[0]e^{-j\pi} + x[1]e^{-j\pi} + x[2]e^{-j\pi} + x[3]e^{-j9\pi/2}
\]

\[
= 1 + j + 0 + 0 = \sqrt{2}e^{-j\pi/4}
\]

\[
X[k] = \{2, \sqrt{2}e^{-j\pi/4}, 0, \sqrt{2}e^{j\pi/4} \}
\]
Another Example: Method 1

• What is the DFT for the $x[n] = \{1, 1, 1, 0, 0, 0\}$ assuming $N=6$

$$X[k] = \sum_{n=0}^{5} x[n] e^{-\frac{j2\pi kn}{6}} = \sum_{n=0}^{5} x[n] e^{-\frac{j\pi kn}{3}}$$

$$= x[0] e^{-\frac{j\pi k0}{3}} + x[1] e^{-\frac{j\pi k1}{3}} + x[2] e^{-\frac{j\pi k2}{3}} + x[3] e^{-\frac{j\pi k3}{3}} + x[4] e^{-\frac{j\pi k4}{3}} + x[5] e^{-\frac{j\pi k5}{3}}$$

$$= 1 + e^{-\frac{j\pi k}{3}} + e^{-\frac{j\pi k2}{3}} + 0e^{-\frac{j\pi k3}{3}} + 0e^{-\frac{j\pi k4}{3}} + 0e^{-\frac{j\pi k5}{3}}$$

$$= 1 + e^{-\frac{j\pi k}{3}} + e^{-\frac{j\pi k2}{3}}$$

$$X[0] = 1 + e^{-\frac{j\pi 0}{3}} + e^{-\frac{j\pi 02}{3}} = 3$$

$$X[1] = 1 + e^{-\frac{j\pi 1}{3}} + e^{-\frac{j\pi 2}{3}} = 1 + 0.5 - j0.86 - 0.5 - j0.86 = 1 - j2(0.86) = 2e^{-\frac{j\pi}{3}}$$

$$X[2] = 1 + e^{-\frac{j\pi 2}{3}} + e^{-\frac{j\pi 4}{3}} = 1 - 0.5 - j0.86 + 0.5 + j0.86 = 0$$

$$X[3] = 1 + e^{-\frac{j\pi 3}{3}} + e^{-\frac{j\pi 6}{3}} = 1 - 1 + 1 = 1$$

$$X[4] = 1 + e^{-\frac{j\pi 4}{3}} + e^{-\frac{j\pi 8}{3}} = 0$$

$$X[5] = 1 + e^{-\frac{j\pi 5}{3}} + e^{-\frac{j\pi 10}{3}} = 1 + 0.5 + j0.86 - 0.5 + j0.86 = 2e^{\frac{j\pi}{3}}$$
Method 2

- What is the DFT for the \( x[n] = \{1, 1, 1, 0, 0, 0\} \) assuming \( N=6 \)

\[
X[k] = \sum_{n=0}^{5} x[n] e^{-j \frac{2\pi}{6} kn} = \sum_{n=0}^{2} e^{-j \frac{\pi}{3} kn} = 1 + e^{-j \frac{\pi}{3} k1} + e^{-j \frac{\pi}{3} k2} = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}}
\]

\[
X[0] = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 3
\]

\[
X[1] = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 2e^{-j \frac{\pi}{3}}
\]

\[
X[2] = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{4\pi}{3}} = 0
\]

\[
X[3] = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 1 + e^{-j \pi} + e^{-j 2\pi} = 1
\]

\[
X[4] = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 1 + e^{-j \frac{4\pi}{3}} + e^{-j \frac{8\pi}{3}} = 0
\]

\[
X[5] = 1 + e^{-j \frac{\pi}{3}} + e^{-j \frac{2\pi}{3}} = 1 + e^{-j \frac{5\pi}{3}} + e^{-j \frac{10\pi}{3}} = 2e^{j \frac{\pi}{3}}
\]
What is the right hand side of the FFT.vi spectrum?

- Note that \( N \) and \( L \) are usually taken to be the same in order to easily calculate \( X[k] \).

