Sampling and Aliasing

Lecture #6

Chapter 4

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.
- To Understand How Computers Process Signals and Systems

Discrete-time Signals and Computers

- Up to now we have been studying continuous-time signals (also called analog signals) such as

$$x(t) = A\cos(\omega_o t + \theta)$$

- However, digital computers and computer programs can not process analog signals.
- Instead they store discrete-time versions of analog signals

$$x[n] = x(nT_s)$$

- This is because digital computers can only store discrete numbers.
  - There are computers called analog computers which do process continuous-time signals
- Since the computer only stores numbers, how does one know what continuous-time signal it represents?

Sampling

- We can obtain a discrete-time signal by sampling a continuous-time signal at equally spaced time instants, $t_n = nT_s$

$$x[n] = x(nT_s) \quad -\infty < n < \infty$$

- The individual values $x[n]$ are called the samples of the continuous time signal, $x(t)$.
- The fixed time interval between samples, T_s, is also expressed in terms of a sampling rate f_s (in samples per second) such that:

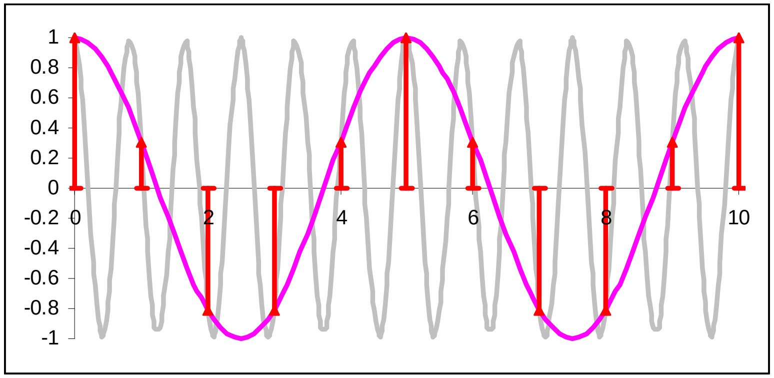
$$f_s = 1/\ T_s \ \text{ samples/sec.}$$

Continuous-to-Discrete Conversion

- By using a Continuous-to-Discrete (C-to-D) converter, we can take continuous-time signals and form a discrete-time signal.
- There are devices called Analog-to-Digital converters (A-to-D)
- The books chooses to distinguish an C-to-D converter from an A-to-D converter by defining a C-to-D as an ideal device while A-to-D converters are practical devices where real world problems are evident.
  - Problems in sampling the amplitudes accurately
  - Problems in sampling at the proper times

Discrete-Time Signals

- A discrete-time signal is a sequence of numbers and carry no information about the time-sequence.
- Looking at the following diagram, which (gray or solid) waveform are these (red) samples associated with?



Discrete-Time Sinusoidal Signals

- Since a Fourier series can be written for any continuous-time signal, let's concentrate on sinusoids
- We define a normalized frequency for the discrete sinusoidal signal.

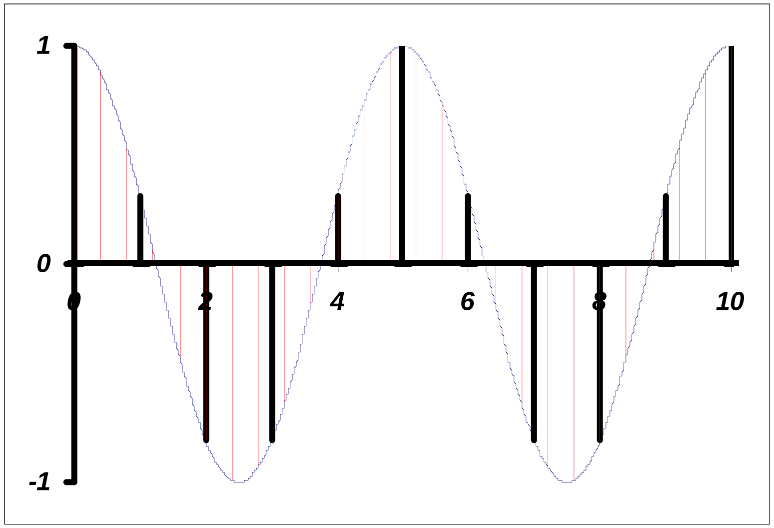
$$x[n] = x(nT_s) = A\cos(\omega n T_s + \theta)$$
$$= A\cos(\hat{\omega} n + \theta)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

- $\hat{\omega}$ is the normalized or discrete-time frequency
- Since we can have different signals with the same $\hat{\omega}$, then there can be an infinite number of continuous-time signal which yield the same discrete-time sinusoid!

