Spectrum Representation

Lecture #5
Chapter 3
Mathematical Forms of a Decomposed Signal

Single Sided Spectrum

\[ x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k) = X_o + \Re\{\sum_{k=1}^{N} X_k e^{j2\pi f_k t}\} \]

Where \( X_o = \) DC component and \( X_k = A_k e^{j\theta_k} \) is the phasor for frequency \( k \)

Two Sided Spectrum

\[ x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{N}\left\{ \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t} \right\} \]

\[ = X_o + \sum_{k=1}^{N}\left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\} = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

Using the substitution \( (k = -k) \) for the second summation to combine into a single summation

where \( a_o = A_o \) for \( k = 0; \) \( a_k = \frac{X_k}{2} = \frac{1}{2} A_k e^{j\theta_k} \) for \( k \neq 0 \)
Periodic Waveforms

• A periodic waveform must satisfy this condition:

\[ x(t + T_o) = x(t) \]

where \( T_o \) is the period of \( x(t) \).
  – It can be shown that periodical waveforms are made up of sinusoidal functions which are harmonically related frequencies.
  – Harmonic frequencies are frequencies which are multiples of each other.

• If a periodic signal, \( x(t) \), has a period of \( T_o \) then it has a fundamental frequency \( f_o = 1 / T_o \).

• Furthermore, we say that \( x(t) \) has frequencies which are harmonics of \( f_o \).
  – That is, the \( k \)th harmonic of \( x(t) \) is \( kf_o \)
Periodic Vs Nonperiodic Signals

• We say that \( x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \) is nonperiodic since there is no assumption about the individual frequencies.

• We say that \( x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_o t} \) is periodic since there is a fundamental period associated with the signal.

• The fundamental frequency \( f_o \) is the greatest common divisor of \( f_k \).
Periodic Vs Nonperiodic Signals

Spectrum of a Periodic Signal
Fundamental Frequency = 100Hz

Spectrum of a Nonperiodic Signal
No Fundamental Frequency
Fourier Analysis

• We have shown that a signal can be formulated in terms of a sum of sinusoids which defines its spectrum
  \[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

• What we like to find are the \( a_k \)'s so we can completely determine an signal's spectrum

• If a signal is periodic, (i.e., the \( f_k \)'s are multiples of \( f_0 \)) then the determination of these the \( a_k \)'s are easily achieved using Fourier Analysis.
  \[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t} \]
Another Mathematical Form of a Decomposed Signal

Two Sided Spectrum with $N \to \infty$

\[ x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{\infty} \left\{ \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t} \right\} \]

\[ = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t} \]

where $f_k \to k f_o$; $a_k = A_o$ for $k = 0$; $a_k = \frac{X_k}{2} = \frac{1}{2} A_k e^{j\theta_k}$ for $k \neq 0$

\[ = a_o + \sum_{k=1}^{\infty} a_k e^{j2\pi f_o t} + a_k^* e^{-j2\pi f_o t} = a_o + \sum_{k=1}^{\infty} 2\text{Re}\{a_k e^{j2\pi f_o t}\} \]

since $s + s^* = 2\text{Re}\{s\}$

\[ = a_o + \sum_{k=1}^{\infty} 2\text{Re}\{a_k e^{j\theta_k} e^{j2\pi f_o t}\} = a_o + \sum_{k=1}^{\infty} 2|a_k| \cos(2\pi f_o t + \theta_k) \]
Summary of Mathematical Forms of a Decomposed Signal

Single Sided Spectrum with $N \to \infty$

$$x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \theta_k) = X_o + \Re\{\sum_{k=1}^{\infty} X_k e^{j2\pi f_k t}\}$$

Where $X_o = \text{DC component}$ and $X_k = A_k e^{j\theta_k}$ is the phasor for frequency $k$

Two Sided Spectrum with $N \to \infty$

$$x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{\infty} \left\{\frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t}\right\}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t} \text{ where } a_k = A_o \text{ for } k = 0; a_k = \frac{X_k}{2} = \frac{1}{2} A_k e^{j\theta_k} \text{ for } k \neq 0$$

$$= a_o + \sum_{k=1}^{\infty} 2\Re\{a_k e^{j2\pi f_o t}\}$$

$$= a_o + \sum_{k=1}^{\infty} 2|a_k| \cos(2\pi f_o t + \theta_k)$$
Fourier Analysis and Synthesis