- Note that due to this fact \( X[N - k] = X[-k] = X^*[k] \)

\[
X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}
\]

\[
X[N - k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(N-k)n} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}Nn} e^{j\frac{2\pi}{N}kn}
\]

\[
= \sum_{n=0}^{N-1} x[n] e^{-j2\pi n} e^{j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}kn}
\]

\[
= X[-k] = X^*[k]
\]

- Therefore, \( X[-k] \) will show up as \( X[N - k] \)
Take a look at our Example

- What is the DFT for the $x[n] = \{1, 1, 0, 0\}$ assuming $N=4$

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4}kn}$$

$$X[-1] = \sum_{n=0}^{3} x[n] e^{j\frac{2\pi}{4}1n}$$

$$= x[0]e^{-j0} + x[1]e^{j\frac{\pi}{2}} + x[2]e^{j\pi} + x[3]e^{j\frac{3\pi}{2}}$$

$$= 1 + (j) + 0 + 0 = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$= X[4-1] = X[3]$$
Another Example

Here we have a signal whose digitized frequency is $\hat{\omega}_o$

$$x_1[n] = e^{j(\hat{\omega}_o n + \phi)} \quad \text{for } n = 0, 1, 2, \ldots, N - 1$$

We now want to obtain the spectrum of our signal using the DFT. If the DFT is done correctly, we would expect a single component at $\hat{\omega}_o$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} = \sum_{n=0}^{N-1} e^{j(\hat{\omega}_o n + \phi)} e^{-j (2\pi / N) kn}$$

$$= e^{j\phi} \sum_{n=0}^{N-1} e^{- j [(2\pi k / N) - \hat{\omega}_o] n}$$
Another Example Continued

Using the partial sum of a geometric series: \[ \sum_{k=0}^{L-1} \alpha^k = \frac{1 - \alpha^L}{1 - \alpha} \]

\[ X[k] = e^{j\phi} \left( \frac{1 - e^{-j[(2\pi k/N) - \hat{\omega}_o]N}}{1 - e^{-j(2\pi k/N) - \hat{\omega}_o}} \right) = e^{j\phi} \left( \frac{e^{-j[(2\pi k/N) - \hat{\omega}_o]N/2} (e^{j[(2\pi k/N) - \hat{\omega}_o]N/2} - e^{-j[(2\pi k/N) - \hat{\omega}_o]N/2})}{e^{-j[(2\pi k/N) - \hat{\omega}_o]/2} (e^{j[(2\pi k/N) - \hat{\omega}_o]/2} - e^{-j[(2\pi k/N) - \hat{\omega}_o]/2})} \right) \]

\[ = e^{j\phi} \left( \frac{e^{-j[(2\pi k/N) - \hat{\omega}_o]N/2}}{\sin([2\pi k/N - \hat{\omega}_o]N/2)} \right) \]

\[ = e^{j\phi} e^{-j[(2\pi k/N) - \hat{\omega}_o](N-1)/2} \frac{\sin([2\pi k/N - \hat{\omega}_o]N/2)}{\sin([2\pi k/N - \hat{\omega}_o]/2)} = e^{j\phi} e^{-j[(2\pi k/N) - \hat{\omega}_o](N-1)/2} D_N(e^{j[(2\pi k/N) - \hat{\omega}_o]N}) \]

where

\[ D_N(e^{j[(2\pi k/N) - \hat{\omega}_o]N}) = \frac{\sin([2\pi k/N - \hat{\omega}_o]N/2)}{\sin([2\pi k/N - \hat{\omega}_o]/2)}, \] the Dirichlet function
Another Example Continued

\[ X[k] = e^{j\phi} e^{-j[(2\pi k/N) - \omega_o](N-1)/2} \frac{\sin(\[(2\pi k/N) - \omega_o\]N / 2)}{\sin(\[(2\pi k/N) - \omega_o\] / 2)} \]

\[ \omega_o = 2\pi k_o / N \Rightarrow \omega_o = \omega_o f_s = \frac{2\pi k_o}{N} f_s \Rightarrow f_0 = \frac{k_o}{N} f_s \]

Case 1: \( k_o \) is NOT an integer;
that is, \( \omega_o \) is NOT a multiple of \( 2\pi / N \) and therefore \( f_0 \) is NOT a multiple of \( f_s \),
which is the resolution of the sampled spectrum obtained using DFT.
\[ \therefore k - k_o \neq 0 \text{ for any value of } k \]

\[ = e^{j\phi} e^{-j[(2\pi k/N) - 2\pi k_o/N](N-1)/2} \frac{\sin(\[(2\pi k/N) - 2\pi k_o/N\]N / 2)}{\sin(\[(2\pi k/N) - 2\pi k_o/N\] / 2)} \]

\[ = e^{j\phi} e^{-j[2\pi N(k-k_o)(N-1)/2]} \frac{\sin(\[2\pi / N (k-k_o)\]N / 2)}{\sin(\[2\pi / N (k-k_o)\] / 2)} \]

\[ = e^{j\phi} e^{-j[2\pi / N (k-k_o)(N-1)/2]} \frac{\sin[\pi (k-k_o)]}{\sin[\pi / N (k-k_o)]}; \text{ for all } k \]
Two Cases for our Example

say let $k_o = 2.5$ and $N = 40$; $\hat{\omega}_o = \frac{2\pi(2.5)}{40} = 2.5 \frac{\pi}{20} = \frac{\pi}{8} = 0.125\pi$