Two Problems with Sampling

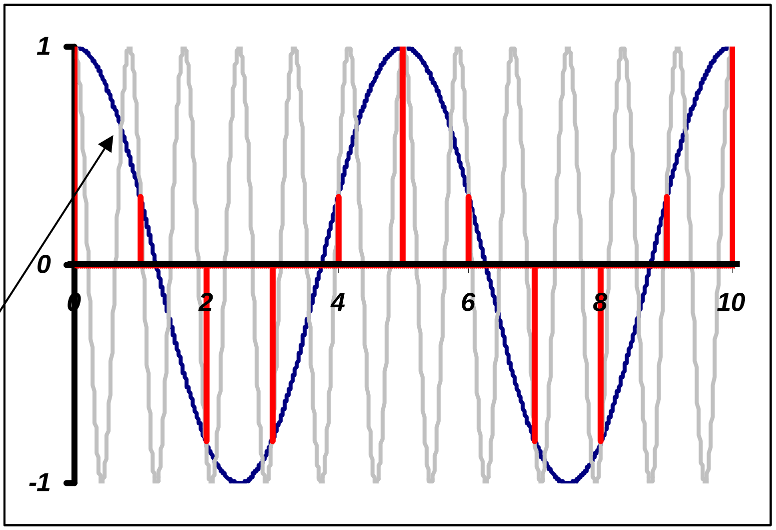
- Problem 1: How many samples are enough to have to represent a continuous time signal?



- In this figure, we have a continuous-time signal sampled every .4 seconds (red samples) and every 1 second (black samples).

Discrete-Time Sinusoidal Signals

- Problem 2: Can a set of samples be represent more that one continuous-time signal
- The discrete-time sinusoid shown in the figure has which can be obtain from, for example, either a 1 second sampled continuous-time sinusoid with $f = 0.2$ Hz or 1.2 Hz.
- In the first case, where $f = 0.2$ Hz, we have:



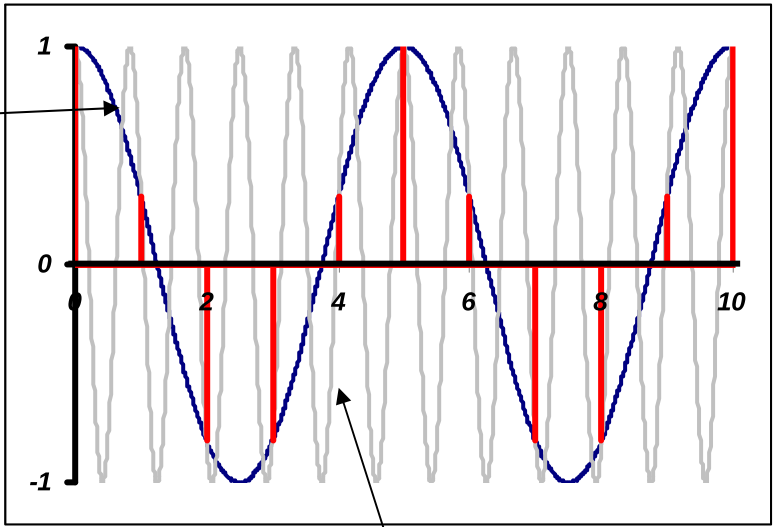
$$\hat{\omega} = 2\pi(0.2)(1) = .4\pi$$

Discrete-Time Sinusoidal Signals

- In the first case, where $f = 0.2$ Hz, we have:

$$\hat{\omega} = 2\pi(0.2)(1) = .4\pi$$

- Since a sinusoid is periodic in 2π, then for the case where f=1.2 Hz



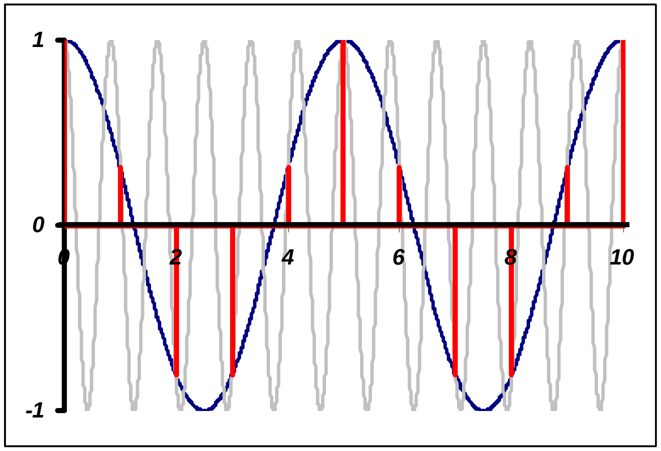
$$x[n] = A\cos(\hat{\omega}n + \theta)$$

$$\hat{\omega} = 2\pi(1.2)(1) = 2.4\pi \Rightarrow 2.4\pi - 2\pi = .4\pi$$

$$x[n] = A\cos(2.4\pi n + \theta) = A\cos(2\pi n + 0.4\pi n + \theta) = A\cos(0.4\pi n + \theta)$$

