Systems

Lecture #3

1.3
Representation of a System

• How do represent a system mathematically?
  – Since a system transforms a signal into another we write an equation:
    \[ y(t) = \mathcal{T}\{x(t)\} \]
  – where \( \mathcal{T} \) is an operator to symbolize a system,
  – \( x(t) \) is the signal that goes into the system: input signal (or source)
  – And \( y(t) \) is transformed signal or output signal (or solution of the equation)

• We can also represent it by a flow diagram

\[ x(t) \xrightarrow{\mathcal{T}} y(t) \]
Example of a Continuous-Time System

• A squarer system: \( y(t) = \{x(t)\}^2 \)
  - The output equals the square of the input.
  - This is the result of putting the sine wave into the squarer

\[ \begin{align*}
\text{Input: } x(t) & \\
\text{Output: } y(t) &= \{x(t)\}^2 
\end{align*} \]

• This is an example of a continuous-time system
• We might be able to build this using an electronic circuit
Discrete-Time Systems

• If we put a discrete-time signal into a system the output may be a discrete-time signal
• This is called a Discrete-time system.

\[ y[n] = \mathcal{T}\{x[n]\} \]

• Using our squarer example: \( y[n] = \{x[n]\}^2 \)
Mixed Systems

- Continuous-to-Discrete systems
  \[ y[n] = \mathcal{T}\{x(t)\} \]
  - Example: a sampler: \[ y[n] = x(nt_s) \]
    - This is also called a A-to-D converter
- Discrete-to-Continuous systems
  \[ y(t) = \mathcal{T}\{x[n]\} \]
  - Example: An D-to-A converter
    - The opposite of a sampler
    - Takes the samples a recreates the Continuous Signal
An Example

• Example: A music CD

Music $\rightarrow$ Recorder $\Rightarrow x(t)$

A-to-D Converter $\rightarrow$ Optical Disk Writer $\rightarrow$ Optical Disk Reader $\Rightarrow x[n]$

CD $x[n]$ $\Rightarrow$ D-to-A Converter $\Rightarrow x(t)$

Stereo $\rightarrow$ Listener
Some Basic Properties of Linear Systems

• If a system is Linear, or better yet Linear and Time Invariant (LTI), it is easier to analyze and understand than systems that are non-linear and/or vary with time.

• All LTI systems must be
  – Linear and support superposition
  – Causal
  – Time Invariant
Linearity for Continuous Signals

$x_1(t)$

$x_2(t)$

$x_3(t) = a_1x_1(t)$

$x_3(t) = a_1x_1(t) + a_2x_2(t)$

$y_1(t)$

$y_2(t)$

$y_3(t) = a_1y_1(t)$

$y_3(t) = a_1y_1(t) + a_2y_2(t)$

SCALAR

SUPERPOSITION
**Shorthand**

\[ x_k(t) \rightarrow y_k(t) \]

\[ \sum_k a_k x_k(t) \rightarrow \sum_k a_k y_k(t) \]
Same for Discrete Signals

\[ x_k[n] \rightarrow y_k[n] \]

\[ \sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n] \]
Causality

• A system is causal if the output at any time depends only on the input values up to that time
• $y(t_o)$ does not depend on $x(t_i)$ that occur at times after $t_o$, $t_i > t_o$.
• True for all real time physical systems
• Not true for system-processed recorded signals or spatial varying signal
  – Such systems can look ahead or left, right, up & down
  – E.g., a Morphing System
Causality

• Not Causal

\[ y(t) \]  \( \rightarrow \)  \[ x(t) \]

\[ y(t_0) \]  \( \rightarrow \)  \[ x(t_i) \]
Causality

- Causal

Allowable Values of $t_i$

$y(t)$

$x(t)$

$y(t_0)$

$t_0$
Time Invariance

Continuous Signals

\[ x_k(t) \rightarrow y_k(t) \]

Delay \( x(t) \) by \( t_0 \) yields same response only later

\[ x_k(t-t_0) \rightarrow y_k(t-t_0) \]

Discrete Signals

\[ x_k[n] \rightarrow y_k[n] \]

\[ x_k[n-n_0] \rightarrow y_k[n-n_0] \]
A Non-LTI System

A multiplier which is a function of time

\[ x(t) \quad y(t) = g(t) \cdot x(t) \]

Check Superposition:
\[ x_1(t) \text{ yields } y_1(t) = g(t) \cdot x_1(t) \]
\[ x_2(t) \text{ yields } y_2(t) = g(t) \cdot x_2(t) \]

let \( x_3(t) = a_1 x_1(t) + a_2 x_2(t) \) then
\[ y_3(t) = g(t) \cdot x_3(t) = g(t) \left[ a_1 x_1(t) + a_2 x_2(t) \right] \]
\[ = a_1 y_1(t) + a_2 y_2(t) \]
OK