- Given a Fourier Series (note the limits go to \(-\infty\) to \(\infty\))

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_o k t} = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_o kt + \theta_k) \]

Where \(A_o = a_o\); \(a_k = \frac{1}{2} A_k e^{j\theta_k}\); \(a_{-k} = a_k^*\); \(f_o = \frac{1}{T_o}\)

- Fourier Analysis is the technique to determine the \(a_k\)'s from \(x(t)\).
- Fourier Synthesis is the (opposite) technique to determine \(x(t)\) from the \(a_k\)'s.
Fourier Analysis

• The way to perform Fourier Analysis is by using the Fourier Series Integral:

\[
a_k = \frac{1}{T_o} \int_{0}^{T_o} x(t)e^{-j\left(\frac{2\pi}{T_o}\right)kt} \, dt
\]

• This states to obtain the \(a_k\)'s
  1. Take the periodic function \(x(t)\) and multiply it by \(e^{-j2\pi/T_o kt}\)
  2. Integrate the product over 1 period, \(T_o\), of \(x(t)\)
  3. Divide the result by \(T_o\)
Derivation of the Fourier Integral

- Some background: The Orthogonality Property
- This is a property of certain functions
- Orthogonality means perpendicular
- So if two functions are orthogonal they are perpendicular as defined by the following:

\[ \int_0^{T_o} v_k(t) v_l^*(t) dt = \begin{cases} 
0 & \text{if } k \neq l \\
T_o & \text{if } k = l
\end{cases} \]
Sinusoids and Orthogonality

- Note that sinusoidal and complex exponential functions are orthogonal.

- First recall:

\[
\int_{0}^{T} e^{j \frac{2\pi}{T_0} t} dt = \left. \frac{e^{j \frac{2\pi}{T_0} kT}}{j \frac{2\pi}{T_0} k} \right|_{0}^{T_0} = \frac{e^{j \frac{2\pi}{T_0} kT_0} - 1}{j \frac{2\pi}{T_0} k} = \frac{e^{j \frac{2\pi}{T_0} kT_0} - 1}{j \frac{2\pi}{T_0} k} = 0; \text{ for } k \neq 0
\]

since \( e^{j \frac{2\pi}{T_0} kT_0} = \cos 2\pi k + j \sin 2\pi k = 1 + j0 = 1 \)
Sinusoids and Orthogonality

• Note that sinusoidal and complex exponential functions are orthogonal.

• First recall:

\[
\int_0^{T_o} e^{j\frac{2\pi}{T_o}kt} \, dt = \left. \frac{e^{j\frac{2\pi}{T_o}kt}}{j\frac{2\pi}{T_o}k} \right|_0^{T_o} = \frac{e^{j\frac{2\pi}{T_o}kT_o} - 1}{j\frac{2\pi}{T_o}k} = \frac{e^{j2\pi k} - 1}{j\frac{2\pi}{T_o}k} = 0; \text{ for } k \neq 0
\]

since \(e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k = 1 + j0 = 1\)
Sinusoids and Orthogonality

• Note that sinusoidal and complex exponential functions are orthogonal.

• For $k = 0$:

$$
\int_{0}^{T_o} e^{j\frac{2\pi}{T_o}kt} dt = \int_{0}^{T_o} e^{j\frac{2\pi}{T_o}0t} dt = \int_{0}^{T_o} 1 dt = t \bigg|_0^{T_o} = T_o - 0 = T_o
$$
Sinusoids and Orthogonality

• Another way

\[
2) \int_{0}^{T_o} e^{\frac{2\pi j k t}{T_o}} dt = \int_{0}^{T_o} \cos \frac{2\pi}{T_o} ktdt + j \int_{0}^{T_o} \sin \frac{2\pi}{T_o} ktdt
\]

\[
= \sin \frac{2\pi}{T_o} kT_o - \sin \frac{2\pi}{T_o} k0
\]

\[
= \frac{2\pi}{T_o} k - j \frac{2\pi}{T_o} k
\]

\[
= 0 - 0 - j \frac{1-1}{\frac{2\pi}{T_o} k} = 0 \text{ for } k \neq 0
\]
Sinusoids and Orthogonality