Aliasing

- This example illustrates that two sampled sinusoids can produce the same discrete-time signal.
  1. $\cos [2\pi(0.2)\, t]$
  2. $\cos [2\pi(1.2)\, t]$



- When this occurs we say that that these signals are aliases of each other.

Aliasing

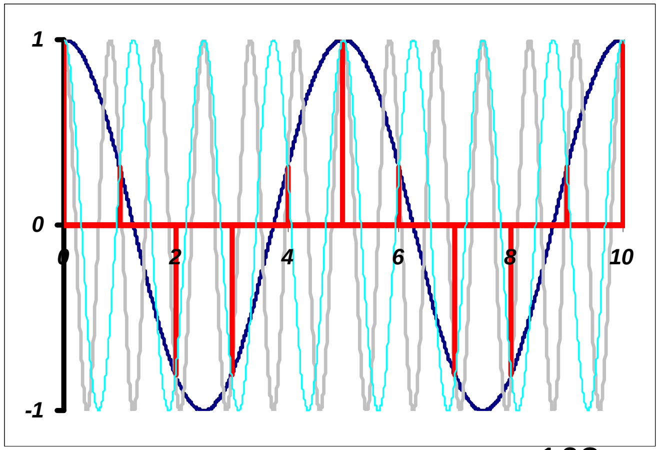
- There are more alias signals for this example:
  1. $x(t) = \cos(2\pi(0.2)\,t) \Rightarrow x[n] = \cos(2\pi(0.2)\,1n) = \cos(0.4\pi\,n)$
  2. $x(t) = \cos(2\pi(1.2)\,t) \Rightarrow x[n] = \cos(2\pi(1.2)\,1n) = \cos(2.4\pi\,n) =$ $\cos(0.4\,\pi\,n + 2\,\pi\,n) = \cos(0.4\pi\,n)$

     $\hat{\omega}_l = 0.4\pi + 2\pi l \quad \text{for } l = 0,1,2,3,\ldots$

     Since $\cos(2\pi - \theta) = \cos(\theta)$, $\hat{\omega}_l = -0.4\pi + 2\pi l \ \text{ for } l = 0,1,2,3,\ldots$
  3. $x(t) = \cos(2\pi(.8)\,t) \Rightarrow x[n] = \cos(2\pi(.8)\,1n) = \cos(1.6\pi\,n) =$ $\cos(2\,\pi\,n - 0.4\,\pi\,n) = \cos(0.4\pi\,n)$
- In summary, (for l = positive or negative integer):

  $\hat{\omega}_o, \quad \hat{\omega}_o + 2\pi l, \quad 2\pi l - \hat{\omega}_o$

  where $\hat{\omega}_o$ is called the principal alias



Aliasing

- Let's look at signals of the form: $\cos(\omega_l t)$

$$\cos(\omega_l t) \underset{sampled}{\Rightarrow} \cos(\hat{\omega}_l n)$$

where $\omega_l = \dfrac{\hat{\omega}_l}{T_s} = \hat{\omega}_l f_s$ and $\hat{\omega}_l = \hat{\omega}_o + 2\pi l$, $\hat{\omega}_o$ is the principal alias, and l is an integer.

Therefore, $\omega_l = \hat{\omega}_l f_s = (\hat{\omega}_o + 2\pi l) f_s$ and $f_l = \dfrac{(\hat{\omega}_o + 2\pi l) f_s}{2\pi}$

since $\cos(\theta) = \cos(-\theta) = \cos(2\pi - \theta)$, then we can have $\hat{\omega}_l = 2\pi l - \hat{\omega}_o$ and $f_l = \dfrac{(2\pi l - \hat{\omega}_o) f_s}{2\pi}$

or $\omega_l = 2\pi l \pm \hat{\omega}_o$ and $f_l = \dfrac{(2\pi l \pm \hat{\omega}_o) f_s}{2\pi}$