Check Time Invariance:
\[ x_1(t) = x(t) \text{ yields } y(t) = g(t) \cdot x(t) \]
\[ x_2(t) = x(t-t) \text{ yields } y_2(t) = g(t) \cdot x_2(t) \]
\[ = g(t)x(t-t) \]

But to be TI
\[ x_2(t) = x(t-t) \text{ yields } y_2(t) = y(t-t) \]
\[ = g(t-t) \cdot x(t-t) \]
Not OK
Another Non-LTI System

A system with an additive constant

\[ y(t) = x(t) + K \]

Check Superposition:

For Superposition to hold, we need to have:

let \( x(t) = a_1 x_1(t) + a_2 x_2(t) \) then \( y(t) = a_1 x_1(t) + a_2 x_2(t) + K \)

But for this system:

\[ y(t) = y_1(t) + y_2(t) = a_1 x_1(t) + K + a_2 x_2(t) + K \]

Not OK
How Does One Describe LTI Systems

- For Continuous Systems – By Using Ordinary Differential Equations (ODE)
- For Discrete Systems – By Using Difference Equations
**1st Order Linear ODE: Simple Electrical Circuit**

\[ V_s = i(t)R + L \frac{di(t)}{dt} \]

\[ \frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L} \]

Solve for \( i(t) \) assuming: \( i(t) = K_1 e^{-At} + K_2 \) with the initial condition that \( i(0) = 0 \). The 2 terms are need due to the following: Since the source \( V_s \) is a constant (battery), we assume that the output must a component which is a constant, \( K_2 \). Since the differential equation is requires that the output and its derivative be proportional to each other, we assume that the output must have a component which is proportional to an exponential function, \( K_1 e^{-At} \).
**1st Order Linear ODE: Simple Electrical Circuit**

\[ V_s = i(t)R + L \frac{di(t)}{dt} \]
\[ \frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L} \]

Substituting \( i(t) = K_1e^{-At} + K_2 \) in the equation, we get

\[ -AK_1e^{-At} + \frac{R}{L}K_1e^{-At} + \frac{R}{L}K_2 = \frac{V_s}{L} \]

Resorting we have

\[ -AK_1e^{-At} + \frac{R}{L}K_1e^{-At} + \frac{R}{L}K_2 = \frac{V_s}{L} \]

This implies

\[ -AK_1e^{-At} + \frac{R}{L}K_1e^{-At} = 0 \]

\[ \frac{R}{L}K_2 = \frac{V_s}{L} \]

\[ -AK_1e^{-At} + \frac{R}{L}K_1e^{-At} = 0 \]
\[ -A + \frac{R}{L} = 0; A = \frac{R}{L} \]

\[ \frac{R}{L}K_2 = \frac{V_s}{L}; K_2 = \frac{V_s}{R} \]

Therefore,

\[ i(t) = K_1e^{-\frac{R}{L}t} + \frac{V_s}{R} \]

But the initial condition states that \( i(0) = 0 \)

\[ i(0) = K_1e^{-\frac{R}{L} \cdot 0} + \frac{V_s}{R} = K_1 + \frac{V_s}{R} = 0 \]

\[ K_1 = -\frac{V_s}{R} \]

\[ i(t) = \frac{V_s}{R} \left(1 - e^{-\frac{R}{L}t}\right) \]
1st Order Linear ODE: Simple Electrical Circuit

\[ i(t) = \frac{V_S}{R} \left( 1 - e^{-\frac{R}{L}t} \right) = \frac{V_S}{R} \left( 1 - e^{-\frac{t}{L/R}} \right) \]

\( \frac{L}{R} \) is called the time constant and we see that within 3 time constants
95% of its final value is reached.
Another 1st Order LODE: Drug Concentration in Blood Being Removed by the Liver

\[ \dot{D} + K_L D = \frac{R_D}{V_c} \]

Where \( K_L \) = drug loss rate

\( V_c \) = Volume of circulatory system in liters

\( R_D \) is the rate of drug input (mg/min)

In a similar way as in the RL circuit, we can solve this for

\[ D(t) = \frac{R_D}{V_c K_L} \left( 1 - e^{-K_L t} \right) \]
2nd Order LODE

\[ M \ddot{x} + B \dot{x} + Kx = F(t) \]

- \( M \) = Mass
- \( B \) = Friction
- \( K \) = Spring constant

\[ L \dddot{i} + R \dot{i} + \frac{1}{C} i = 0 \]

- \( R \) = Resistance
- \( L \) = Inductance
- \( C \) = Capacitance
Homework

- Linear Systems
  - Is $y(t) = x(t)^2$ a linear system? Prove your point.
  - Is $y(t) = t^2$ a linear system? Prove your point.
  - CT.1.3.1

- ODE
  - Solve and plot the solution to the equation: $dx/dt + 6 \times \quad x = 0; \quad x(0) = 5$; use Matlab to obtain the plot
  - Solve and plot the solution to the equation: $dx/dt + 6 \times \quad x = 6; \quad x(0) = 0$; use Matlab to obtain the plot