• Now let’s check sinusoids and complex exponential function for orthogonality:

\[
\int_{0}^{T_o} v_k(t)v_l^*(t)dt = \int_{0}^{T_o} e^{j\frac{2\pi}{T_o}kt}e^{-j\frac{2\pi}{T_o}lt} dt = \int_{0}^{T_o} e^{j\frac{2\pi}{T_o}kt} dt = \int_{0}^{T_o} e^{j\frac{2\pi}{T_o}(k-l)t} dt = \int_{0}^{T_o} e^{j\frac{2\pi}{T_o}(k-l)t} dt = \int_{0}^{T_o} e^{j\frac{2\pi}{T_o}(k-l)t} dt = \int_{0}^{T_o} (\cos(0) + j \sin(0)) dt = \int_{0}^{T_o} (1 + j0) dt = \int_{0}^{T_o} dt = T_o
\]

if \(k \neq l\)

if \(k = l\)
Proof of Fourier Series Integral

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T_o}}
\]

\[
x(t)e^{-\frac{j2\pi lt}{T_o}} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T_o}} \times e^{-\frac{j2\pi lt}{T_o}} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi (k-l)t}{T_o}}
\]

\[
\int_{0}^{T_o} x(t)e^{-\frac{j2\pi lt}{T_o}} dt = \int \left\{ \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi (k-l)t}{T_o}} \right\} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k \int_{0}^{T_o} e^{\frac{j2\pi (k-l)t}{T_o}} dt
\]

\[
= \cdots + a_{l-1} \int_{0}^{T_o} e^{\frac{j2\pi (l-1-l)t}{T_o}} dt + a_l \int_{0}^{T_o} e^{\frac{j2\pi (l-l)t}{T_o}} dt + a_{l+1} \int_{0}^{T_o} e^{\frac{j2\pi (l+1-l)t}{T_o}} dt + \cdots
\]

\[
= \cdots + a_{l-1} 0 + a_l T_o + a_{l+1} 0 + \cdots
\]

\[
= a_l T_o \quad \Rightarrow \quad a_k = \frac{1}{T_o} \int_{0}^{T_o} x(t)e^{-\frac{j2\pi (k-l)t}{T_o}} dt
\]
Fourier Series

• Fourier Analysis:

\[ a_k = \frac{1}{T_o} \int_{0}^{T_o} x(t) e^{-j\left(\frac{2\pi}{T_o}\right)kt} \, dt \]

• Fourier Synthesis:

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\left(\frac{2\pi}{T_o}\right)kt} \]
Example

• Find the spectrum of: \( \sin^3(3\pi t) \)

\[
a_k = \frac{1}{T_o} \int_0^{T_o} \sin^3(3\pi t) e^{-\frac{j2\pi kt}{T_o}} \, dt
\]

\[
= \frac{1}{T_o} \int_0^{T_o} \left( \frac{e^{j3\pi t} - e^{-j3\pi t}}{2j} \right)^3 e^{-\frac{j2\pi kt}{T_o}} \, dt
\]

\[
= \frac{1}{T_o} \int_0^{T_o} \left( \frac{e^{j9\pi t} - 3e^{j6\pi t} e^{-j3\pi t} + 3e^{j3\pi t} e^{-j6\pi t} - e^{-j9\pi t}}{-8j} \right) e^{-\frac{j2\pi kt}{T_o}} \, dt
\]

\[
= \frac{1}{T_o} \int_0^{T_o} \left( \frac{e^{j9\pi t} - 3e^{j3\pi t} + 3e^{-j3\pi t} - e^{-j9\pi t}}{-8j} \right) e^{-\frac{j2\pi kt}{T_o}} \, dt
\]

\[
\omega_o = 3\pi; \quad f_o = \frac{3\pi}{2\pi} = \frac{3}{2}; \quad T_o = \frac{2}{3}
\]
Example

