Sinusoids

Lecture #2

Chapter 2

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.
- To Understand How Computers Process Signals and Systems

Sinusoidal Signal

- Sinusoidal Signals are **periodic** functions which are based on the sine or cosine function from trigonometry.
- The general form of a Sinusoidal Signal

$$x(t)=A\cos(\omega_o t+\phi)$$

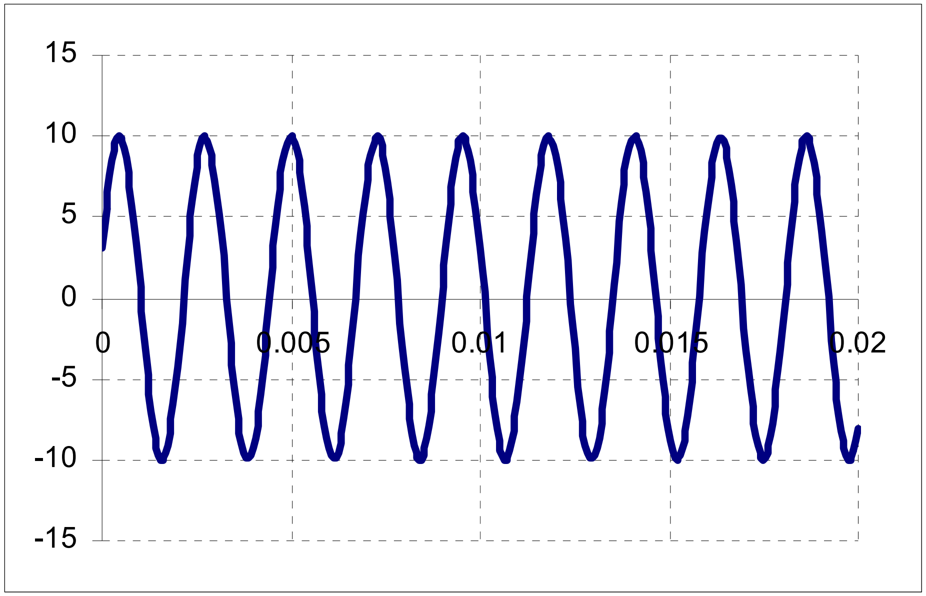
Or

$$x(t)=A\cos(2\pi f_o t+\phi)$$

- where cos (·) represent the cosine function
  - We can also use sin(·), the sine function
- $\omega_o t+\phi$ or $2\pi f_o t+\phi$ is angle (in radians) of the cosine function
  - Since the angle depends on time, it makes $x(t)$ a signal
- ω_o is the radian frequency of the sinusoidal signal
  - f_o is called the cyclical frequency of the sinusoidal signal
- ϕ is the phase shift or phase angle
- A is the amplitude of the signal

Example

$x(t) = 10\cos(2\pi(440)t - 0.4\pi)$

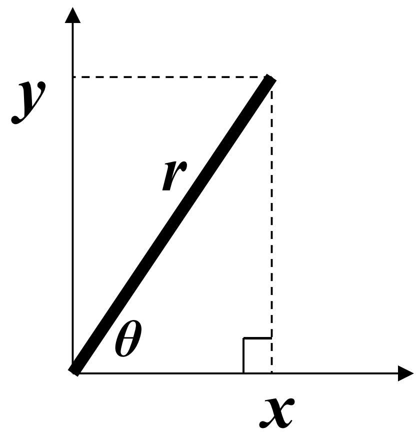


One cycle takes 1/440 = .00227 seconds

This is called the period, T, of the sinusoid and is equal to the inverse of the frequency, f

Sine and Cosine Functions

- Definition of sine and cosine



$$\sin\theta = \frac{y}{r}$$

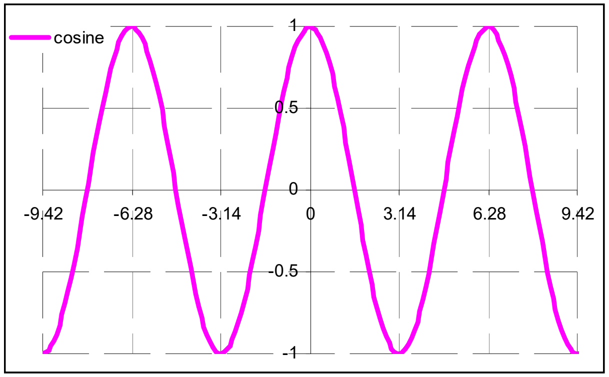
$$\Rightarrow y = r\sin\theta$$

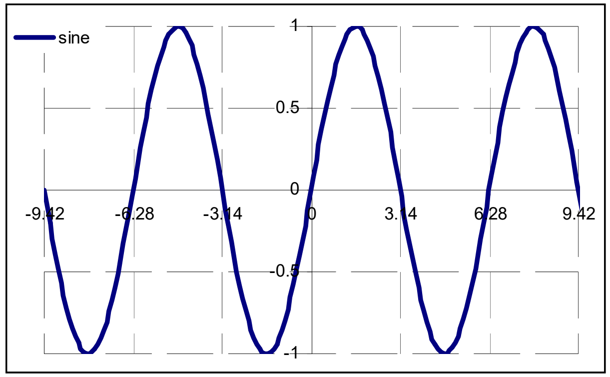
$$\cos\theta = \frac{x}{r}$$

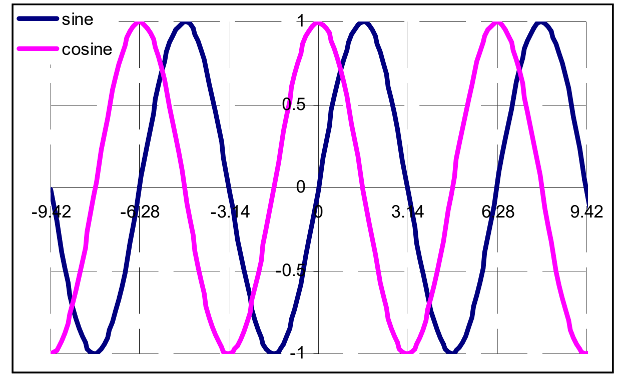
$$\Rightarrow x = r\cos\theta$$

- Depending upon the quadrant of θ the sine and cosine function changes
  - As the θ increases from 0 to $\pi/2$, the cosine decreases from 1 to 0 and the sine increases from 0 to 1
  - As the θ increases beyond $\pi/2$ to π, the cosine decreases from 0 to -1 and the sine decreases from 1 to 0
  - As the θ increases beyond π to $3\pi/2$, the cosine increases from -1 to 0 and the sine decreases from 0 to -1
  - As the θ increases beyond $3\pi/2$ to 2π, the cosine increases from 0 to 1 and the sine increases from -1 to 0

Properties of Sinusoids







Property	Equation
Equivalence	$\sin\theta = \cos(\theta - \pi/2)$ or $\cos\theta = \sin(\theta + \pi/2)$
Periodicity	$\cos(\theta + 2\pi k) = \cos\theta$ or $\sin(\theta + 2\pi k) = \sin\theta$ where k is an integer
Evenness of cosine	$\cos\theta = \cos(-\theta)$
Oddness of sine	$\sin\theta = -\sin(-\theta)$
Zeros of sine	$\sin\pi k = 0$, when k is an integer
Zeros of cosine	$\cos[\pi(k+1)/2] = 0$, when k is an even integer; odd multiples of $\pi/2$
Ones of the cosine	$\cos 2\pi k = 1$, when k is an integer; even multiples of π
Ones of the sine	$\sin[\pi(k+1/2)] = 1$, when k is an even integer; alternate odd multiples of $\pi/2$
Negative ones of the cosine	$\cos[2\pi(k+1)/2] = -1$, when k is an integer; odd multiples of π
Negative ones of the sine	$\sin[\pi(k+1/2)] = -1$, when k is an odd integer; alternate odd multiples of $3\pi/2$

Properties of Sinusoids

K	(K+1)/2	X pi()	cosine
0	0.5	1.571	0
1	1	3.142	-1
2	1.5	4.712	0
3	2	6.283	1
4	2.5	7.854	0
5	3	9.425	-1
6	3.5	10.996	0
7	4	12.566	1
8	4.5	14.137	0