In our example, $\hat{\omega}_o$ is 0.4π and $f_s = 1$. Then,

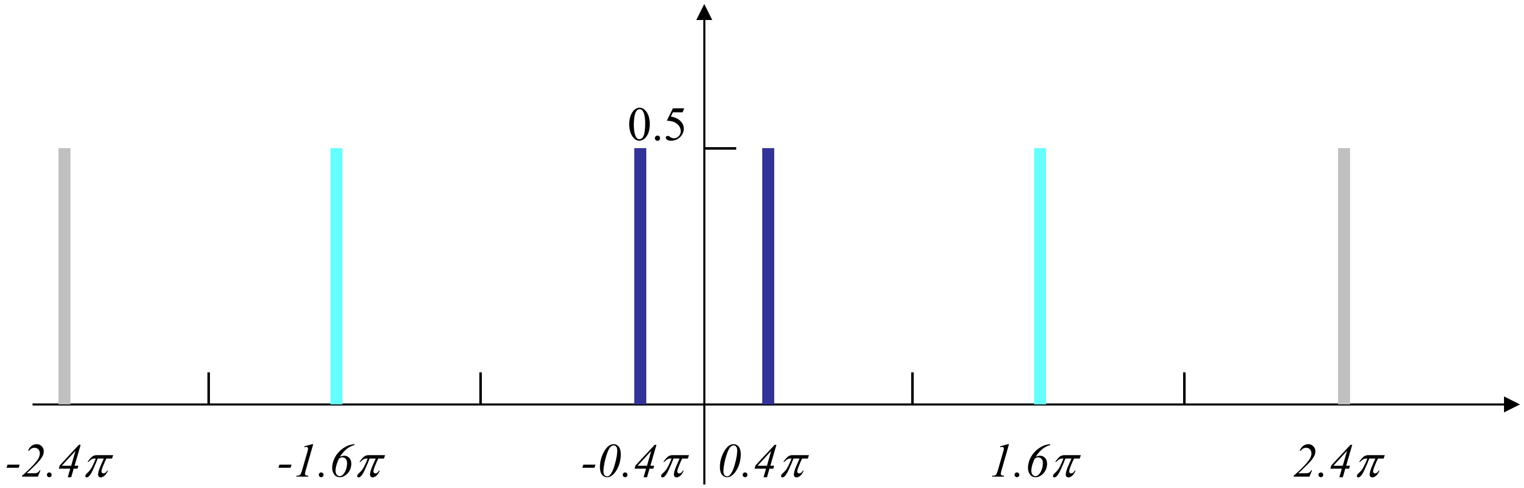
$$\omega_l = 2\pi l \pm \hat{\omega}_o = 2\pi l \pm 0.4\pi = 0.4\pi, 1.6\pi, 2.4\pi, 3.6\pi, \text{....rad/sec}$$

$$f_l = \frac{(2\pi l \pm \hat{\omega}_o) f_s}{2\pi} = \frac{2\pi l \pm 0.4\pi}{2\pi} = l \pm 0.2 = 0.2, 0.8\ 1.2,\ 1.8,\ \ldots\ldots\ \text{Hz}$$

Shannon's Sampling Theorem

- How frequently do we need to sample?
- The solution: Shannon's Sampling Theorem: A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken a rate $f_s = 1 / T_s$ that is greater than $2 f_{max}$.
- Note that the minimum sampling rate, $2 f_{max}$, is called the Nyquist rate.

Spectrum of the Discrete-time Signal



- There are an infinite number of frequency components of discrete-time signal
- They consists of the principal along with the other aliases (an infinite number of them).

Nyquist Rate

- Shannon's theorem tell us that if we have at least 2 samples per period of a sinusoid, we have enough information to reconstruct the sinusoid.
- What happens if we sample at a rate which is less than the Nyquist Rate?
  - Aliasing will occur!!!!

Ideal Reconstruction

- The sampling theorem suggests that a process exists for reconstructing a continuous-time signal from its samples.
- If we know the sampling rate and know its spectrum then we can reconstruct the continuous-time signal by **scaling** the **principal alias** of the discrete-time signal to the frequency of the continuous signal.
- The normalized frequency will always be in the range between $0 \sim \pi$ and be the principal alias if the sampling rate is greater than the Nyquist rate.

