System Control Basic Problem

• Say you have a system which you can not alter but its response is not optimal
• Examples
  – Motor control for exoskeletons
  – Robotic control
• Problems that can occur
  – Unstable Transient Response
  – Oscillations
  – Overshoots
  – Rise Time
  – Output Errors
**Uncontrolled System**

**Unit Step Response**

\[ G_{open}(s) = \frac{1}{s^2 + 10s + 20} \]

\[ \text{Output}(s) = \frac{1}{s^2 + 10s + 20} \frac{1}{s(s+7.2)(s+2.8)} = \frac{K_1}{s+7.2} + \frac{K_2}{s+2.8} + \frac{K_3}{s} \]

\[ K_3 = \frac{1}{s^2 + 10s + 20} \bigg|_{s=0} = \frac{1}{20} = 0.05 \]

\[ K_1 = \frac{1}{s(s+2.8)} \bigg|_{s=-7.2} = \frac{1}{-7.2(-7.2+2.8)} = 0.03 \]

\[ K_2 = \frac{1}{s(s+7.2)} \bigg|_{s=-2.8} = \frac{1}{-2.8(-2.8+7.2)} = -0.08 \]

\[ \text{Output}(t) = (0.03e^{-0.08e^{-2.8t}} + 0.05)u(t) \]

Output value is in error: 1-0.05=95% error

Rise Time: 1 second

---

\[ G_{open}(s) = \frac{1}{s^2 + 10s + 200} \]

\[ \text{Output}(s) = \frac{1}{s^2 + 10s + 200} \frac{1}{s(s+5+j13.2)(s+5-j13.2)} = \frac{K_1}{s+5+j13.2} + \frac{K_2}{s+5-j13.2} + \frac{K_3}{s} \]

\[ K_3 = \frac{1}{s^2 + 10s + 200} \bigg|_{s=-5} = \frac{1}{200} = 0.005 \]

\[ K_1 = \frac{1}{s(s+5-j13.2)} \bigg|_{s=-j5-j13.2} = \frac{1}{-5-j13.2(-5-j13.2+5-j13.2)} = 0.0027 \angle -2.8 \]

\[ \text{Output}(t) = (0.0054\cos(13.2t-2.8)e^{-5t} + 0.005)u(t) \]

Problems

Output value is in error: 1-0.005=99.5% error

Overshoot & Oscillations: 0.006-0.005=0.001 \Rightarrow 20%

Rise time: 1 second
Uncontrolled System

\[ G_{open}(s) = \frac{1}{as^2 + bs + c} \]

\[ Output(s) = \frac{1}{as^2 + bs + c} \cdot \frac{1}{s + r_1 + s + r_2 + \frac{K_1}{s} + \frac{K_2}{s + r_1} + \frac{K_3}{s + r_2}} \]

\[ Output(t) = (K_1 e^{-r_1t} + K_2 e^{-r_2t} + K_3)u(t) \] where roots are real and unequal

\[ = [(K_1 t + K_3) e^{-r_1t} + K_3]u(t) \] where roots are real and equal \( r = r_1 = r_2 \)

\[ = [2K_1 e^{-\alpha t} \cos(\omega t + \theta) + K_3]u(t) \] where roots are imaginary and unequal \( r_1 = \alpha + j\omega; r_2 = \alpha - j\omega \)

Final Output Value

As \( t \to \infty \), \( Output(t) \to K_3 \) = \( \frac{1}{as^2 + bs + c} \bigg|_{s=0} = \frac{1}{c} \neq 1 \)

Roots

\[ r_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a} \]

Let’s choose \( a = 1 \), \( b = 2\xi \omega_n \), and \( c = \omega_n^2 \)

\[ r_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - 1} \]

For the underdamped case, \( \xi^2 > 1 \), \( r_{1,2} = -\xi \omega_n \pm j\omega_n \sqrt{1 - \xi^2} \).
Take our system, $G_{open}(s)$, add a PID controller, $G_{PID}(s)$, in series and put them a unity negative feedback arrangement
**PID Controller Elements**

\[
PIDs_{\text{output}}(s) = (K_p + \frac{K_I}{s} + K_Ds) \times \text{dif}(s)
\]