K	K+1/2	X pi()	sine
0	0.5	1.571	1
1	1.5	4.712	-1
2	2.5	7.854	1
3	3.5	10.996	-1
4	4.5	14.137	1
5	5.5	17.279	-1
6	6.5	20.420	1
7	7.5	23.562	-1
8	8.5	26.704	1

Identities and Derivatives

Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2 \sin\theta \cos\theta$
4	$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
5	$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
6	$\cos a \cos b = [\cos(a + b) + \cos(a - b)]/2$
7	$\sin a \sin b = [\cos(a - b) - \cos(a + b)]/2$
8	$\cos^2\theta = [1 + \cos 2\theta]/2$
9	$\sin^2\theta = [1 - \cos 2\theta]/2$
10	$d \sin\theta / d\theta = \cos\theta$
11	$d \cos\theta / d\theta = -\sin\theta$

Sinusoidal Signal

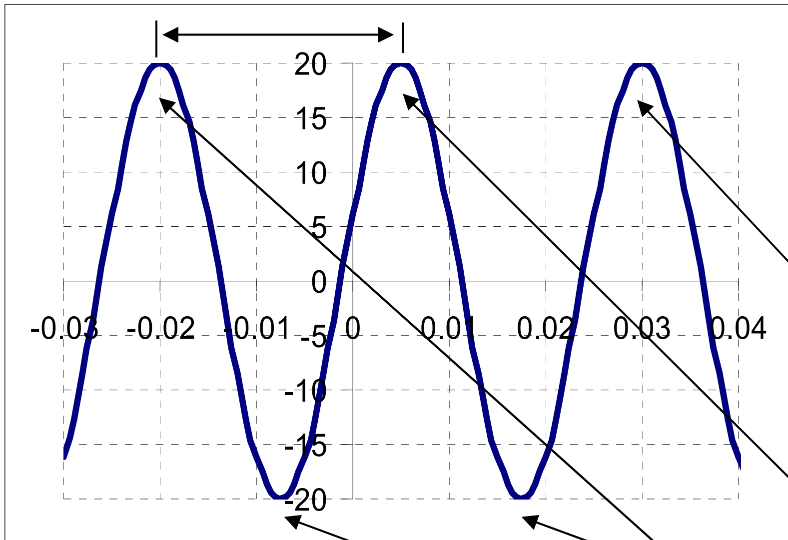
- The general form of a Sinusoidal Signal

$$x(t)=A\cos(\omega_o t+\theta) = A\cos(2\pi f_o t + \theta)$$

  - $\omega_o = 2\pi f_o$ is the radian frequency of the sinusoidal signal
    - Since $\omega_o t$ has units of radians which is dimensionless, ω_o has units of rad/sec
  - f_o is called the cyclical frequency of the sinusoidal signal
    - Has units of sec^{-1} or Hz (formerly, cycles per second)
  - θ is the phase shift or phase angle
    - Has units of radians
  - A is the amplitude of the signal and is the scaling factor that determines how large the signal will be.
    - Since the cosine function varies from -1 to +1 then our signal will vary from $-A$ to $+A$.
    - A is sometimes called the peak of the signal and $2A$ is called the peak-to-peak value

Sinusoid Signals

$x(t) = 20\cos(2\pi(40)t - 0.4\pi)$



$A = 20,\ \omega_o = 2\pi(40),\ f_o = 40,\ \theta = -0.4\pi$

Maxima at $2\pi(40)t - 0.4\pi = 2\pi k$ *or when* $t = \ldots, -0.02, 0.005, 0.03, \ldots$

Minima at $2\pi(40)t - 0.4\pi = 2\pi(k+1)0.5$ *or when* $t = \ldots -0.0075, 0.0175$

Time Period ($1/f_o$) between = $0.005 - (-0.02) = 0.025$ *sec*

Relation of Period to Frequency

- Period of a sinusoid, T_o, is the length of one cycle and

$$T_o = 1/f_o$$

- The following relationship must be true for all Signals which are periodic (not just sinusoids)

$$x(t + T_o) = x(t)$$

- So

$$A \cos(\omega_o(t + T_o) + \theta) = A \cos(\omega_o t + \theta)$$
$$A \cos(\omega_o t + \omega_o T_o + \theta) = A \cos(\omega_o t + \theta)$$

Relation of Period to Frequency Continued

- Since a sinusoid is periodic in 2π, this means:

$$\text{since } T_o = 1/f_o$$

$$\text{Then } \omega_o T_o = 2\pi f_o T_o$$

$$\omega_o T_o = 2\pi$$

$$\text{therefore, } T_o = 2\pi/\omega_o$$

$$A\cos(\omega_o t + \omega_o T_o + \theta) = A\cos(\omega_o t + \theta)$$

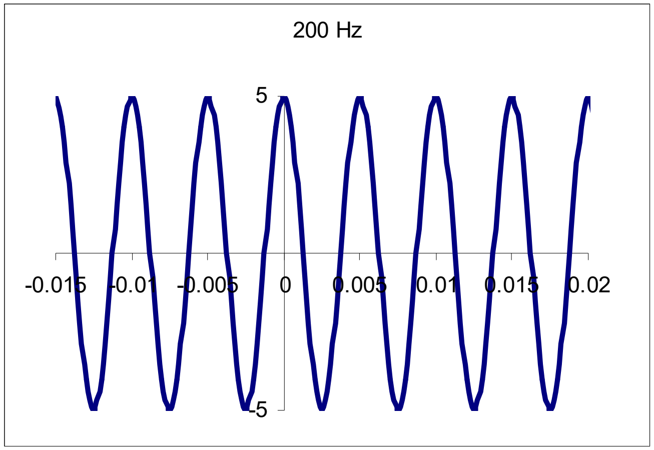
$$A\cos(\omega_o t + 2\pi + \theta) = A\cos(\omega_o t + \theta)$$

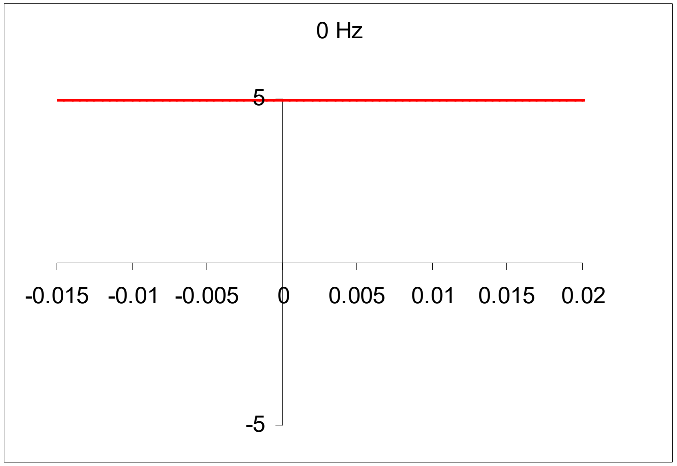
- The period is in units of seconds
- The frequency is in units of sec^{-1} or Hz (formerly, cycles per second)