Ideal Reconstruction Continued

- If continuous-time signal has a frequency of ω, then the discrete-time signal will have a principal alias of

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

- So we can use this equation to determine the frequency of the continuous-time signal from the principal alias:

$$\omega = \hat{\omega} f_s = \frac{\hat{\omega}}{T_s}$$

- Note that the normalized frequency must be less than π if the Nyquist rate is used

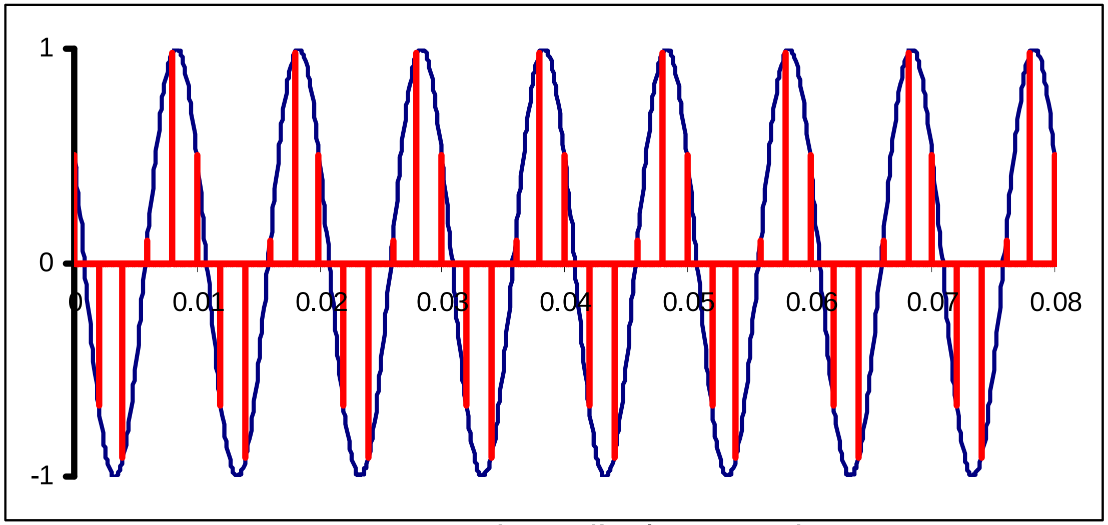
$$\hat{\omega} = \omega_{MAX} T_s = 2\pi f_{MAX} T_s = \frac{2\pi f_{MAX}}{f_s} = \frac{2\pi f_{MAX}}{f_s(\geq 2 f_{MAX})} \leq \pi$$

- And the reconstructed continuous-time frequency must be

$$\omega = 2\pi f = \hat{\omega} f_s \Rightarrow f = \frac{\hat{\omega} f_s}{2\pi} \leq \frac{\pi f_s}{2\pi} = \frac{f_s}{2} = f_{\max}$$