- Find the spectrum of: \( \sin^3(3\pi t) \)

\[
a_k = \frac{3}{2} \int_0^{2/3} \left( \frac{e^{j9\pi t} - 3e^{j3\pi t} + 3e^{-j3\pi t} - e^{-j9\pi t}}{-8j} \right) e^{-j\pi t/2} dt
\]

\[
= \frac{3}{2} \int_0^{2/3} \left( \frac{e^{j9\pi t} - 3e^{j3\pi t} + 3e^{-j3\pi t} - e^{-j9\pi t}}{-8j} \right) e^{-j3\pi t} dt
\]

\[
= \frac{3}{2} \int_0^{2/3} \left( \frac{e^{j(9-3k)\pi t} - 3e^{j(3-3k)\pi t} + 3e^{-j(3+3k)\pi t} - e^{-j(9+3k)\pi t}}{-8j} \right) dt
\]

\[
= -\frac{3}{j16} \int_0^{2/3} e^{j(3-k)3\pi t} dt + \frac{9}{j16} \int_0^{2/3} e^{j(1-k)3\pi t} dt - \frac{9}{j16} \int_0^{2/3} e^{-j(1+k)3\pi t} dt + \frac{3}{j16} \int_0^{2/3} e^{-j(3+k)3\pi t} dt
\]
Example

- Find the spectrum of: \( \sin^3(3\pi t) \)

\[
a_k = -\frac{3}{j16} \int_0^{\frac{2}{3}} e^{j(3-k)3\pi t} \, dt = -\frac{3}{j16} \int_0^{\frac{2}{3}} dt = -\frac{3}{j16} \times \frac{2}{3} = -\frac{1}{j8} = \frac{j}{8} \quad \text{for } k = 3; \Rightarrow a_3 = \frac{j}{8} = \frac{1}{8} e^{j}\\
\]

\[
= \frac{9}{j16} \int_0^{\frac{2}{3}} e^{j(1-k)3\pi t} \, dt \quad \text{for } k = 1; \Rightarrow a_1 = -\frac{j^3}{8} = \frac{3}{8} e^{-j}\\
\]

\[
= -\frac{9}{j16} \int_0^{\frac{2}{3}} e^{j(1+k)3\pi t} \, dt \quad \text{for } k = -1; \Rightarrow a_{-1} = \frac{j^3}{8} = \frac{3}{8} e^{j}\\
\]

\[
= \frac{3}{j16} \int_0^{\frac{2}{3}} e^{j(3+k)3\pi t} \, dt \quad \text{for } k = -3; \Rightarrow a_{-3} = -\frac{j}{8} = \frac{1}{8} e^{-j}\\
\]
Fourier Series of a Square Wave

\[ s(t) = \begin{cases} 
1 & \text{for } 0 \leq t < \frac{1}{2}T_o \\
0 & \text{for } \frac{1}{2}T_o \leq t \leq T_o 
\end{cases} \]
Fourier Series of a Square Wave

Analysis

\[ a_k = \frac{1}{T_o} \int_0^{T_o} s(t) e^{-j(\frac{2\pi}{T_o})kt} \, dt \]

\[ = \frac{1}{T_o} \int_0^{T_o/2} 1 e^{-j(\frac{2\pi}{T_o})kt} \, dt + \frac{1}{T_o} \int_{T_o/2}^{T_o} 0 e^{-j(\frac{2\pi}{T_o})kt} \, dt \]

\[ = \frac{1}{T_o} \int_0^{T_o/2} 1 e^{-j(\frac{2\pi}{T_o})kt} \, dt = \frac{1}{T_o} \frac{e^{-j(\frac{2\pi}{T_o})k(T_o/2)}}{- j (\frac{2\pi}{T_o})k} \bigg|_0^{T_o/2} \]

\[ = \frac{1}{T_o} \frac{e^{-j(\frac{2\pi}{T_o})k(T_o/2)} - e^{-j(\frac{2\pi}{T_o})k0}}{- j (\frac{2\pi}{T_o})k} = \frac{1 - e^{-j\frac{2\pi}{T_o}k}}{j2\pi k} \]

\[ = \frac{1 - (-1)^k}{j2\pi k}; k \neq 0 \]

\[ a_k = \frac{1 - (-1)^k}{j2\pi k} \]

\[ a_k = \frac{1 - (-1)}{j2\pi k} = \frac{1}{j\pi k} = \frac{1}{\pi k} e^{-j\frac{\pi}{2}}; k \text{ odd} \]

\[ = \frac{1 - (1)}{j2\pi k} = 0; k \text{ even} \]

\[ a_0 = \frac{1}{T_o} \int_0^{T_o} s(t) \, dt = \frac{1}{T_o} \int_0^{T_o/2} 1 \, dt = \frac{1}{2} \]