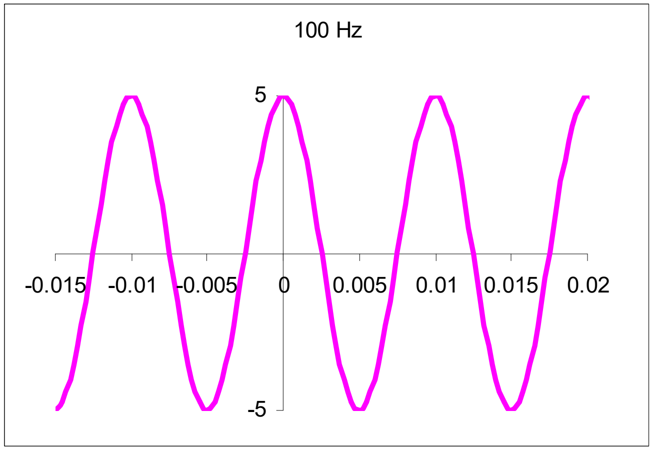
Frequencies

$$A\cos(2\pi f_o t + \theta)$$

for 200 Hz, 100 Hz, 0 Hz







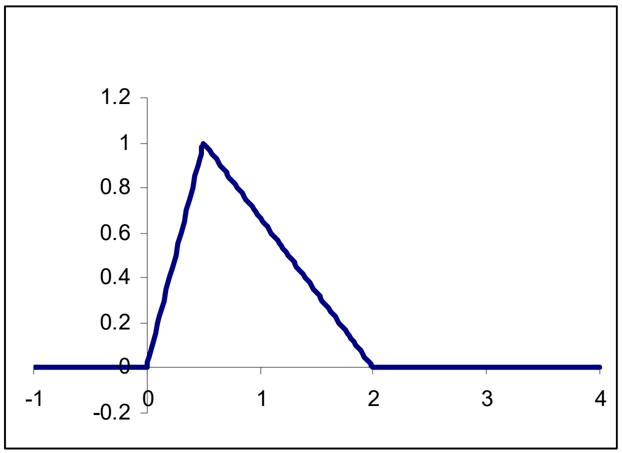
Phase Shift and Time Shift

- The phase shift parameter θ (with frequency) determines the time locations of the maxima and minima of the sinusoid.
- When $\theta = 0$, then for positive peak at $t = 0$.
- When $\theta \neq 0$, then the phase shift determines how much the maximum is shifted from $t = 0$.
- However, delaying a signal by t_1 seconds, also shifts its waveform.

Time Shifting

- Look at the following waveform:

$$
s(t) = \begin{cases} = 2t & 0 \le t \le \frac{1}{2} \\ = \frac{1}{3}(4-2t) & \frac{1}{2} \le t \le 2 \\ = 0 & \text{elsewhere} \end{cases}
$$

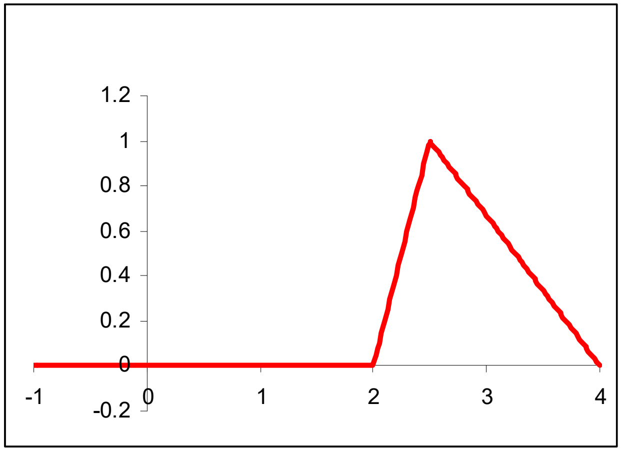


Time Shifting Continued

- Now let's time shifted it by 2 seconds (delay),

$$x(t)=s(t - 2)$$

$$x(t) = \begin{cases} = 2(t - 2) \Rightarrow 2t - 4 & 0 \le (t-2) \le \frac{1}{2} \Rightarrow 2 \le t \le 2\frac{1}{2} \\ \frac{1}{3}(4-2(t-2)) \Rightarrow \frac{1}{3}(8-2t) & \frac{1}{2} \le (t-2) \le 2 \Rightarrow 2\frac{1}{2} \le t \le 4 \\ = 0 & \text{elsewhere} \end{cases}$$



Time Shifting Continued

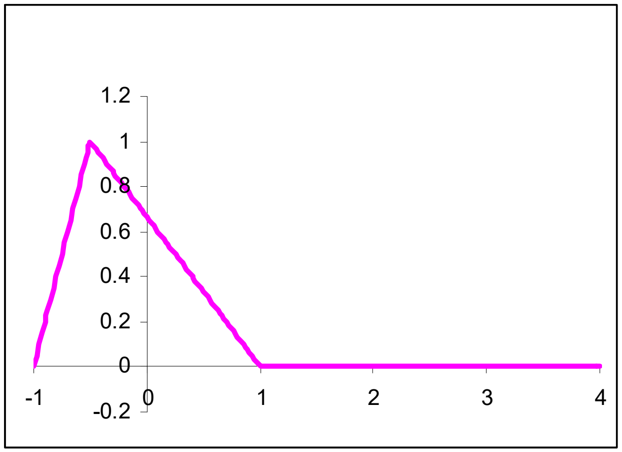
- Now let's time shifted it by -1 seconds (advance),

$$x(t)=s(t+1)$$

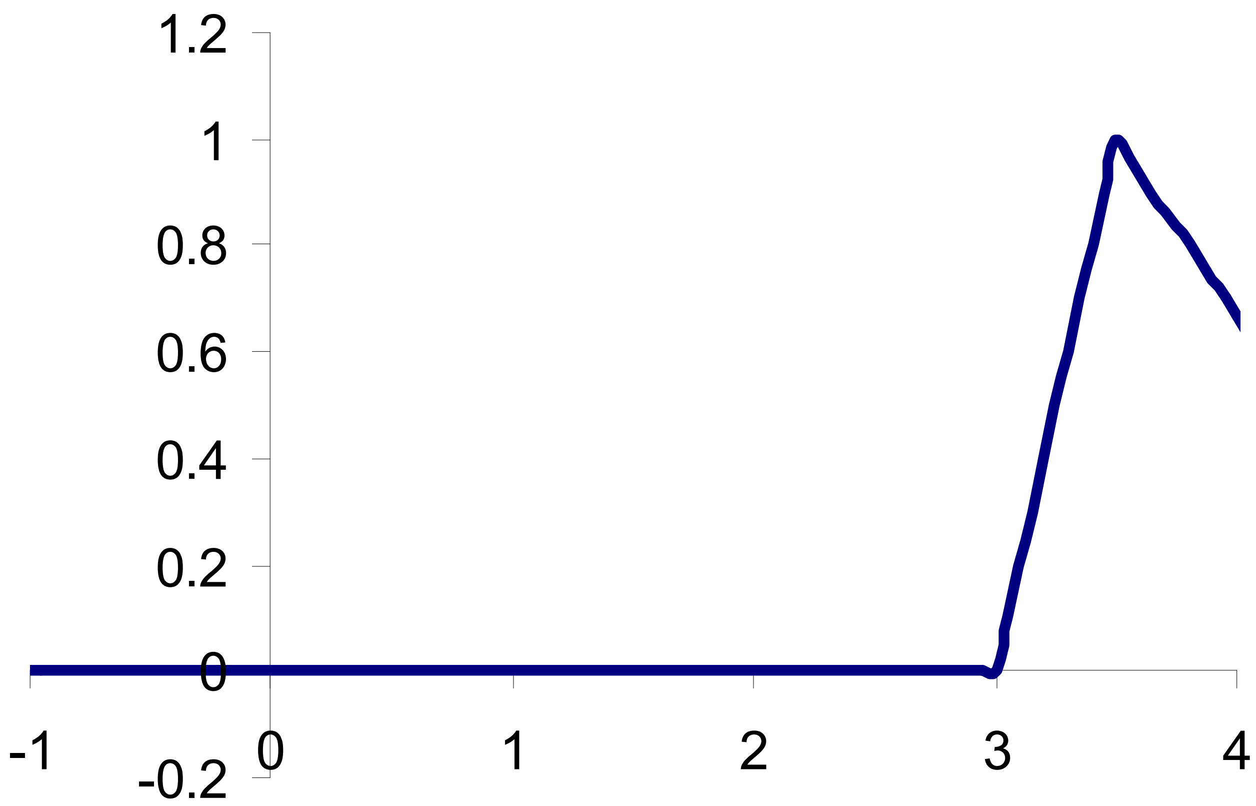
$$= 2(t+1) \Rightarrow 2t+2 \qquad 0 \le (t+1) \le \frac{1}{2} \Rightarrow -1 \le t \le -\frac{1}{2}$$

$$x(t) = \frac{1}{3}(4-2(t+1)) \Rightarrow \frac{1}{3}(2-2t) \qquad \frac{1}{2} \le (t+1) \le 2 \Rightarrow -\frac{1}{2} \le t \le 1$$