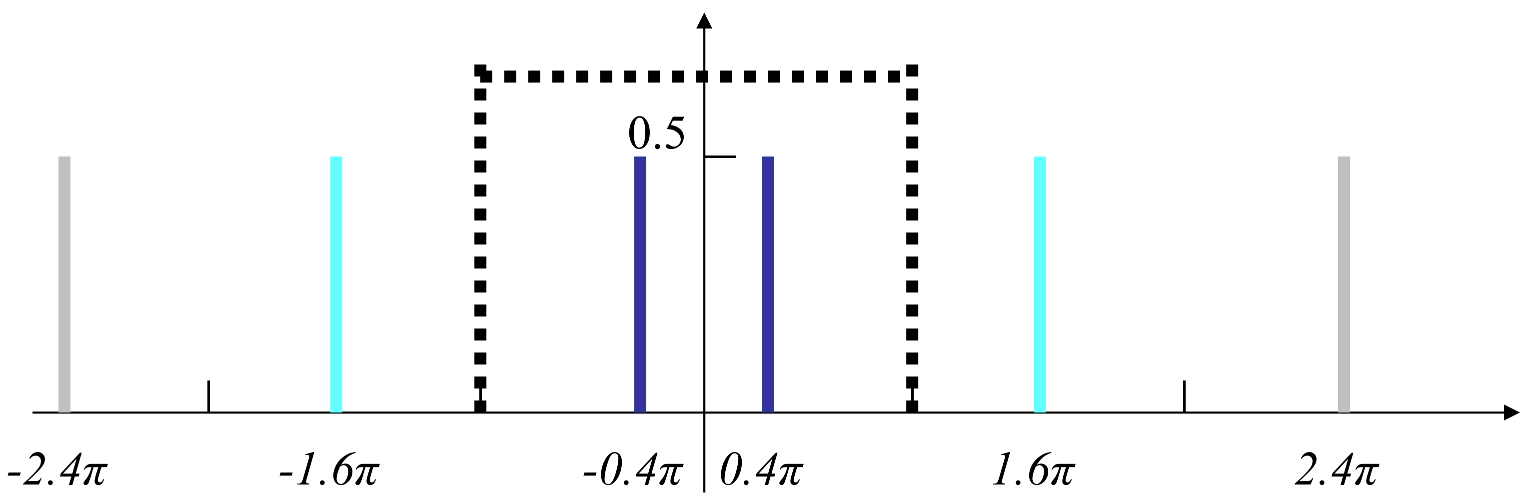
Oversampling

- When we sample at a rate which is greater than the Nyquist rate, we say we are oversampling.
- If we are sampling a 100 Hz signal, the Nyquist rate is 200 samples/second => $x(t)=\cos(2\pi(100)t+\pi/3)$
- If we sample at 2.5 times the Nyquist rate, then $f_s = 500$ samples/sec
- This will yield a normalized frequency at $2\pi(100/500) = 0.4\pi$



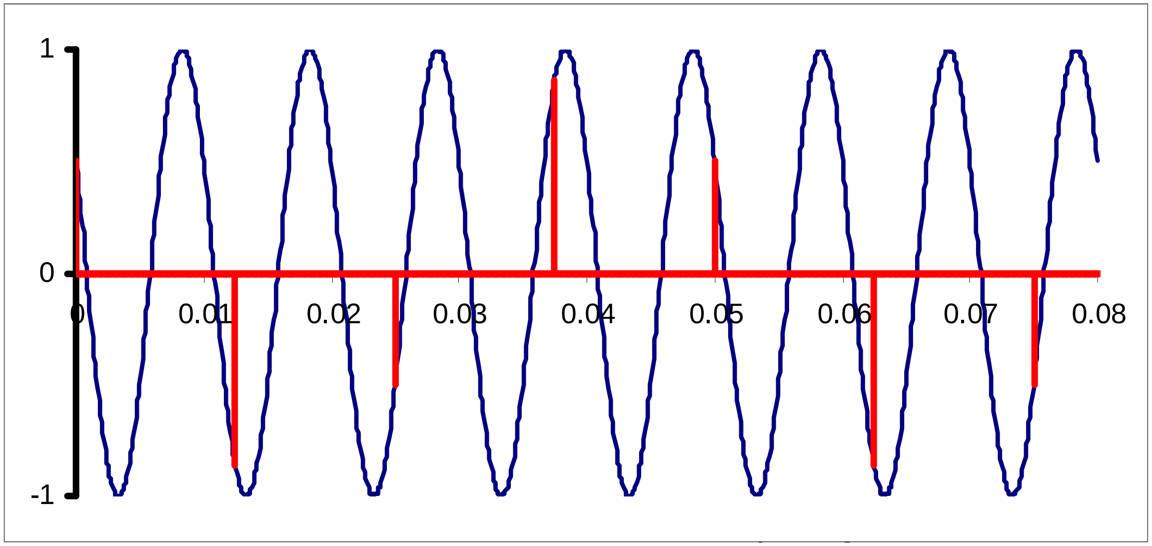
Oversampling

- Since we are greater than the Nyquist rate, the normalized frequency will be $< \pi$ which means it is the principal alias.
- And we get back the original continuous frequency when we do the reconstruction
- $f = 0.4\pi f_s / 2\pi = 0.4\pi 500 / 2\pi = 0.2\ (500) = 100$ Hz



Undersampling and Aliasing

- When we sample at a rate which is less than the Nyquist rate, we say we are undersampling and aliasing will yield misleading results.
- If we are sampling a 100 Hz signal, the Nyquist rate is 200 samples/second => $x(t)=\cos(2\pi(100)t+\pi/3)$
- If we sample at .4 times the Nyquist rate, then f_s = 80 s/sec
- This will yield a normalized frequency at $2\pi(100/80) = 2.5\pi$