\[
= \frac{K_Ds^2 + K_ps + K_I}{s} \times \text{dif}(s)
\]
PID

\[ G_{open}(s) = \frac{1}{as^2 + bs + c} \]
\[ G_{PID}(s) = K_P + \frac{K_I}{s} + K_Ds = \frac{K_Ds^2 + K_Ps + K_I}{s} \]
\[ G_{CLOSED}(s) = \frac{G_{open}(s)G_{PID}(s)}{1 + H(s)G_{open}(s)G_{PID}(s)} \]
\[ H(s) = 1 \]
\[ G_{open}(s)G_{PID}(s) = \frac{1}{as^2 + bs + c} \frac{K_Ds^2 + K_Ps + K_I}{s} = \frac{K_Ds^2 + K_Ps + K_I}{as^3 + bs^2 + cs} \]
\[ G_{CLOSED}(s) = \frac{\frac{K_Ds^2 + K_Ps + K_I}{as^3 + bs^2 + cs}}{1 + \frac{K_Ds^2 + K_Ps + K_I}{as^3 + bs^2 + cs}} = \frac{K_Ds^2 + K_Ps + K_I}{as^3 + bs^2 + cs + K_Ds^2 + K_Ps + K_I} \]
PID P Control only

\[ G_{\text{CLOSED}}(s) = \frac{K_D s^2 + K_P s + K_I}{s^3 + (b + K_D)s^2 + (c + K_P)s + K_I} \]

\[ K_I = K_D = 0 \]

\[ G_{\text{CLOSED}}(s) = \frac{K_P}{s^2 + bs + (c + K_P)} \]

\[ \text{Output}(s) = \frac{K_P}{s^2 + bs + (c + K_P)} \left( \frac{1}{s} + \frac{K_1}{s + r_1} + \frac{K_2}{s + r_2} + \frac{K_3}{s} \right) \]

\[ \text{Output}(t) = (K_1 e^{-rt} + K_2 e^{-r' t} + K_3)u(t) \text{ where roots are real and unequal} \]

\[ = [(K_1 + K_2) e^{-rt} + K_3]u(t) \text{ where roots are real and equal } r = r_1 = r_2 \]

\[ = [2K_1 e^{-\alpha t} \cos(\omega t + \theta) + K_3]u(t) \text{ where roots are imaginary and unequal } r_i = \alpha + j\omega; r_2 = r_1^* \]
**PID P Control only**

**P CONTROLLER**

\[
G_{CLOSED}(s) = \frac{K_p}{as^2 + bs + (c + K_p)}
\]

\[
Output(s) = \frac{K_p}{as^2 + bs + (c + K_p)} \cdot \frac{1}{s} = \frac{K_1}{s + r_1} + \frac{K_2}{s + r_2} + \frac{K_3}{s}
\]

Final Output Value

As \( t \to \infty \), \( Output(t) \to K_3 = \left. \frac{K_p}{as^2 + bs + (c + K_p)} \right|_{s=0} = \frac{K_p}{c + K_p} \to 1 \) as \( K_p \to \infty \)

Roots

\[
r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a(c + K_p)}}{2a}
\]

Let's choose \( a = 1 \), \( b = 2\xi \omega_n \), and \( c = \omega_n^2 \)

\[
r_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - \left(1 + \frac{K_p}{\omega_n^2}\right)}
\]

For the underdamped case,

\[
\xi^2 < \frac{1}{1 + \frac{K_p}{\omega_n^2}}, \quad r_{1,2} = -\xi \omega_n \pm j\omega_n \sqrt{\left(1 + \frac{K_p}{\omega_n^2}\right) - \xi^2},
\]

\( K_p \) increases the frequency of damped oscillations.

For the critically damped or over-damped case,

\[
\xi^2 > \frac{1}{1 + \frac{K_p}{\omega_n^2}}, \quad r_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - \left(1 + \frac{K_p}{\omega_n^2}\right)},
\]

\( K_p \) increases one of the poles and may decrease rise time.
PID P Control only

P Controller: Open Loop Red; P Blue

\[
\begin{align*}
\text{Open: } & -7.2361, -2.7639 \\
\text{P: } & -5+18.5742i, -5-18.5742i \\
\frac{1}{s^2 + 10s + 20} & \quad \frac{350}{s^2 + 10s + 370}
\end{align*}
\]

P CONTROLLER

\[
G_{\text{closed}}(s) = \frac{K_p}{as^2 + bs + (c + K_p)}
\]

\[
\text{Output}(s) = \frac{K_p}{as^2 + bs + (c + K_p)} \frac{1}{s}
\]