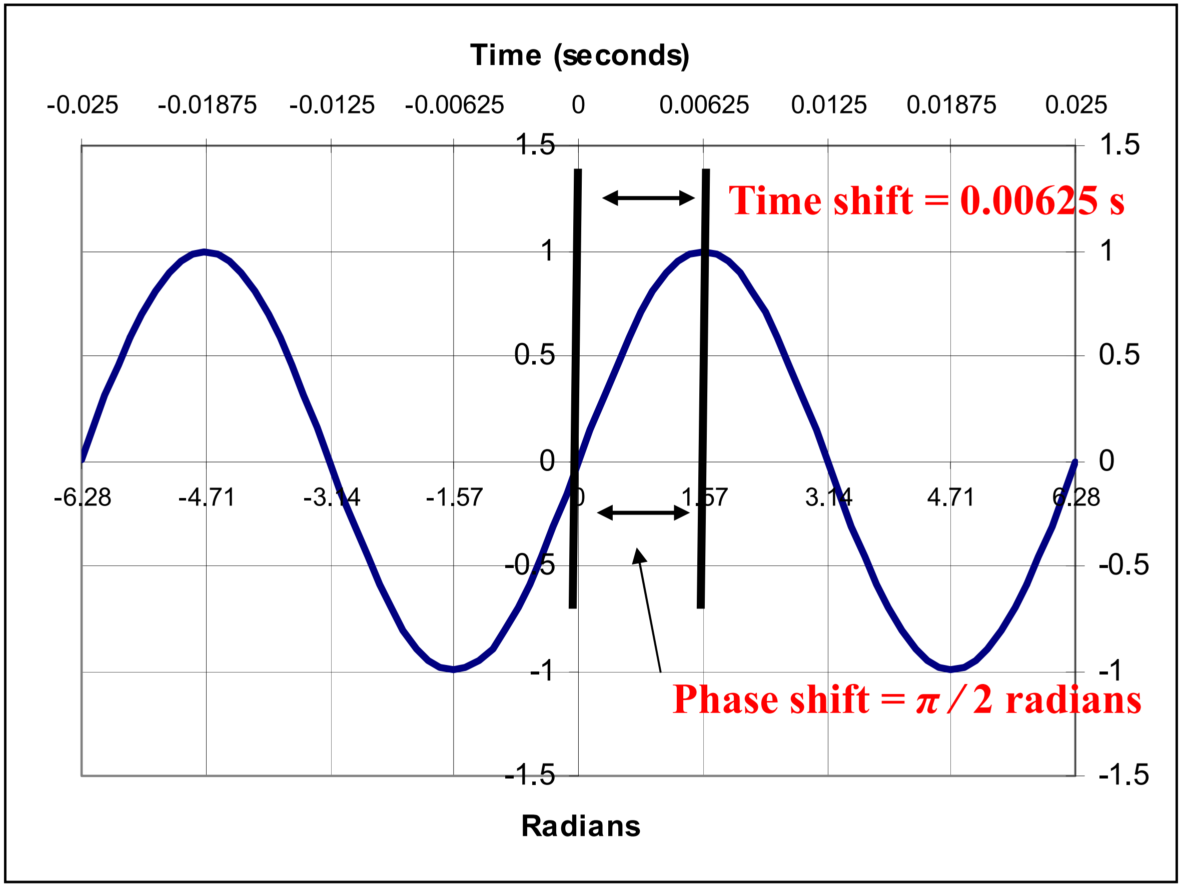
$$= 0 \qquad \text{elsewhere}$$



Time Shifting



Phase shift and Time Shift



$$x(t) = \cos(2\pi 40t - \frac{\pi}{2})$$

$$f = 40Hz;$$

$$T = \frac{1}{40} = 0.025 \text{ sec}$$

phase shift:

$$\theta = -\frac{\pi}{2}$$

time shift:

$$t_s = -\frac{-\frac{\pi}{2}}{2\pi 40} = \frac{1}{160} = 0.00625 \text{ sec}$$

$$x(t) = \cos(2\pi 40(t - 0.00625))$$

Phase and Time Shift

$$x(t\text{-}t_1) = A\cos(\omega_o(t\text{-}t_1)) = A\cos(\omega_o t + \theta)$$
$$A\cos(\omega_o t\text{-}\omega_o t_1) = A\cos(\omega_o t + \theta)$$
$$\omega_o t\text{-}\omega_o t_1 = \omega_o t + \phi$$
$$-\omega_o t_1 = \theta$$
$$t_1 = -\theta / \omega_o = -\phi / 2\pi f_o$$

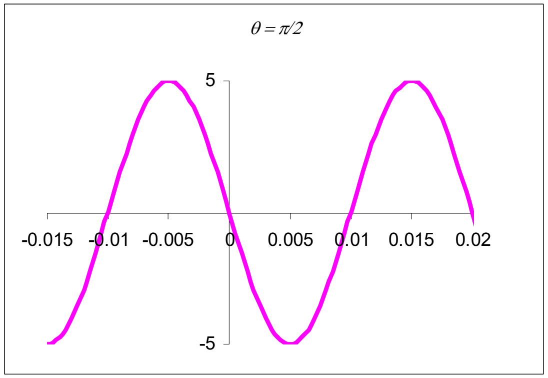
$$\theta = -2\pi f_o t_1 = -2\pi(t_1 / T_o)$$

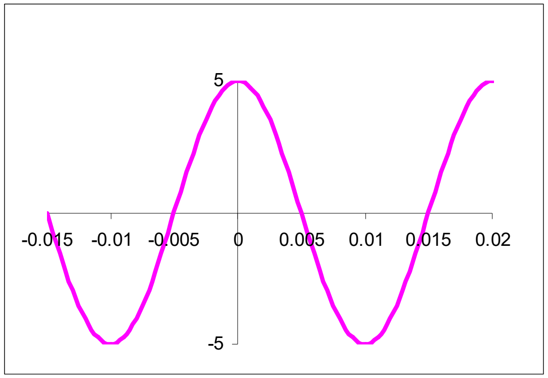
- Note that a positive (negative) value of t_1 equates to a delay (advance)
- And a a positive (negative) value of θ equates to an advance (delay)

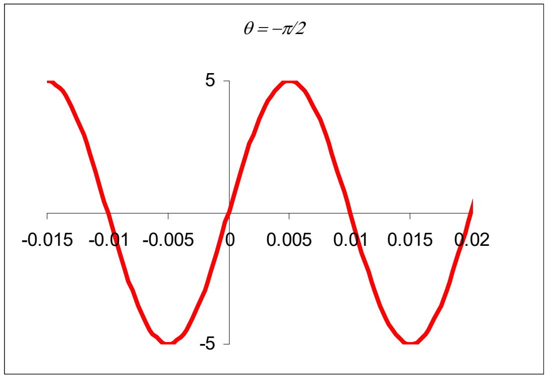
Phase and Time Shift

- Note that a positive (negative) value of t_1 equates to a delay (advance)
- And a a positive (negative) value of θ equates to an advance (delay)

$$x(t) = 5\cos(2\pi\, 50t + \theta)$$
$$\theta = \pi / 2; -\pi / 2$$
$$t_1 = -\pi / 2 / (2\pi\ 50) = -.005 \text{ sec};\ +.005 \text{ sec}$$