\[ e^{-j\pi k} = \cos(-\pi k) + j \sin(-\pi k) = (-1)^k + j0 = (-1)^k \]
Fourier Series of a Square Wave Synthesis

\[ x(t) = \sum_{k=\text{odd}}^{\infty} a_k e^{j\frac{2\pi kt}{T_o}} = a_0 + \sum_{k=1}^{\infty} 2\Re\{a_k e^{j\frac{2\pi kt}{T_o}}\} \]

\[ = \frac{1}{2} + 2 \sum_{k=\text{odd}}^{\infty} \Re\left\{\frac{e^{-j\frac{\pi}{2}}}{\pi k} e^{j\frac{2\pi kt}{T_o}}\right\} \]

\[ = \frac{1}{2} + 2 \sum_{k=\text{odd}}^{\infty} \Re\left\{\frac{1}{\pi k} e^{j\frac{2\pi (kt - \frac{\pi}{2})}{T_o}}\right\} \]

\[ = \frac{1}{2} + 2 \sum_{k=\text{odd}}^{\infty} \frac{1}{\pi k} \cos\left(\frac{2\pi}{T_o}kt - \frac{\pi}{2}\right) \]
Spectrum of a Square Wave

\[ a_k = \frac{1}{j \pi k} = \frac{1}{\pi k} e^{-j \frac{\pi}{2}}; \ k \ \text{odd} \]

\[ a_k = 0; \ k \ \text{even} \]

\[ a_0 = \frac{1}{2} \]

\[ a_{-5} = \frac{j}{5 \pi} \]
\[ |a_{-5}| = \frac{1}{5 \pi} \]

\[ a_{-3} = \frac{j}{3 \pi} \]
\[ |a_{-3}| = \frac{1}{3 \pi} \]

\[ a_1 = \frac{j}{\pi} \]
\[ |a_1| = \frac{1}{\pi} \]

\[ a_{-1} = \frac{j}{3 \pi} \]
\[ |a_{-1}| = \frac{1}{3 \pi} \]

\[ a_3 = \frac{j}{3 \pi} \]
\[ |a_3| = \frac{1}{3 \pi} \]

\[ a_{-3} = \frac{j}{5 \pi} \]
\[ |a_{-3}| = \frac{1}{5 \pi} \]

\[ a_5 = \frac{j}{5 \pi} \]
\[ |a_5| = \frac{1}{5 \pi} \]

\[ f_0 = 0.25 \text{ Hz} \]

\[ T_0 = 4 \]

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Fourier Series of a Sawtooth Wave

\[ s(t) = \frac{t}{T_o} \quad \text{for } 0 \leq t \leq T_o \]
Fourier Series of a Sawtooth Wave Analysis

\[ a_k = \frac{1}{3} \int_{0}^{3} s(t) e^{-j \frac{2\pi kt}{3}} dt = \frac{1}{3} \int_{0}^{3} t e^{-j \frac{2\pi kt}{3}} dt \]

\[ u = \frac{t}{3}, \quad du = \frac{1}{3} dt \]

\[ dv = e^{-j \frac{2\pi kt}{3}} dt, \quad v = \frac{1}{-j 2\pi k/3} e^{-j \frac{2\pi kt}{3}} \]

\[ a_k = \frac{1}{3} \int_{0}^{3} t e^{-j \frac{2\pi kt}{3}} dt \]

\[ = \frac{1}{9} \left[ \frac{t}{-j 2\pi k/3} e^{-j \frac{2\pi kt}{3}} \right]_{0}^{3} - \frac{1}{-j 2\pi k/3} \int_{0}^{3} e^{-j \frac{2\pi kt}{3}} dt \]

\[ = \frac{1}{9} \left[ \frac{1}{-j 2\pi k/3} \right] \{3 e^{-j 2\pi k} - 0\} - \left( \frac{1}{-j 2\pi k/3} \right) \left( \frac{1}{-j 2\pi k/3} \right) \{e^{-j 2\pi k} - 1\} \]

since \( e^{-j 2\pi k} = \cos(-2\pi k) + j \sin(-2\pi k) = 1 + j0 \)
Fourier Series of a Sawtooth Wave Analysis

