Sinusoidal Response and Discrete Systems

Lecture #5
3CT.2
**Steady State Sinusoidal Response**

- When the source-free or transient response ends in a stable system, what remains is the Steady State Response.
- For systems where the source is a Sinusoid, it is called the Steady State Sinusoidal Response.
- Therefore, we look at the solution of the system for the source response.
Steady State Sinusoidal Response

\[ ay + by + cy = A \cos \omega t \]

Source response is sinusoidal since the Source is Sinusoidal:

\[ y(t) = A_1 \sin \omega t + B_1 \cos \omega t \]

First Derivative: \( \dot{y}(t) = A_1 \omega \cos \omega t - B_1 \omega \sin \omega t \)

Second Derivative: \( \ddot{y}(t) = -A_1 \omega^2 \sin \omega t - B_1 \omega^2 \cos \omega t \)

Substitute the source response and its derivatives into the system equation:

\[ a(-A_1 \omega^2 \sin \omega t - B_1 \omega^2 \cos \omega t) + b(A_1 \omega \cos \omega t - B_1 \omega \sin \omega t) + c(A_1 \sin \omega t + B_1 \cos \omega t) = A \cos \omega t \]

Sort the coefficients of the sine and cosine time dependency functions

\[ ([c - a\omega^2]A_1 - b\omega B_1) \sin \omega t + ([c - a\omega^2]B_1 + b_1 \omega A_1) \cos \omega t = A \cos \omega t \]

Compare the coefficients of the left side of the equation to the right side.
**Steady State Sinusoidal Response**

The coefficients of the cosine function

\[ A = (c - a\omega^2)B_i + b\omega A_i \quad \text{Eqn. 1} \]

The coefficients of the sine function

\[ 0 = (c - a\omega^2)A_i - b\omega B_i \quad \text{Eqn. 2} \]

Solve for the unknown coefficients of the source response, \( A_i \) and \( B_i \)

From Eqn. 2 ⇒ \( B_i = \frac{(c - a\omega^2)}{b\omega} A_i \)

Substituting this into Eqn. 1 and solve for \( A_i \) ⇒ \( A_i = \frac{b\omega}{[(c - a\omega^2)^2 + (b\omega)^2]} A \)

Then for \( B_i \) ⇒ \( B_i = \frac{(c - a\omega^2)}{b\omega} \frac{b\omega}{[(c - a\omega^2)^2 + (b\omega)^2]} \frac{A}{A} = \frac{(c - a\omega^2)}{[(c - a\omega^2)^2 + (b\omega)^2]} A \)

Substituting \( A_i \) and \( B_i \) into \( y(t) \)

\[ y(t) = \frac{A}{[(c - a\omega^2)^2 + (b\omega)^2]} [b\omega \sin \omega t + (c - a\omega^2)\cos \omega t] \]

And combine \( y(t) \)

\[ y(t) = \frac{A}{\sqrt{[(c - a\omega^2)^2 + (b\omega)^2]}} \cos [\omega t - \tan^{-1}\left( \frac{b\omega}{(c - a\omega^2)} \right)] \]

BME 333 Biomedical Signals and Systems
- J. Schesser
A Simpler Approach to Steady State Sinusoidal Systems – Frequency Response

• For systems where the source is a sinusoid, we can replace $p$ with $j\omega$ in the system function $H(p)$ to yield in a complex function of $j\omega$, $H(j\omega)$, or phasor form of $j\omega$,

$$H(j\omega) = A(\omega) + jB(\omega)$$

$$= \sqrt{A(\omega)^2 + B(\omega)^2} \angle \tan^{-1}\left[\frac{B(\omega)}{A(\omega)}\right]$$

• We call $H(j\omega)$ the Frequency Response.

$$\{ j\omega \rightarrow j2\pi F; H(j\omega) \rightarrow H(F) \}$$
**Frequency Response**