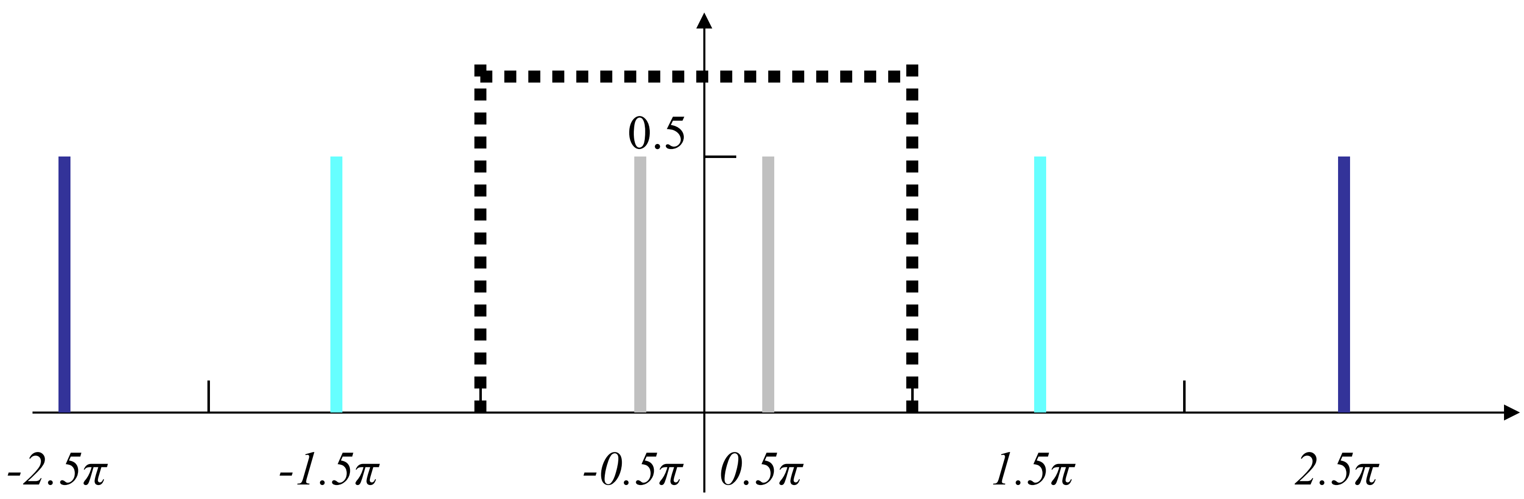


Undersampling

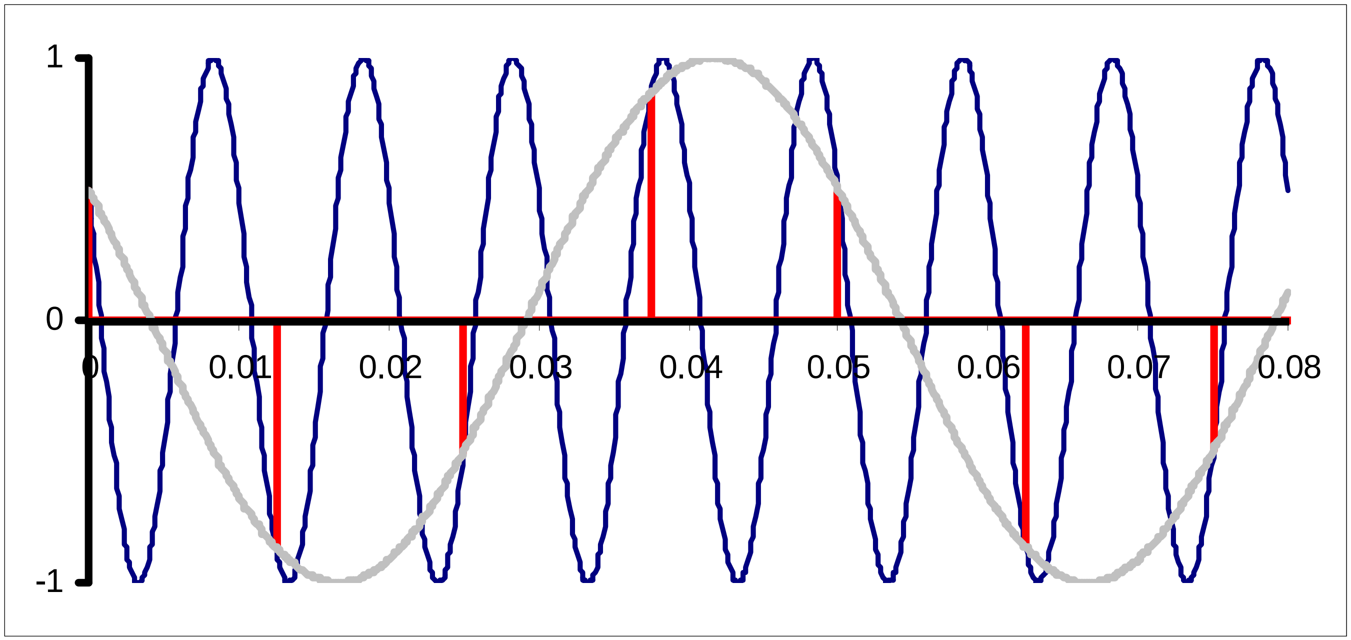
- Since it is > π, 2.5π is NOT the principal alias
- The principle alias is $2.5\pi - 2\pi = 0.5\pi$

- Using 0.5π as the principal alias and performing a reconstruction we then have:

  $f = 0.5\pi f_s / 2\pi = 0.5\pi\ 80 / 2\pi = 0.5\ (40) = 20$ Hz and we have reconstructed the ***wrong*** signal!!!