If we do this, the figure below shows that do not have a single component at $\hat{\omega}_o$

$k_o = 2.5; N = 40$

\[
|X[k]| = \left| \frac{\sin(\pi(k - 2.5))}{\sin(\pi/40(k - 2.5))} \right|
\]

\[
|X[0]| = \left| \frac{\sin(\pi(0 - 2.5))}{\sin(\pi/40(0 - 2.5))} \right| = 5.1
\]

\[
|X[1]| = \left| \frac{\sin(\pi(1 - 2.5))}{\sin(\pi/40(1 - 2.5))} \right| = \left| \frac{\sin(\pi(-1.5))}{\sin(\pi/40(-1.5))} \right| = 8.5
\]

\[
|X[2]| = \left| \frac{\sin(\pi(2 - 2.5))}{\sin(\pi/40(2 - 2.5))} \right| = \left| \frac{\sin(\pi(-0.5))}{\sin(\pi/40(-0.5))} \right| = 25.5
\]

\[
|X[3]| = \left| \frac{\sin(\pi(3 - 2.5))}{\sin(\pi/40(3 - 2.5))} \right| = \left| \frac{\sin(\pi(0.5))}{\sin(\pi/40(0.5))} \right| = 25.5
\]

\[
|X[4]| = \left| \frac{\sin(\pi(4 - 2.5))}{\sin(\pi/40(4 - 2.5))} \right| = \left| \frac{\sin(\pi(1.5))}{\sin(\pi/40(1.5))} \right| = 8.5
\]

\vdots

etc.
Another Example Continued

\[ X[k] = e^{j\phi} e^{-j((2\pi k/N - \hat{\omega}_o)(N-1)/2)} \frac{\sin([2\pi k/N - \hat{\omega}_o]N/2)}{\sin([2\pi k/N - \hat{\omega}_o]/2)} \]

Case 2: \( k \) is an integer; that is, \( \hat{\omega}_o \) is an integer multiple of \( 2\pi/N \) and \( f_o \) is a multiple of \( f_s \), the resolution of the sampled spectrum obtained from the DFT.

\[ \therefore k - k_o \text{ will be a non-zero integer } l \text{ except when } k = k_o \]

\[ = e^{j\phi} e^{-j([2\pi k/N - 2\pi k_o/N](N-1)/2)} \frac{\sin([2\pi k/N - 2\pi k_o/N]N/2)}{\sin([2\pi k/N - 2\pi k_o/N]/2)} \]

\[ = e^{j\phi} e^{-j([2\pi/N(k-k_o)](N-1)/2)} \frac{\sin([2\pi/N(k-k_o)]N/2)}{\sin([2\pi/N(k-k_o)]/20)} \]

\[ = e^{j\phi} e^{-j([2\pi/N(k-k_o)](N-1)/2)} \frac{\sin[\pi(k-k_o)]}{\sin[\pi/N(k-k_o)]} = e^{j\phi} e^{-j(2\pi l/N)(N-1)/2} \frac{\sin[\pi l]}{\sin[\pi l/N]} \]

\[ = 0 \quad k \neq k_o \text{ since the } \sin(\pi l) = 0 \]

\[ = Ne^{j\phi} \quad k = k_o \ (l = 0) \text{ since } X[k_o] = e^{j\phi}1 \ 0; \]

using L'Hopital's rule \( \lim_{l \to 0} \frac{\sin[\pi l]}{\sin[\pi l/N]} = \lim_{l \to 0} \frac{\pi \cos[\pi l]}{N \cos[\pi l/N]} = N \).
Two Cases for our Example

Case 2: when the fundamental frequency of our signal is a multiple of \( \frac{f_s}{N} \), which is the resolution of the sampled spectrum obtained from the DFT.

that is say \( k_o = 2 \) and \( N = 40 \);
\[ \hat{\omega}_o = \frac{2\pi(2)}{40} = \frac{2\pi}{20} = 0.1\pi \]

If we do this, the figure below shows that we have a single component at \( \hat{\omega}_o \) or at \( k = k_o = 2 \).