Kp: 350

\[
\text{Output}(t) = [2K_p e^{-\alpha t} \cos(\omega t + \theta) + K_s]u(t) \text{ where roots are imaginary and unequal } r_1 = \alpha + j\omega, r_2 = r_1^*
\]

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**PID PD CONTROL**

**PD CONTROLLER**

\[
G_{\text{CLOSED}}(s) = \frac{K_D s^2 + K_P s + K_I}{s^3 + (b + K_D)s^2 + (c + K_P)s + K_I}
\]

\[K_I = 0\]

\[
G_{\text{CLOSED}}(s) = \frac{K_P s^2 + K_P s}{s^3 + (b + K_D)s^2 + (c + K_P)s} = \frac{K_P s + K_P}{s^2 + (b + K_D)s + (c + K_P)}
\]

\[\text{Output}(s) = \frac{K_P s + K_P}{s^2 + (b + K_D)s + (c + K_P)} \cdot \frac{1}{s} = \frac{K_1}{s + r_1} + \frac{K_2}{s + r_2} + \frac{K_3}{s}\]

\[\text{Output}(t) = (K_I e^{-rt} + K_P e^{-rt}) + K_3 u(t)\]

where roots are real and unequal

\[\text{Output}(t) = [(K_I + K_P)e^{-rt} + K_3]u(t)\]

where roots are real and equal \(r_1 = r_2\)

\[\text{Output}(t) = [2K_I e^{-rt} \cos(\alpha t + \theta) + K_3]u(t)\]

where roots are imaginary and unequal \(r_1 = \alpha + j\omega; r_2 = \alpha \ast\)

**Final Output Value**

As \(t \to \infty\), \(\text{Output}(t) \to K_1 = \frac{K_P s^2 + K_P s}{s^3 + (b + K_D)s^2 + (c + K_P)} \bigg|_{s=0} = \frac{K_P}{c + K_P} \to 1\) as \(K_p \to \infty\)

**Roots**

\[r_{1,2} = \frac{-b + K_D \pm \sqrt{(b + K_D)^2 - 4ac(b + K_P)}}{2a}\]

Let's choose \(a = 1,\ b = 2\xi\omega_n,\ \text{and}\ c = \omega_n^2\)

\[r_{1,2} = \frac{-2\xi\omega_n + K_D \pm \sqrt{(2\xi\omega_n + K_D)^2 - 4(\omega_n^2 + K_P)}}{2}\]

\[r_{1,2} = \frac{-\xi\omega_n + \frac{K_D}{2} \pm \omega_n \sqrt{\left(\xi + \frac{K_D}{2\omega_n}\right)^2 - \left(1 + \frac{K_P}{\omega_n}\right)}}{2}\]

\(K_D\) has the potential to make the transient become critically damped or overdamped.
**PID PD CONTROL**

\[ G_{CLOSED}(s) = \frac{K_p s^2 + K_i s}{s^2 + (b + K_p) s^2 + (c + K_p) s} = \frac{K_p s + K_i}{s^2 + (b + K_p) s^2 + (c + K_p) s} \]

Output(s) = \frac{K_p s + K_i}{s^2 + (b + K_p) s^2 + (c + K_p) s} \frac{1}{s} = \frac{K_1}{s + r_1} + \frac{K_2}{s + r_2} + \frac{K_3}{s}

Output(t) = (K_1 e^{-r_1 t} + K_2 e^{-r_2 t} + K_3) u(t) \text{ where roots are real and unequal}

Kp: 350  Kd: 50

---


\[
\begin{align*}
\frac{1}{s^2 + 10s + 20} & \quad 350 \quad \frac{50s + 350}{s^2 + 10s + 370} \\
\quad & \quad \quad \quad \quad \frac{50s + 350}{s^2 + 60s + 370}
\end{align*}
\]
PID I Control Only