\[ a_k = \frac{1}{9} \left[ \frac{1}{-j 2\pi k/3} \{3 \times 1\} - \left( \frac{1}{-j 2\pi k/3} \right) \left( \frac{1}{-j 2\pi k/3} \right) \{1 - 1\} \right] \]

\[ = \frac{1}{9} \left[ \frac{1}{-j 2\pi k/3} \{3 \times 1\} - 0 \right] \]

\[ = \frac{1}{3} \frac{1}{(-j 2\pi k/3)} = j \frac{1}{2\pi k} = \frac{1}{2\pi k} e^{j\frac{\pi}{2}}; k \neq 0 \]

\[ a_0 = \frac{1}{3} \int_0^3 f(t) dt = \frac{1}{3} \int_0^3 \frac{1}{3} t dt = \left( \frac{1}{2} \right) \frac{9}{2} = 0.5 \]

\[ s(t) = \frac{1}{2} + 2 \sum_{k=1}^{\infty} \frac{1}{2\pi k} \cos \left( \frac{2\pi}{3} kt + \frac{\pi}{2} \right) \]
Fourier Series of a Sawtooth Wave Synthesis

2 terms $T_o=3$

3 terms $T_o=3$

11 terms $T_o=3$

58 terms $T_o=3$
Spectrum of a Sawtooth Wave

\[ a_k = j \frac{1}{2\pi k} \frac{1}{2\pi k} e^{j\frac{\pi}{2}}; \quad k \neq 0 \]
\[ a_0 = \frac{1}{2} \]

\[ k=0 \]
\[ a_0 = 1/2 \]

\[ T_0 = 3 \]
\[ f_0 = 0.33 \text{ Hz} \]

\[ k=1 \]
\[ a_1 = j/2\pi \]
\[ |a_1| = 1/2\pi \]

\[ k=-1 \]
\[ a_{-1} = -j/2\pi \]
\[ |a_{-1}| = 1/2\pi \]

\[ k=-2 \]
\[ a_{-2} = -j/4\pi \]
\[ |a_{-2}| = 1/4\pi \]

\[ k=2 \]
\[ a_3 = j/4\pi \]
\[ |a_3| = 1/4\pi \]
Time-Frequency Spectrum

- Up till now we have seen signals whose frequencies do not change with time.

- However, in reality, signals can produce different frequencies at different times
  - Music
  - Voice
  - Frequency Modulation

- To plot the spectrum of such signals, we can use a 3-D plot called a spectrogram.
  - Plots time on the x-axis, frequency on the y-axis and magnitude on the z-axis (out of the plane of the paper)
Example of a Spectrogram

- Three sinusoids (0.1 Hz, 0.2 Hz, 0.4Hz) in sequence
Frequency Modulation

- We have signals of the type:

\[ x(t) = \Re \{ Ae^{j(\omega_o t + \theta)} \} = A \cos(\omega_o t + \theta) \]

where the angle (of the cosine) varies linearly with time and the time derivative of the angle is \( \omega_o \), constant.

- However, we can generalize this signal such that the angle varies with time such that its derivative is not constant.

\[ x(t) = \Re \{ Ae^{j\psi(t)} \} = A \cos(\psi(t)) \]

\[ \omega(t) = \frac{d}{dt} \psi(t) \quad \text{Instantaneous Frequency} \]
FM Radio

• Frequency Modulation is the scheme used in FM broadcast.

• For example, if \( \omega(t) = \omega_c + m(t) \) where \( \omega_c \) is called the carrier frequency and \( m(t) \) contains the “information” (voice or music), then FM broadcast is:

\[
\psi(t) = \int \{\omega_c + m(t)\} dt = \omega_c t + \int m(t) dt
\]
Homework

• Exercises:
  – 3.4 – 3.8

• Problems:

– 3.5 Instead use \( x(t) = 10 + 20\cos(2\pi(200)t + \frac{1}{4}\pi) + 10\cos(2\pi(250)t) \)

– 3.8,

– 3.9 Use Matlab for plotting the spectrum; submit your code

\[
0 \quad \text{for } 0 < t \leq 2.5
\]

– 3.12 Instead use \( x(t) = \begin{cases} 
0 & \text{for } 0 < t \leq 2.5 \\
2 & \text{for } 2.5 < t \leq 5 
\end{cases} \)

Use Matlab for plotting the spectrum; submit your code

– 3.13, –3.14