In order words: \( X[k] = Ne^{i\hat{\omega}k} \delta(k - 2) \) which corresponds to \( 2\times2\pi / 40 = 0.1\pi = \hat{\omega}_o \)
Time Domain & Frequency Domain

- Time Domain

**LTI System**

CT: Differential Eq

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = d\dot{x}(t) + e\times(t)$$

DT: Difference Eq

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Input
Time signal
CT: $x(t)$
DT: $x[n]$

Output
Time Signal
CT: $y(t)$
DT: $y[n]$
Time Domain & Frequency Domain

Time Domain
• Impulse Response

LTI System
CT: Differential Eq
DT: Difference Eq

Input
Impulse signal
CT: δ(t)
DT: δ[n]

Output
Impulse Response
CT: h(t)
DT: h[n]

CT: \( a\dot{y}(t) + b\dot{y}(t) + cy(t) = d\dot{x}(t) + ex(t) \)
DT: \( y[n] = \sum_{k=0}^{M} b_k x[n-k] \)
Time Domain & Frequency Domain

- Time Domain

LTI System
CT: Differential Eq
\[ a\ddot{y}(t) + b\dot{y}(t) + cy(t) = d\dot{x}(t) + ex(t) \]

DT: Difference Eq
\[ y[n] = \sum_{k=0}^{M} b_k x[n-k] \]

Convolution
CT: \[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]
DT: \[ y[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l] \]

Input
Time signal
CT: \( x(t) \)
DT: \( x[n] \)

Output
Time signal
CT: \( y(t) \)
DT: \( y[n] \)
Time Domain & Frequency Domain

- Frequency Domain

**LTI System Frequency Response**

**CT:**

\[ H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt = M(\omega)e^{-j\phi(\omega)} \]

**DT:**

\[ H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} = M(\hat{\omega})e^{-j\phi(\hat{\omega})} \]

**Input Signal Spectrum**

\[ CT: x(t) = A_0 + \sum_{k=0}^{M} A_k \cos(2\pi f_k t + \theta_k) \]

\[ DT: x[n] = A_0 + \sum_{k=0}^{M} A_k \cos(\omega_k T_n + \theta_k) \]

\[ = A_0 + \sum_{k=0}^{M} A_k \cos(\hat{\omega}_k n + \theta_k) \]

**Output Signal Spectrum**

\[ CT: y(t) = A_0 M(0)e^{-j\psi(0)} + \sum_{k=0}^{M} A_k M(2\pi f_k) \cos(2\pi f_k t + \theta_k - \psi(2\pi f_k)) \]

\[ DT: y[n] = A_0 M(0)e^{-j\psi(0)} + \sum_{k=0}^{M} A_k M(\hat{\omega}_k) \cos(\hat{\omega}_k n + \theta_k - \psi(\hat{\omega}_k)) \]
Fourier Transform
Signals and Systems

Signals

• The FT of a signal transforms it from the time domain, \( x(t) \) or \( x[n] \), to the frequency domain to yield its spectral representation, \( X(j\omega) \) or \( X[k] \).

• The inverse FT transforms the signal’s spectrum, \( X(j\omega) \) or \( X[k] \), in the frequency domain to the time domain, \( x(t) \) or \( x[n] \).

• The FT can also be applied to systems.

Systems

• The FT of a system’s impulse response, \( h(t) \) or \( h[n] \), transforms it into the frequency response, \( H(j\omega) \) or \( H[k] \).

• The inverse FT transforms a system’s frequency response, \( H(j\omega) \) or \( H[k] \) to the impulse response, \( h(t) \) or \( h[n] \).