We have $A(p)y(t) = B(p)x(t)$

And if $x(t) = X(\omega)\cos \omega t = \Re\{X(j\omega)e^{j\omega t}\}$

where $e^{j\omega t} = \cos \omega t + j \sin \omega t$

then $y(t) = \Re\{Y(j\omega)e^{j\omega t}\}$

If we put $x(t)$ in the system we can get $y(t)$

but instead we can use $Y(j\omega)e^{j\omega t}$ and $X(j\omega)e^{j\omega t}$

Therefore, $A(p)Y(j\omega)e^{j\omega t} = B(p)X(j\omega)e^{j\omega t}$
**Frequency Response**

Therefore, \( A(p)Y(j\omega)e^{j\omega t} = B(p)X(j\omega)e^{j\omega t} \)

which becomes \( Y(j\omega)A(j\omega)e^{j\omega t} = X(j\omega)B(j\omega)e^{j\omega t} \)

\[ A(j\omega)Y(j\omega) = B(j\omega)X(j\omega) \]

Or

\[ Y(j\omega) = \frac{B(j\omega)}{A(j\omega)}X(j\omega) \]

\[ Y(j\omega) = H(j\omega)X(j\omega) \]

\[ H(j\omega) = \frac{B(j\omega)}{A(j\omega)} \]

[The text uses \( \omega = 2\pi F \ y(t) = \Re\{H(F)X(F)e^{j2\pi Ft}\} \) ]
**Frequency Response Using Phasors**

Example:

In our example: \( x(t) = A \cos(\omega t) \Rightarrow X(j\omega) = A \angle 0 \)

\[(ap^2 + bp + c)y(t) = A \cos \omega t \Rightarrow (a(j\omega)^2 + bj\omega + c)Y(j\omega) = A \angle 0 \]

\[
Y(j\omega) = \frac{A \angle 0}{c - a\omega^2 + jb\omega} = \frac{A \angle 0}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}} \angle \tan^{-1}\left(\frac{b\omega}{c - a\omega^2}\right)
\]

\[
Y(j\omega) = \frac{A}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}} \angle -\tan^{-1}\left(\frac{b\omega}{c - a\omega^2}\right)
\]

\[
y(t) = \frac{A}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}} \cos[\omega t - \tan^{-1}\left(\frac{b\omega}{c - a\omega^2}\right)]
\]
Bode Plots

• Plot of the log of the magnitude and angle of the frequency response $H(j\omega)$ on a single logarithmic chart

• Sanity Checks: at $\omega = 0$, $\omega \to \infty$, at other $\omega$’s (e.g., at poles or zero break frequencies or resonance frequencies)

• From the previous second order example:

$$H(j\omega) = \frac{1}{(c - a\omega^2) + jb\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}}$$

$$\phi(j\omega) = -\tan^{-1}\left(\frac{b\omega}{c - a\omega^2}\right)$$
2nd Order ODE Bode Plot
2nd Order ODE Bode Plot

- Undamped
- Underdamped
- Critically Damped
- Overdamped
**Our old Example**

Let us look back again at our RL circuit:

\[
V \cos \omega t = i(t)R + L \frac{di(t)}{dt}
\]

\[
L \frac{di(t)}{dt} + Ri(t) = V \cos \omega t
\]

\[
(pL + R)i(t) = V \cos \omega t
\]

\[
(j\omega L + R)I(j\omega) = V \angle 0
\]

\[
I(j\omega) = \frac{V \angle 0}{(j\omega L + R)} = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \angle -\tan^{-1} \frac{\omega L}{R}
\]
Discrete Time Equations

• Source Response:
  \[ a \, y[n+2] + b \, y[n+1] + c \, y[n] = x[n] \]

• Characteristic/Homogeneous Response
  \[ a \, y[n+2] + b \, y[n+1] + c \, y[n] = 0 \]

• Eigenfunctions: \( y[n] = A \, z^n \)
  Eigenvalues: \( z \)
A Discrete Time Example: Mortgage Loan Calculation