The Alias Problem due to Undersampling



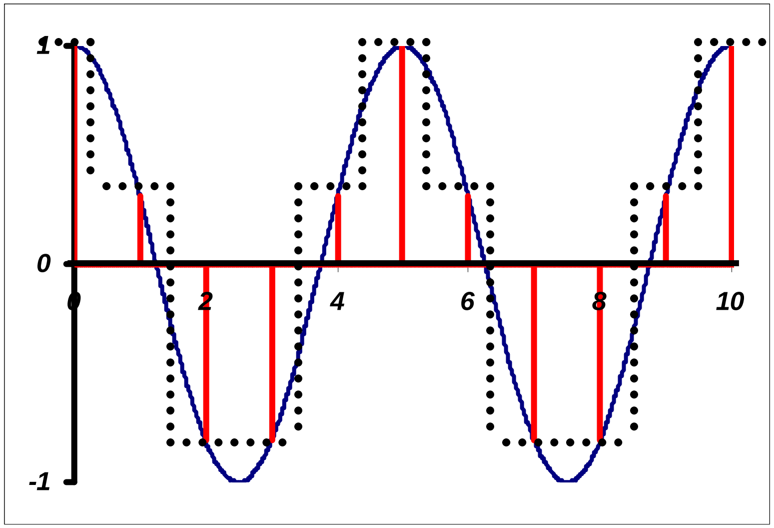
Aliasing and Folding

- Your book treats undersampling in terms of aliasing and folding
- During reconstruction, both of these phenomenon will produce erroneous results.
- The difference between aliasing and folding has to do with which part of the spectrum created the alias.

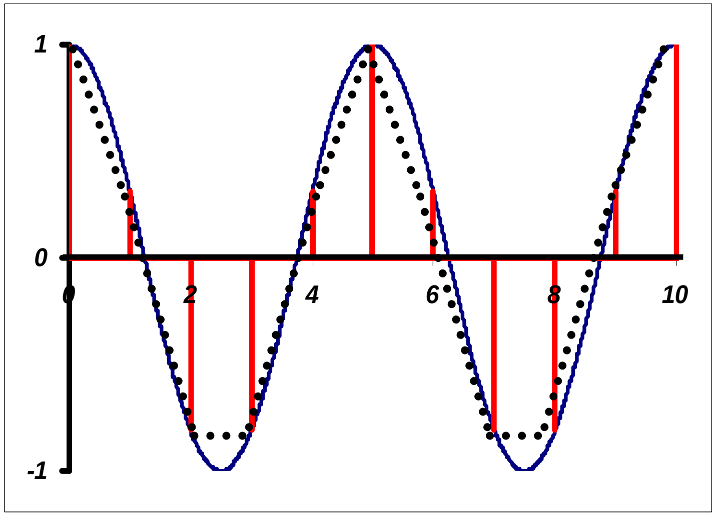
Discrete-to-Continuous Conversion

- An D-to-C converter uses the samples to reconstruct the continuous-time signal by interpolation.
- There are various interpolation algorithms which may be used:
  - Zero-Order Hold
  - Linear
  - Cubic Spline

Interpolation



Zero-Order Hold



Linear

- Oversampling always improves the reconstruction
- Best reconstruction is Low Pass Filter or what the text calls: Ideal Bandlimited Interpolation

Non-sinusoidal Signals

- Since a Fourier series can be written for any continuous-time signal, the sampling and reconstruction processes for any continuous-time signal is the same
  - Shannon's Sampling theorem
  - Nyquist Rate $f_s \geq 2f_{max}$ to eliminate aliasing
  - Oversampling to improve interpolation
  - Ideal (low pass filter) Bandlimited interpolation

Homework

- Exercises:
  - 4.1 – 4.5
- Problems:
  - 4.1, 4.2, 4.3,
  - 4.4, Use Matlab to plot the signal in part a.
  - 4.5, 4.8, 4.11, 4.19