• Note that the impulse response is really the output signal of a system when the input signal is the impulse function, \( \delta(t) \) or \( \delta[n] \).
Fourier Transform
Signals and Systems
The Math is the Same

**Signals**
- For continuous signals, its spectral representation is:
  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]
- For discrete signals, its spectral representation is discrete frequency function of \( k \) and is calculated as:
  \[ X[k] = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k n}{N}} \]

**Systems**
- For systems which process continuous signals, the frequency response is calculated as:
  \[ H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} \, dt \]
- For systems which process discrete signals, the frequency response is a continuous frequency function of \( \hat{\omega} \) and is calculated as:
  \[ H(j\hat{\omega}) = H(e^{j\hat{\omega}}) = \sum_{k=0}^{M-1} h[k]e^{-j\hat{\omega}k} = \sum_{k=0}^{M-1} b_k e^{-j\hat{\omega}k} \]
Frequency Response is a continuous function

$$H(e^{j \hat{\omega}}) = \sum_{k=0}^{M-1} b_k e^{-j k \hat{\omega}}$$

$$b_k = \{1, 0, 0, 1\}$$

$$H(e^{j \hat{\omega}}) = \sum_{k=0}^{3} b_k e^{-j k \hat{\omega}} = 1 + 0 + 0 + e^{-j 3 \hat{\omega}} = 1 + e^{-j 3 \hat{\omega}}$$

$$= e^{-j \frac{3 \hat{\omega}}{2}} (e^{-j \frac{3 \hat{\omega}}{2}} + e^{j \frac{3 \hat{\omega}}{2}}) = 2 \cos \left( \frac{3 \hat{\omega}}{2} \right) e^{-j \frac{3 \hat{\omega}}{2}}$$
Fourier Transform
Signals and Systems
The Math is the Same

Signal Spectral Density is a discrete function

\[
X[k] = \sum_{n=0}^{L-1} x[n] e^{-j \frac{2\pi k n}{N}}
\]

\[x[n] = \{1, 0, 0, 1\}\]

\[
X[k] = \sum_{n=0}^{3} x[n] e^{-j \frac{2\pi k n}{4}} = 1 + 0 + 0 + e^{-j \frac{\pi k}{2}} = 1 + e^{-j \frac{\pi k}{2}}
\]

\[
X[k] = 1 + e^{-j \frac{3\pi k}{2}}
\]

\[
X[0] = 1 + e^{-j \frac{3\pi 0}{2}} = 2
\]

\[
X[1] = 1 + e^{-j \frac{3\pi}{2}} = 1 + j = \sqrt{2} e^{j \frac{\pi}{4}}
\]

\[
X[2] = 1 + e^{-j \frac{3\pi}{2}} = 1 - 1 = 0
\]

\[
X[3] = 1 + e^{-j \frac{3\pi}{2}} = 1 + j = \sqrt{2} e^{j \frac{\pi}{4}}
\]

OR

\[
X[k] = \sum_{n=0}^{3} x[n] e^{-j \frac{2\pi k n}{4}} = 1 + 0 + 0 + e^{-j \frac{\pi k}{2}} = 1 + e^{-j \frac{\pi k}{2}}
\]

\[
= e^{-j \frac{3\pi k}{4}} \left( e^{j \frac{3\pi k}{4}} + e^{-j \frac{3\pi k}{4}} \right) = 2 \cos \left( \frac{3\pi k}{4} \right) e^{-j \frac{3\pi k}{4}}
\]

\[
X[k] = 2 \cos \left( \frac{3\pi k}{4} \right) e^{-j \frac{3\pi k}{4}}
\]

\[
X[0] = 2 \cos \left( \frac{3\pi 0}{4} \right) e^{-j \frac{3\pi 0}{4}} = 2
\]

\[
X[1] = 2 \cos \left( \frac{3\pi}{4} \right) e^{-j \frac{3\pi}{4}} = -\sqrt{2} e^{-j \frac{3\pi}{4}} = \sqrt{2} e^{-j \frac{3\pi}{4}} e^{j \pi} = \sqrt{2} e^{j \frac{\pi}{4}}
\]

\[
X[2] = 2 \cos \left( \frac{3\pi}{2} \right) e^{-j \frac{3\pi}{2}} = 0
\]

\[
X[3] = 2 \cos \left( \frac{9\pi}{4} \right) e^{-j \frac{9\pi}{4}} = 2 \cos \left( \frac{\pi}{4} \right) e^{-j \frac{\pi}{4}} = \sqrt{2} e^{-j \frac{\pi}{4}}
\]
# Time Domain to Frequency Domain Transformations

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### Fourier Series

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T} t} dt$$

### Continuous Time Fourier Transform CTFT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

### Discrete Time Fourier Transform DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

### Discrete Fourier Transform DFT

$$X[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi nk}{N}}$$
Homework

- **Exercises:**
  - 13.2- 13.4, 13.6 – 13.8

- **Problems:**
  - 13.1 Use Matlab to plot the spectrum; submit your code
  - 13.4, 13.5, 13.6