• Assumptions:
  – Let $P[i]$ = remaining principal at period $i$
  – Let $r$ = the interest rate per period
  – $N$ = point at which the loan is paid off (i.e., $P[N] = 0$)
  – $Pc$ = constant periodic payment $= \Delta_P^i + \Delta_I^i$
  – where $\Delta_P^i$ is the portion of $Pc$ associated with the payout of the principal for period $i$
  – where $\Delta_I^i = rP[i]$ is the portion of $Pc$ associated with the payout of the interest for period $i$
Mortgage Loan Calculation

Problem Formulation

• The principal remaining at period $i$ equals the principal at period $i-1$ less the principal payout at period $i-1$

$$ P[i] = P[i-1] - \Delta^i_p = P[i-1] - \{P_c - rP[i-1]\} $$

OR

$$(1+r)P[i-1] - P[i] = P_c$$
Mortgage Loan Calculation
Solution

• Using the eigenfunction = $a^i$, we test the solution: $P[i] = A_1 a^i + A_2$ and we have

$$(1+r)\{A_1 a^{i-1} + A_2\} - \{A_1 a^i + A_2\} = Pc$$

$\{(1+r)A_1 a^{i-1} - A_1 a^i\} + (1+r)A_2 - A_2 = Pc$

1) $(1+r)A_1 a^{i-1} - A_1 a^i = 0 \Rightarrow a = (1+r)$

2) $\{(1+r)-1\}A_2 = Pc \Rightarrow A_2 = \frac{Pc}{r}$

$P[i] = A_1 (1+r)^i + \frac{Pc}{r} \quad ; \quad P[N] = 0 \Rightarrow A_1 = -\frac{Pc}{r(1+r)^N}$

$\therefore P[i] = \frac{Pc}{r(1+r)^N} \{(1+r)^N - (1+r)^i\}$
Mortgage Loan Calculation Solution

• z operators

\[ z^1 \Rightarrow \text{advance of 1 sample time} \]
\[ z^2 \Rightarrow \text{advance of 2 sample times} \]
\[ \vdots \]
\[ z^{-1} \Rightarrow \text{delay of 1 sample time} \]
\[ z^{-2} \Rightarrow \text{delay of 2 sample times} \]
\[ \vdots \]
Mortgage Loan Calculation
Solution

\[(1 + r)P[i - 1] - P[i] = Pc\]

\[
[(1 + r)z^{-1} - 1]P[i] = Pc : \text{Source equation}
\]

\[
[(1 + r)z^{-1} - 1]P[i] = 0 : \text{Source Free Homogeneous equation}
\]

\[(1 + r)z^{-1} - 1 = 0 : \text{Characteristic equation}\]

\[
\therefore z_1 = 1 + r
\]

\[
P[i] = K_1 + K_2z_1^i = K_1 + K_2(1 + r)^i
\]
Mortgage Loan Calculation Solution

Source equation

\[(1 + r)z^{-1} - 1]K_1 = Pc\]

Note: \(z^{-1}K_1 = K_1\) since \(K_1\) is a constant

\[((1 + r) - 1)K_1 = Pc\]

\[K_1 = \frac{Pc}{r}\]

\[P[i] = \frac{Pc}{r} + K_2(1 + r)^i\]

Note: Final condition: \(P[N] = 0\)

\[P[N] = \frac{Pc}{r} + K_2(1 + r)^N = 0\]

\[K_2 = -\frac{Pc}{r(1 + r)^N}\]

\[P[i] = \frac{Pc}{r} - \frac{Pc}{r(1 + r)^N}(1 + r)^i = \frac{Pc}{r(1 + r)^N}[(1 + r)^N - (1 + r)^i]\]
Homework

- Sinusoidal Steady State
  - Calculate the Sinusoidal Steady State Response of the network function for the following networks:

- Bode Plots
  - Draw the Bode Plots for these networks.
  - Use Matlab to plot the Bode Plot, submit your code.

- Discrete ODE
  - Calculate the monthly payment $P_c$

- 3CT.3.1, 3CT.3.2, 3CT.3.4