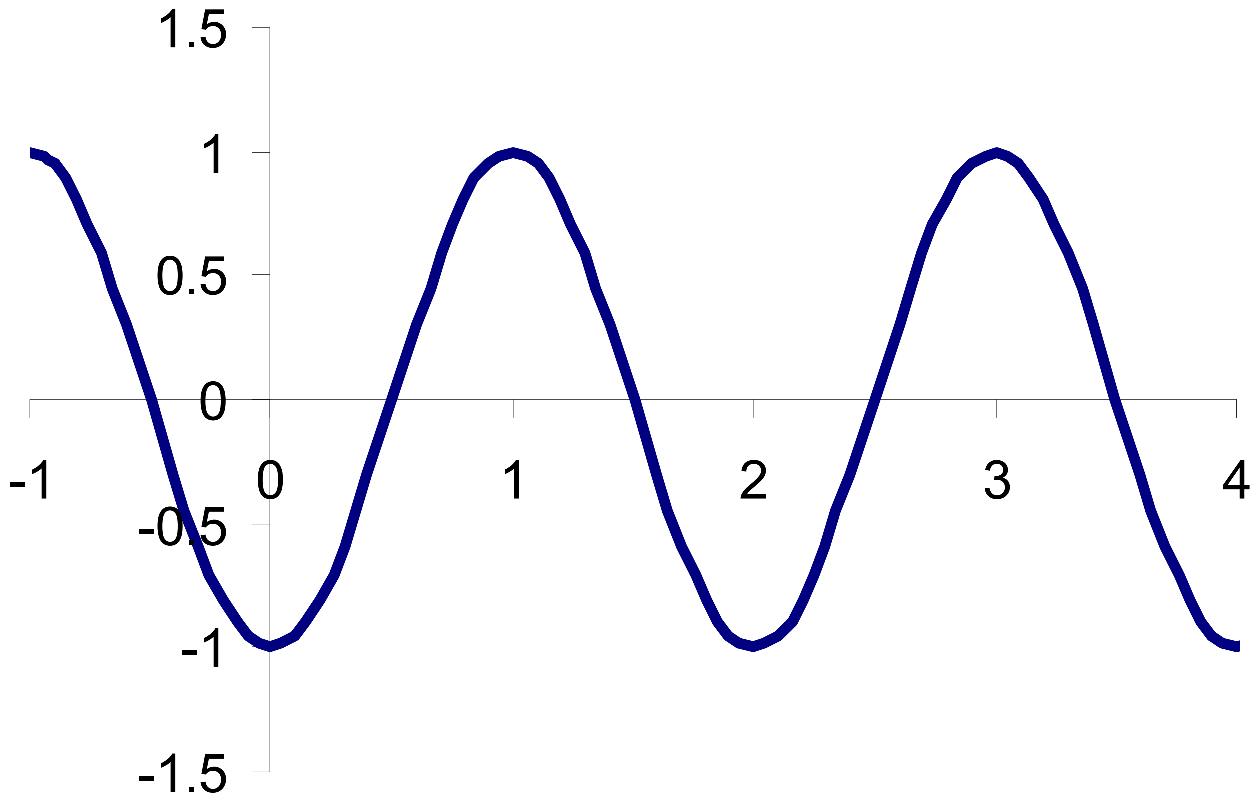






Time shifting

Shift = 0.5 seconds



Periodicity of Sinusoids

What happens when $\theta = 2\pi$?

- When $\theta = 2\ \pi$, then the sinusoidal waveform does not change since sinusoids are periodic in 2π
- Therefore, adding or subtracting multiple of 2π does not change the waveform
- This is called modulo 2π

Plotting Sinusoid Signals
Case #1: Delay

$$x(t) = 20\cos(2\pi(40)t - 0.4\pi) \quad A = 20, f_o = 40Hz, \theta = -0.4\pi$$

Basic Calculations

1)Amplitude is 20

2) Frequency $= 40Hz$

Calculation of the Period

$$T_o = \frac{1}{f_o} = \frac{1}{40}$$

$$= 0.025\sec = 25m\sec$$

3) Phase angle/shift = $-0.4\pi = -1.25664 radians$

Calculation of the time shift $= 2\pi f t_s + \theta = 0$

$$2\pi(40)t_s - 0.4\pi = 0$$

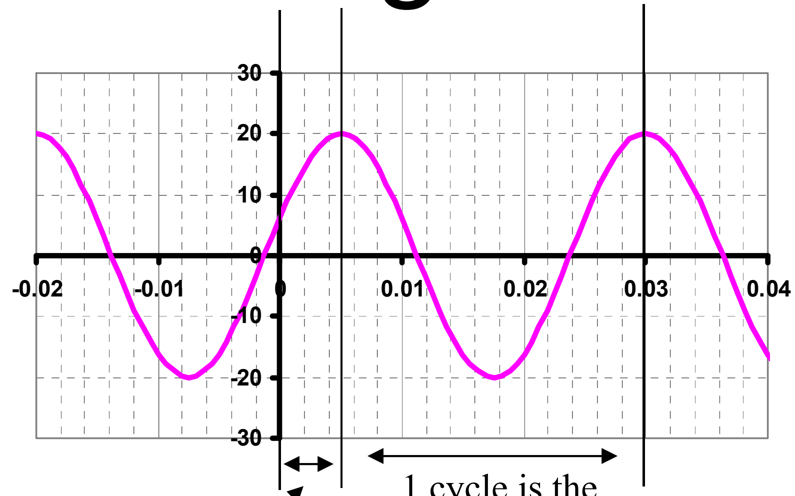
$$t_s = \frac{.4\pi}{2\pi(40)} = 0.005\sec = 5m\sec \text{ delay}$$

Note that $\dfrac{t_s}{T_o} = \dfrac{|5|m\sec}{25m\sec} = 0.2$

which also equals $\dfrac{|\text{phase angle}|}{2\pi} = \dfrac{|-0.4\pi|}{2\pi} = 0.2$

Plotting Sinusoid Signals

t	$2\pi ft + \theta$	$20\cos(2\pi ft + \theta)$
-0.0250	-7.54	6.18
-0.0225	-6.91	16.18
-0.0200	-6.28	20.00
-0.0175	-5.65	16.18
-0.0150	-5.03	6.18
-0.0125	-4.40	-6.18
-0.0100	-3.77	-16.18
-0.0075	-3.14	-20.00
-0.0050	-2.51	-16.18
-0.0025	-1.88	-6.18
0.0000	-1.26	6.18
0.0025	-0.63	16.18
0.0050	0.00	20.00
0.0075	0.63	16.18
0.0100	1.26	6.18
0.0125	1.88	-6.18
0.0150	2.51	-16.18
0.0175	3.14	-20.00
0.0200	3.77	-16.18
0.0225	4.40	-6.18
0.0250	5.03	6.18
0.0275	5.65	16.18
0.0300	6.28	20.00
0.0325	6.91	16.18
0.0350	7.54	6.18
0.0375	8.17	-6.18
0.0400	8.80	-16.18
0.0425	9.42	-20.00
0.0450	10.05	-16.18
0.0475	10.68	-6.18
0.0500	11.31	6.18



1 cycle is the Period = 0.03-0.005 = 0.025 sec

Time to the first (in positive time - delay) maximum is Time Shift = 0.005 sec => Negative Phase shift $\theta = -0.005*2\pi f = -0.4\pi$

Since the period is 0.025, chose to plot the graph with 10 points per cycle or a Δt of 0.0025 seconds.

Plotting Sinusoid Signals Case #1: No Delay

$$x(t) = 20\cos(2\pi(40)t) \quad A = 20, f_o = 40Hz, \theta = 0$$

Basic Calculations

1)Amplitude is 20

2) Frequency $= 40Hz$

Calculation of the Period

$$T_o = \frac{1}{f_o} = \frac{1}{40}$$

$$= 0.025\sec = 25m\sec$$

3) Phase angle/shift = 0 *radians*

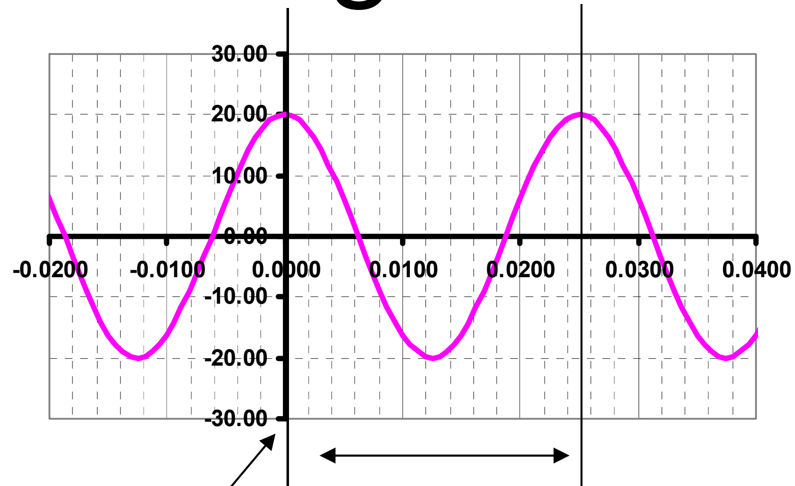
Calculation of the time shift $= 2\pi f t_s + \theta = 0$

$$2\pi(40)t_s - 0 = 0$$

$$t_s = \frac{0}{2\pi(40)} = 0\sec$$

Plotting Sinusoid Signals

t	$2\pi ft + \theta$	$20\cos(2\pi ft + \theta)$
-0.0250	-6.28	20.00
-0.0225	-5.65	16.18
-0.0200	-5.03	6.18
-0.0175	-4.40	-6.18
-0.0150	-3.77	-16.18
-0.0125	-3.14	-20.00
-0.0100	-2.51	-16.18
-0.0075	-1.88	-6.18
-0.0050	-1.26	6.18
-0.0025	-0.63	16.18
0.0000	0.00	20.00
0.0025	0.63	16.18
0.0050	1.26	6.18
0.0075	1.88	-6.18
0.0100	2.51	-16.18
0.0125	3.14	-20.00
0.0150	3.77	-16.18
0.0175	4.40	-6.18
0.0200	5.03	6.18
0.0225	5.65	16.18
0.0250	6.28	20.00
0.0275	6.91	16.18
0.0300	7.54	6.18
0.0325	8.17	-6.18
0.0350	8.80	-16.18
0.0375	9.42	-20.00
0.0400	10.05	-16.18
0.0425	10.68	-6.18
0.0450	11.31	6.18
0.0475	11.94	16.18
0.0500	12.57	20.00