I CONTROLLER

\[ G_{\text{CLOSED}}(s) = \frac{K_p s^2 + K_p s + K_I}{as^3 + (b + K_D)s^2 + (c + K_p)s + K_I} \]

\[ K_p = K_D = 0 \]

\[ G_{\text{CLOSED}}(s) = \frac{K_I}{as^3 + bs^2 + cs + K_I} \]

\[ \text{Output}(s) = \frac{K_I}{as^3 + bs^2 + cs + K_I} \cdot \frac{1}{s + r_1} + \frac{K_2}{s + r_2} + \frac{K_3}{s + r_3} + \frac{K_4}{s} \]

\[ \text{Output}(t) = (K_1 e^{-r_1 t} + K_2 e^{-r_2 t} + K_3 e^{-r_3 t} + K_4) u(t) \text{ where roots are real and unequal} \]

\[ = [(K_1 t + K_2)e^{-r_1 t} + K_3 e^{-r_3 t} + K_4] u(t) \text{ where roots are real and equal } r = r_1 = r_2 \]

\[ = [(K_1 t^2 + K_2 t + K_3)e^{-r_1 t} + K_4] u(t) \text{ where roots are real and equal } r = r_1 = r_2 = r_3 \]

\[ = [2K_1 e^{-\alpha t} \cos(\omega t + \theta) + K_3 e^{-r_3 t} + K_4] u(t) \text{ where 2 roots are imaginary and unequal } r_1 = \alpha + j\omega; r_2 = r_1^* \]

Final Output Value

\[ \text{As } t \to \infty, \text{ Output}(t) \to K_4 = \frac{K_I}{K_I} = 1 \]
**PID**

\[
G_{\text{CLOSED}}(s) = \frac{K_D s^2 + K_P s + K_I}{s^3 + b s^2 + c s + K_D s^2 + K_P s + K_I}
\]

\[
\text{Output}(s) = \frac{K_D s^2 + K_P s + K_I}{s^3 + b s^2 + c s + K_D s^2 + K_P s + K_I} \cdot \frac{1}{s} = \frac{K_1}{s + r_1} + \frac{K_2}{s + r_2} + \frac{K_3}{s + r_3} + \frac{K_4}{s}
\]

Output\( (t) = (K_1 e^{-r_1 t} + K_2 e^{-r_2 t} + K_3 e^{-r_3 t} + K_4) u(t) \) where roots are real and unequal

\[
= [(K_1 t + K_2) e^{-r_1 t} + K_3 e^{-r_2 t} + K_4] u(t) \text{ where roots are real and equal } r = r_1 = r_2
\]

\[
= [(K_1 t^2 + K_2 t + K_3) e^{-r_1 t} + K_4] u(t) \text{ where roots are real and equal } r = r_1 = r_2 = r_3
\]

\[
= [2K_1 e^{-\alpha t} \cos(\omega t + \theta) + K_3 e^{-r_3 t} + K_4] u(t) \text{ where 2 roots are imaginary and unequal } r_1 = \alpha + j \omega; r_2 = r_1^* \]

Final Output Value

As \( t \to \infty \), Output\( (t) \to K_4 = \frac{K_D s^2 + K_P s + K_I}{s^3 + b s^2 + c s + K_D s^2 + K_P s + K_I} \bigg|_{s=0} = \frac{K_I}{K_I} = 1 \)
### PID

![PID Control System Diagram](image)

**PID Controller:** Open Loop Red; I Blue; ID Green; PID Black

**Transfer Functions:**
- **Open Loop:**
  \[
  \frac{1}{s^2 + 10s + 20}
  \]
  \[
  \frac{300}{s^3 + 10s^2 + 20s + 300}
  \]

- **ID Loop:**
  \[
  \frac{-59.7493s - 0.12535s + 2.2372i}{s^3 + 60s^2 + 20s + 300}
  \]
  \[
  \frac{-53.144s - 5.8991i - 0.95694i}{s^3 + 60s^2 + 370s + 300}
  \]

- **PID Loop:**
  \[
  \frac{50s^2 + 300}{s^3 + 60s^2 + 20s + 300}
  \]
  \[
  \frac{50s^2 + 350s + 300}{s^3 + 60s^2 + 370s + 300}
  \]

**PID Parameters:**
- **Kp:** 350
- **Kd:** 50
- **Ki:** 300
PID

PID Controller: Open Loop Red; I Blue; ID Green; PID Black

Open:\[ -7.2361, -2.7639 \]

I: \[ -9.012, -0.49399 + 3.2943i, -0.49399 - 3.2943i \]

\[
\frac{1}{s^2 + 10s + 20} \quad \frac{100}{s^3 + 10s^2 + 20s + 100}
\]

ID: \[ -59.693, -0.15349 + 1.2852i, -0.15349 - 1.2852i \]

\[
\frac{50s^2 + 100}{s^3 + 60s^2 + 20s + 100}
\]

PID: \[ -53.0626, -6.6542, -0.28322 \]

\[
\frac{50s^2 + 350s + 100}{s^3 + 60s^2 + 370s + 100}
\]

Kp: 350  \quad Kd: 50  \quad Ki: 100

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General Tips for Designing a PID Controller

1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error

Adjust each of $K_p$, $K_i$, and $K_d$ until you obtain a desired overall response. It is not necessary to implement all three controllers (proportional, derivative, and integral) into a single system. Keep the controller as simple as possible.

<table>
<thead>
<tr>
<th>CL RESPONSE</th>
<th>RISE TIME</th>
<th>OVERSHOOT</th>
<th>SETTLING TIME</th>
<th>S-S ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_p$</td>
<td>Decrease</td>
<td>Increase</td>
<td>Small Change</td>
<td>Decrease</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Decrease</td>
<td>Increase</td>
<td>Increase</td>
<td>Eliminate</td>
</tr>
<tr>
<td>$K_d$</td>
<td>Small Change</td>
<td>Decrease</td>
<td>Decrease</td>
<td>No Change</td>
</tr>
</tbody>
</table>


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Matlab Control System Toolbox

• Transfer Function tf(num,den)
  – Constructs a transfer function
  – Inputs are arrays which represent the coefficients of the numerator and denominator of the desired transfer function

\[
\frac{s + 3}{5s^3 + 2s + 4}
\]

num=[1 3];
denom=[5 0 2 4];
tf(num,denom)

Transfer function:
\[ \frac{s + 3}{5s^3 + 2s + 4} \]
Matlab Control System Toolbox

- **Step Function Response** `step(tf,t)`
  - Calculates the response due to a step function
  - Inputs are desired transfer function and timeframe

\[
\frac{1}{5s^2 + 2s + 4}
\]

Transfer function:
\[
\frac{1}{5s^2 + 2s + 4}
\]

num=[1];
denom=[5 2 4];
testtf=tf(num,denom)
time=0:0.1:20;
step(testtf,time)
Matlab Control System Toolbox

- Series `series(tf1,tf2)`
  - Constructs a transfer function from 2 transfer functions
  - Inputs are transfer functions

\[
TF1 = \frac{1}{5s^2 + 2s + 4}
\]

\[
TF2 = \frac{s + 3}{5s^3 + 2s + 4}
\]

- num=[1];
  - denom=[5 2 4];
  - testtf1=tf(num,denom);
  - num=[3 1];
  - denom=[5 0 2 4];
  - testtf2=tf(num,denom);
  - series(testtf1,testtf2)

Transfer function:

\[
3s + 1
\]

-----------------------------

\[
25s^5 + 10s^4 + 30s^3 + 24s^2 + 16s + 16
\]
Matlab Control System Toolbox

- Feedback feedback(openlooptf,feedbacktf)
  - Constructs a transfer function of the feedback arrangement
  - Inputs are transfer functions of the openloop system and the feedback system; for unity feedback use 1 as the second input.

\[
\text{Opentf} = \frac{1}{5s^2 + 2s + 4}
\]

\[
\text{Closedloop} = \frac{1}{5s^2 + 2s + 4} \frac{5s^2 + 2s + 4}{1 + \frac{5s^2 + 2s + 4}{5s^2 + 2s + 4}} = \frac{1}{5s^2 + 2s + 5}
\]

num=[1];
denom=[5 2 4];
testtf=tf(num,denom);
feedback(testtf,1)

Transfer function:
\[
\frac{1}{5s^2 + 2s + 5}
\]

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Matlab Control System Toolbox

• Roots roots(array)
  – Calculates the roots of a polynomial
  – Inputs are arrays which represent the coefficients of the polynomial

\[5s^3 + 2s + 4\] roots([5 0 2 4])

\[0.3930 + 0.9292i\]
\[0.3930 - 0.9292i\]
\[-0.7860\]
Matlab Control System Toolbox

- SISO PID GUI Tool
Homework

• Using the Matlab control system toolbox functions, choose $K_D$, $K_P$, and $K_I$ to yield a step response that has small output error, no overshoots, and a rise time of less than 1 second for the following transfer function:

$$\frac{1}{s^2 + 10s + 200}$$

• Chose each PID separately to show what it does to the system. Show the roots for each case.
• Most likely you’ll have to iterate your answer to yield a good response.