1 cycle is the Period = 0.025 sec

Time to the first maximum is Time Shift = 0 sec => Phase shift of $\theta = 0$

Since the period is 0.025, chose to plot the graph with 10 points per cycle or a Δt of 0.0025 seconds.

Plotting Sinusoid Signals
Case #1: Advance

$$x(t) = 20\cos(2\pi(40)t + 0.6\pi) \quad A = 20, f_o = 40Hz, \theta = 0.6\pi$$

Basic Calculations

1) Amplitude is 20

2) Frequency $= 40Hz$

Calculation of the Period

$$T_o = \frac{1}{f_o} = \frac{1}{40}$$

$$= 0.025\sec = 25m\sec$$

3) Phase angle/shift $= +0.6\pi = 1.885 radians$

Calculation of the time shift $= 2\pi f t_s + \theta = 0$

$$2\pi(40)t_s + 0.6\pi = 0$$

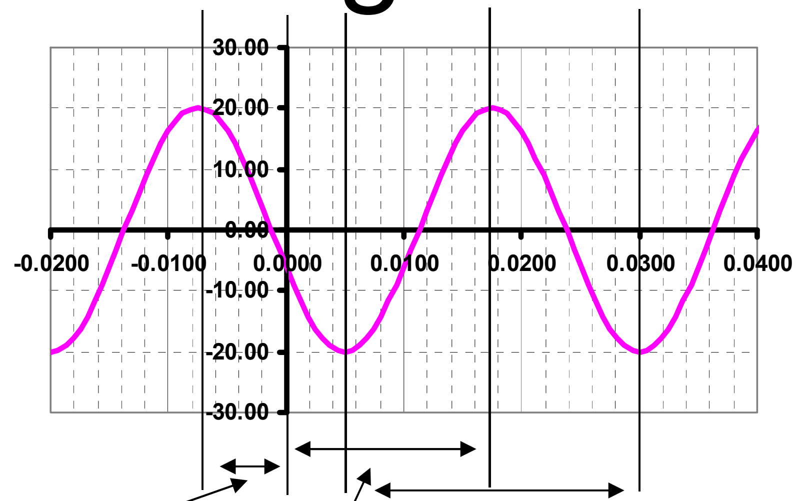
$$t_s = -\frac{.6\pi}{2\pi(40)} = -0.0075\sec = -7.5m\sec \text{ advance}$$

Note that $\dfrac{t_s}{T_o} = \dfrac{|-7.5|m\sec}{25m\sec} = 0.3$

which also equals $\dfrac{|\text{phase angle}|}{2\pi} = \dfrac{|+0.6\pi|}{2\pi} = 0.3$

Plotting Sinusoid Signals

t	$2\pi ft+\theta$	$20\cos(2\pi ft+\theta)$
-0.0250	-4.40	-6.18
-0.0225	-3.77	-16.18
-0.0200	-3.14	-20.00
-0.0175	-2.51	-16.18
-0.0150	-1.88	-6.18
-0.0125	-1.26	6.18
-0.0100	-0.63	16.18
-0.0075	0.00	20.00
-0.0050	0.63	16.18
-0.0025	1.26	6.18
0.0000	1.88	-6.18
0.0025	2.51	-16.18
0.0050	3.14	-20.00
0.0075	3.77	-16.18
0.0100	4.40	-6.18
0.0125	5.03	6.18
0.0150	5.65	16.18
0.0175	6.28	20.00
0.0200	6.91	16.18
0.0225	7.54	6.18
0.0250	8.17	-6.18
0.0275	8.80	-16.18
0.0300	9.42	-20.00
0.0325	10.05	-16.18
0.0350	10.68	-6.18
0.0375	11.31	6.18
0.0400	11.94	16.18
0.0425	12.57	20.00
0.0450	13.19	16.18
0.0475	13.82	6.18
0.0500	14.45	-6.18



Time to the first maximum (in <u>negative</u> time - advance) is Time Shift = -0.0075 sec => Positive Phase Shift

$\theta = 0.0075 * 2\pi f = +0.6\pi$

1 cycle is the Period = 0.03-0.005 = 0.025 sec

Time to the first maximum (in <u>positive</u> time - delay) is Time Shift = +0.0175 sec => Negative Phase Shift $\theta = 0.0075 * 2\pi f = -1.4$

Since the period is 0.025, chose to plot the graph with 10 points per cycle or a Δt of 0.0025 seconds.

Matlab Program to Plot Sinusoids

```
function cosineplotphase(Frequency,Amplitude,Phase,Points,Timestart,Timeend);
omega=2*pi*Frequency;
Period=1/Frequency;
phase=Phase;
Timedelta=Period/Points;
time = (Timestart:Timedelta:Timeend);
y=Amplitude*cos(omega*time+phase);
plot(time,y,'r');
title('Sinusoidal Plot');
xlabel('Seconds');
axis([ Timestart Timeend -1.1*Amplitude +1.1*Amplitude]);
```

What's so important about Sinusoids and periodic signals?

- Are signals is nature periodic?
- Name some:
  - Vibrations
  - Voice
  - Electromagnetic
  - Biomedical
- So are signals is nature periodic?
  - Not always but we may be able to model them with period signals
- Biomedical Research Summer Institute Examples:
  - Measure blood viscosity
  - Measure cell mass

Complex Numbers

- Complex numbers: What are they?
- What is the solution to this equation?

$$ax^2+bx+c=0$$

- This is a second order equation whose solution is:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What is the solution to?

1. $x^2+4x+3=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 3}}{2} = \frac{-4 \pm \sqrt{16-12}}{2}$$

$$= \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2} = -1,-3$$

What is the solution to?

2. $x^2+4x+5=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4\times 5}}{2} = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} \quad ?????$$

What is the Square Root of a Negative Number?

- We define the square root of a negative number as an imaginary number
- We define

$$\sqrt{-1} \Rightarrow j \text{ for engineers } (i \text{ for mathematicans})$$

- Then our solution becomes:

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm j\sqrt{4}}{2} = \frac{-4 \pm j2}{2} = -2 + j1, -2 - j1$$

The Complex Plane

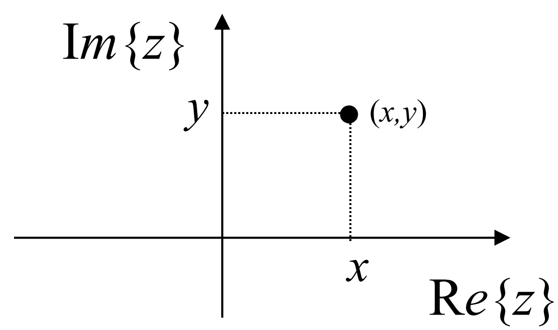
- $z = x + jy$ is a complex number where:

  $x = \mathrm{Re}\{z\}$ is the real part of z

  $y = \mathrm{Im}\{z\}$ is the imaginary part of z

- We can define the complex plane and we can define 2 representations for a complex number:

$$z = x + jy$$



Rectangular Form

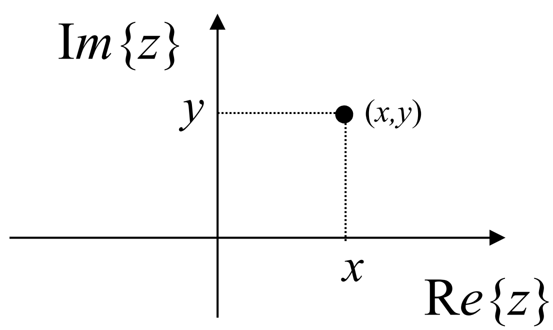
- Rectangular (or cartesian) form of a complex number is given as

$$z = x + jy$$

$x = \mathrm{Re}\{z\}$ is the real part of z

$y = \mathrm{Im}\{z\}$ is the imaginary part of z

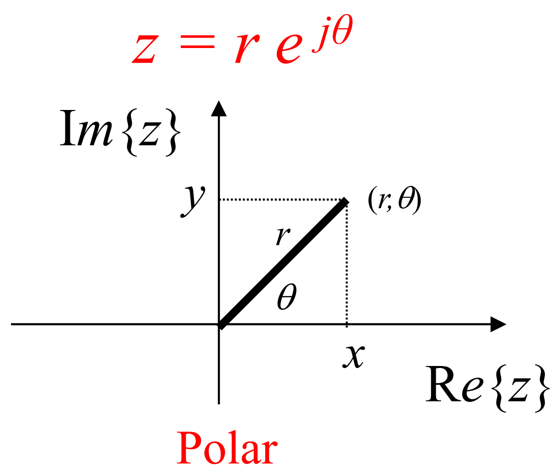
$z = x + jy$



Rectangular or Cartesian

Polar Form

- $z = r\,e^{j\theta} = r\angle\theta$ is a complex number where:
- r is the magnitude of z
- θ is the angle or argument of z (arg z)



Relationships between the Polar and Rectangular Forms

$$z = x + jy = r\,e^{j\theta}$$

- Relationship of Polar to the Rectangular Form:

$$x = \mathrm{R}e\{z\} = r\cos\theta$$

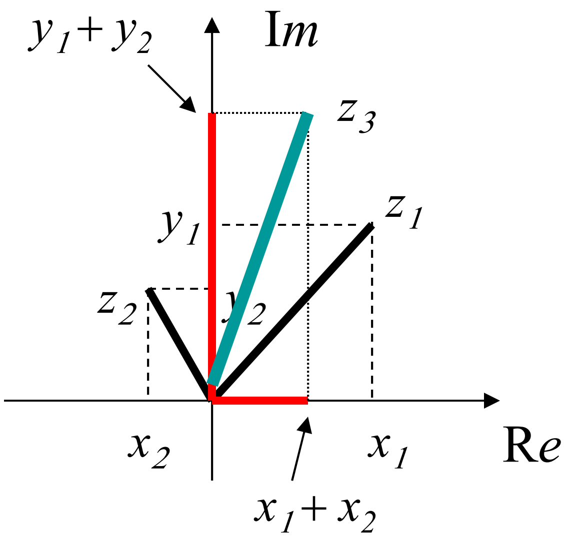
$$y = \mathrm{I}m\{z\} = r\sin\theta$$

- Relationship of Rectangular to Polar Form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(\frac{y}{x})$$

Addition of 2 complex numbers

- When two complex numbers are added, it is best to use the rectangular form.
- The real part of the sum is the sum of the real parts and imaginary part of the sum is the sum of the imaginary parts.
- Example: $z_3 = z_1 + z_2$



$$z_1 = x_1 + jy_1; z_2 = x_2 + jy_2$$

$$z_3 = z_1 + z_2 = x_1 + jy_1 + x_2 + jy_2$$

$$= x_1 + x_2 + jy_1 + jy_2$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication of 2 complex numbers

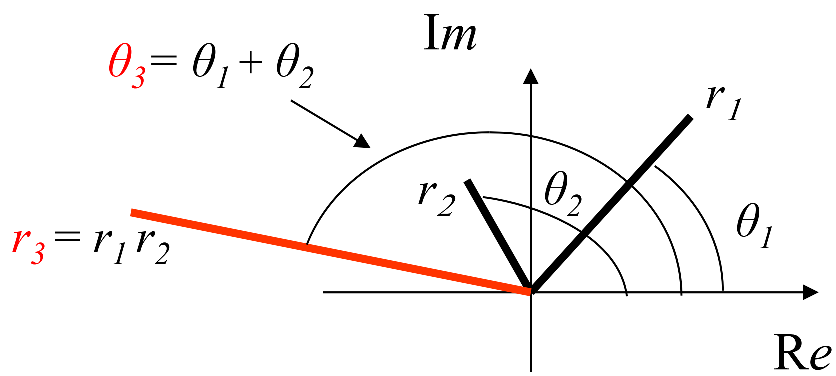
- When two complex numbers are multiplied, it is best to use the polar form:
- Example: $z_3 = z_1$ x z_2

$$z_1 = r_1 e^{j(\theta_1)}; z_2 = r_2 e^{j(\theta_2)}$$

$$z_3 = z_1 \times z_2 = r_1 e^{j(\theta_1)} \times r_2 e^{j(\theta_2)}$$

$$= r_1 r_2 e^{j(\theta_1)} e^{j(\theta_2)} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

- We multiply the magnitudes and add the phase angles



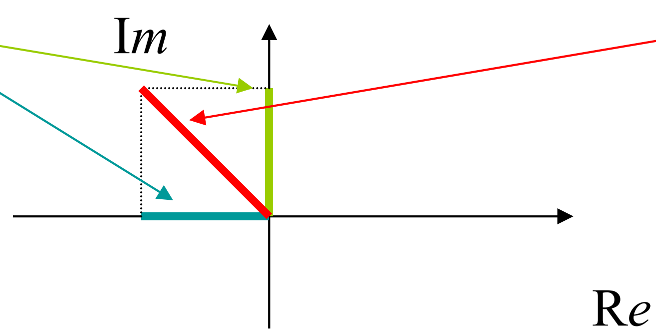
Some examples

$$5e^{j\frac{\pi}{2}} = 5\cos(\frac{\pi}{2}) + j5\sin(\frac{\pi}{2}) = 5\times 0 + j5\times 1 = j5$$

$$5e^{j\pi} = 5\cos(\pi) + j5\sin(\pi) = 5\times -1 + j5\times 0 = -5$$

$$5e^{-j\pi} = 5\cos(-\pi) + j5\sin(-\pi) = 5\cos(\pi) - j5\sin(\pi) = 5\times -1 + j5\times 0 = -5$$

$$5e^{j\frac{\pi}{2}} + 5e^{j\pi} = j5 - 5 = 5(-1+j) = 5\times\sqrt{(-1)^2+1^2}\,e^{j\tan^{-1}(-1)} = 5\sqrt{2}e^{j\frac{3\pi}{4}}$$



Some examples

$$5e^{j\frac{\pi}{4}} + 5e^{j\frac{\pi}{2}}$$

$$5e^{j\frac{\pi}{4}} + 5e^{j\frac{\pi}{2}} = \underbrace{5\cos(\frac{\pi}{4})} + \underbrace{j5\sin(\frac{\pi}{4})} + \underbrace{5\cos(\frac{\pi}{2})} + \underbrace{j5\sin(\frac{\pi}{2})}$$

$$= \frac{5}{\sqrt{2}} + 0 + j\frac{5}{\sqrt{2}} + j5 \qquad = \frac{5}{\sqrt{2}} + j5 \times 1.707$$

$$= 3.536 + j8.536$$

$$= \sqrt{3.536^2 + 8.536^2}\, e^{j\tan^{-1}(\frac{8.536}{3.536})} = \sqrt{12.5 + 72.86}\, e^{j\tan^{-1}(2.41)}$$

$$= \sqrt{84.367}\, e^{j\tan^{-1} 2.41} = 9.24 e^{j1.18}$$

Some examples

$$5e^{j\frac{\pi}{4}} + 5e^{j\frac{\pi}{2}} = 5\cos(\frac{\pi}{4}) + j5\sin(\frac{\pi}{4}) + 5\cos(\frac{\pi}{2}) + j5\sin(\frac{\pi}{2})$$

$$= \frac{5}{\sqrt{2}} + 0 + j\frac{5}{\sqrt{2}} + j5$$

$$= \frac{5}{\sqrt{2}} + j5\times 1.707 = 5\times 0.707 + j5\times 1.707$$

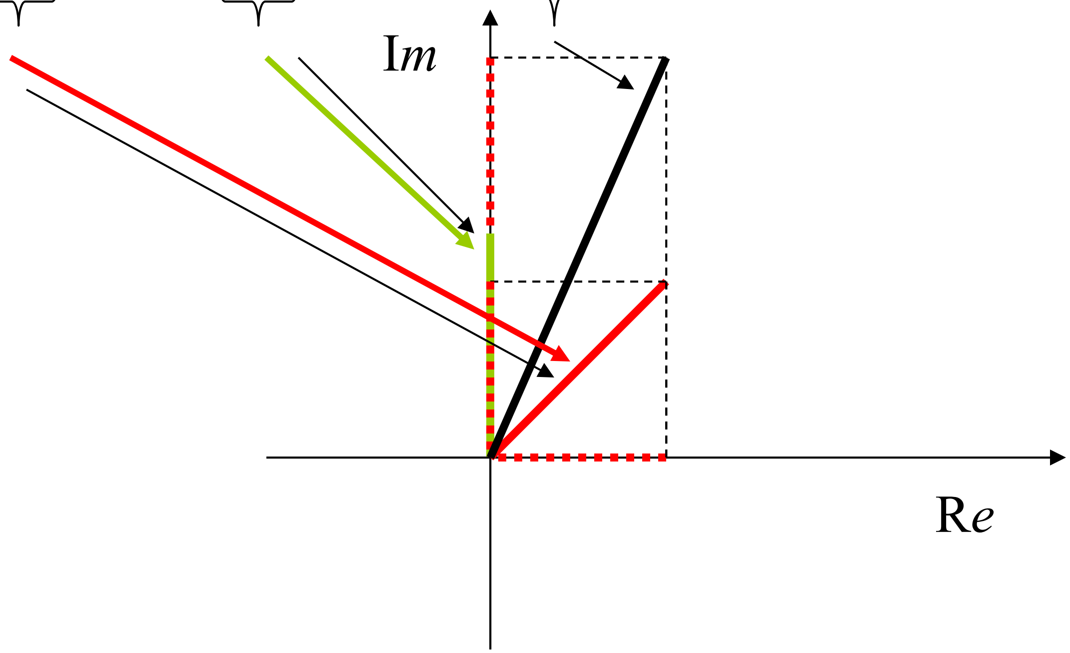
OR

$$= 5\times\sqrt{0.707^2 + 1.707^2}\,e^{j\tan^{-1}(\frac{1.707}{.707})} = 5\times\sqrt{0.5 + 2.91}\,e^{j\tan^{-1}(2.41)}$$

$$= 5\times\sqrt{3.41}\,e^{j1.18} = 5\times 1.85e^{j1.18} = 9.24e^{j1.18}$$

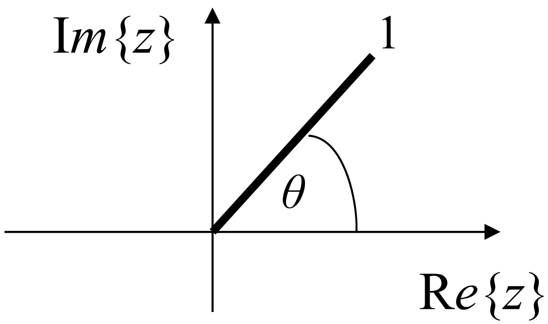
Some examples

$$\underbrace{5e^{j\frac{\pi}{4}}} + \underbrace{5e^{j\frac{\pi}{2}}} = \underbrace{9.24e^{j1.18}}$$



Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$



- We can use Euler's Formula to define complex numbers

$$z = r\,e^{j\theta} = r\cos\theta + j\,r\sin\theta$$
$$= x + j\,y$$

Complex Exponential Signals

- A complex exponential ***signal*** is define as:

$$z(t) = Ae^{j(\omega_o t+\theta)}$$

- Note that it is defined in polar form where
  - the magnitude of $z(t)$ is $|z(t)| = A$
  - the angle (or argument, arg $z(t)$) of $z(t) = (\omega_o t + \theta)$
    - Where ω_o is called the radian frequency and θ is the phase angle (phase shift)

Complex Exponential Signals

- Note that by using Euler's formula, we can rewrite the complex exponential signal in rectangular form as:

$$z(t) = Ae^{j(\omega_o t+\theta)}$$

$$= A\cos(\omega_o t + \theta) + jA\sin(\omega_o t + \theta)$$

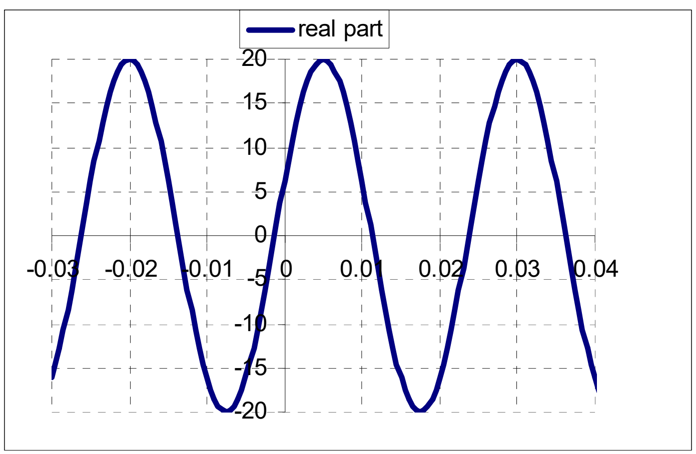
- Therefore real part is the cosine signal and imaginary part is a sine signal both of radial frequency ω_o and phase angle of θ

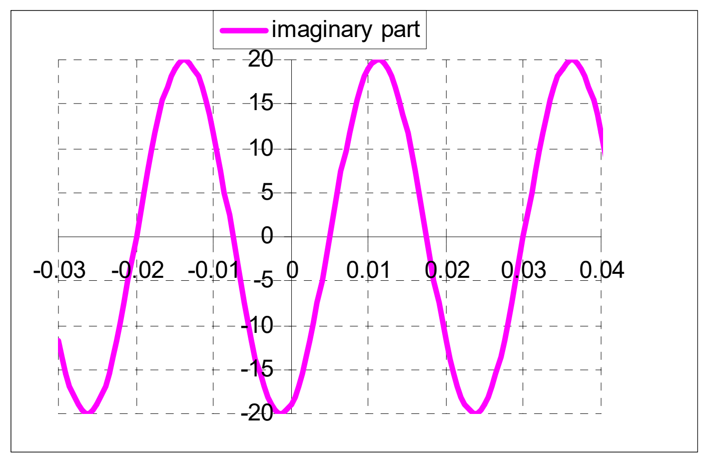
Plotting the waveform of a complex exponential signal

- For an complex signal, we plot the real part and the imaginary part separately.
- Example:

$$z(t) = 20e^{\,j(2\pi(40)t\text{-}0.4\pi)} = 20e^{\,j(80\pi t\text{-}0.4\pi)}$$

$$= 20\cos(80\pi t\text{-}0.4\pi) + j20\sin(80\pi t\text{-}0.4\pi)$$





NOTE!!!!

- The reason why we prefer the complex exponential representation of the real cosine signal:

$$x(t) = \Re e\{z(t)\} = \Re e\{Ae^{j(\omega_o t+\theta)}\}$$

$$= A\cos(\omega_o t + \theta)$$

- In solving equations and making other calculations, it easier to use the complex exponential form and then take the Real Part.

Homework

- Exercises:
  - 2.1 – 2.5
- Problems:

- 2.1 Instead plot $x(t) = 5\cos(\frac{2\pi}{25}t - \frac{\pi}{10})$ for $-25 \le t \le 50$

  Plot $x(t)$ using Matlab; as part of your answer provide your code

- 2.2

- 2.3a, 2.3b ω=0.4π,

  Plot these functions using Matlab; as part of your answer provide your code

- 2.4