Convolution

Lecture #6
2CT.3 – 8
Definition

• When we compute the following integral for \( f_1(\tau) \) and \( f_2(\tau) \) we say that we are convoluting \( f_2 \) with \( f_1 \)

\[
g(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) \, d\tau
\]

• This says: take \( f_2(\tau) \), flip it (convolve) in time \( f_2(-\tau) \), then displace it in time by \( t \) seconds \( f_2(t-\tau) \), and multiply it by \( f_1(\tau) \). Finally, integrate the product over all \( \tau \).
Properties of Convolution

• First some shorthand:
  \[ \int f_1(\tau)f_2(t-\tau)d\tau \Rightarrow f_1(t) \otimes f_2(t) \]

• Commutative:
  \[ f_1(t) \otimes f_2(t) = f_2(t) \otimes f_1(t) \]

• Associative:
  \[ f_1(t) \otimes [f_2(t) \otimes f_3(t)] = [f_1(t) \otimes f_2(t)] \otimes f_3(t) \]

• Distributive:
  \[ f_1(t) \otimes [f_2(t) + f_3(t)] = [f_1(t) \otimes f_2(t)] + [f_1(t) \otimes f_3(t)] \]
A Graphical Example of How to Perform a Convolution

\[ f(t) = A(1 - \tau / T) \]

\[ h(\tau) \]

\[ C = \int f(\tau)h(t - \tau)d\tau \]

We need to look at 5 cases: 1) \( t < 0 \), 2) \( 0 < t < T \), 3) \( T < t < 2T \), 4) \( 2T < t < 3T \), & 5) \( t > 3T \)

For case 1 and case 5, \( C = 0 \) since there is no overlap.
**Graphical Convolution Example Continued**

Case 2: $0 < t < T$

\[
f(t) h(t-\tau), \quad 0 < t < T
\]

\[
C = \int_{0}^{T} AB(1 - \frac{\tau}{T})d\tau = AB(t - \frac{t^2}{2T})
\]

Case 3: $T < t < 2T$

\[
f(t) h(t-\tau), \quad T < t < 2T
\]

\[
C = \int_{0}^{T} AB(1 - \frac{\tau}{T})d\tau = AB(T - \frac{T^2}{2T}) = \frac{ABT}{2}
\]

Case 4: $2T < t < 3T$

\[
f(t) h(t-\tau), \quad 2T < t < 3T
\]

\[
C = \int_{-(2T-t)}^{T} AB(1 - \frac{\tau}{T})d\tau = AB\left[\frac{T}{2} + (2T - t) + \frac{(2T - t)^2}{2T}\right]
\]
Matlab Code

clear all;
endpulse=2;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
mm=-endpoint*2:ts:endpoint*2;
pulser=(n>=0) & (n<=endpulse);
pulse=1*pulser;
tripulse=(n>=0) & (n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
title('Convolution');
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
Integration of Convolution Integral

Case 2: $0 < t < T$

$$C = \int_0^t AB\left(1 - \frac{\tau}{T}\right)d\tau = AB(-T)\frac{1}{2}(1 - \frac{\tau}{T})^2 |_0^t = AB(-T)\frac{1}{2}\left\{(1 - \frac{t}{T})^2 - 1\right\}$$

$$= AB(-T)\frac{1}{2}\{(1 - \frac{2t}{T} + \frac{t^2}{T^2}) - 1\} = AB(-T)\frac{1}{2}\left\{-\frac{2t}{T} + \frac{t^2}{T^2}\right\}$$

$$= AB(-T)\frac{1}{2}\left\{-\frac{2t}{T} + \frac{t^2}{T^2}\right\} = AB\{t - \frac{t^2}{2T}\}$$

Case 3: $T < t < 2T$ (From the integration from Case 2)

$$C = \int_0^T AB\left(1 - \frac{\tau}{T}\right)d\tau = AB(-T)\frac{1}{2}(1 - \frac{\tau}{T})^2 |_0^T = AB\{T - \frac{T^2}{2T}\}$$

$$= AB\{T - \frac{T}{2}\} = AB\frac{T}{2}$$

Case 4: $2T < t < 3T$ (From the integration from Case 2)

$$C = \int_{-(2T-t)}^T AB\left(1 - \frac{\tau}{T}\right)d\tau = AB(-T)\frac{1}{2}(1 - \frac{\tau}{T})^2 |_{-(2T-t)}^T = AB(-T)\frac{1}{2}\left\{(1 - \frac{T}{T})^2 - (1 - \frac{(2T-t)}{T})^2\right\}$$

$$= AB(-T)\frac{1}{2}\{(0)^2 - (1 + \frac{(2T-t)}{T})^2\} = AB(T)\frac{1}{2}\{(1 + \frac{(2T-t)}{T})^2\}$$

$$= AB(T)\frac{1}{2}\{(1 + \frac{2(2T-t)}{T} + \frac{(2T-t)^2}{T^2})\} = AB\{\frac{T}{2} + (2T-t) + \frac{(2T-t)^2}{2T}\}$$
Convolution and Systems

• For an LTI system, let’s define $h(t)$ as the system response to a unit impulse source, $\delta(t)$.
• Then the following must be true:

\[
x(t) \rightarrow y(t) \\
\delta(t) \rightarrow h(t) \\
\delta(t - k\Delta) \rightarrow h(t - k\Delta) \text{ time invariance} \\
x(k\Delta)\delta(t - k\Delta) \rightarrow x(k\Delta)h(t - k\Delta) \text{ scalar} \\
\sum_{k} x(k\Delta)\delta(t - k\Delta) \rightarrow \sum_{k} x(k\Delta)h(t - k\Delta) \text{ superposition}
\]
Convolution and Systems Continued

Construct $x(t)$ as the sum of $k$ unit impulse slices. The first expression represents the $k$ slices of the source totaled as $k$ approaches infinity.

Left Side Equals

$$\lim_{\Delta \to 0} \sum_{k \to \infty} x(k\Delta)\delta(t - k\Delta)\Delta \quad \text{approaches} \quad \int x(\tau)\delta(t - \tau)\,d\tau = x(t)$$

Construct $y(t)$ as the sum of $k$ slices of the response due to an unit impulse function: $x(k\Delta)h(t-k\Delta)$. The integral on the right is the convolution of $x(t)$ and $h(t)$.

Right Side Equals

$$\lim_{\Delta \to 0} \sum_{k \to \infty} x(k\Delta)h(t - k\Delta)\Delta \quad \text{approaches} \quad \int x(\tau)h(t - \tau)\,d\tau$$

$$\sum_{k} x(k\Delta)\delta(t - k\Delta)\Delta \quad \rightarrow \quad \sum_{k} x(k\Delta)h(t - k\Delta)\Delta$$

$$x(t) \rightarrow \int x(\tau)h(t - \tau)\,d\tau = y(t)$$

This result is very important since it says that if one knows the impulse response of a system then the output response for any given input source can be found by convolving the input with the impulse response.
Impulse Response and Causality

- $y(t_o)$ does not depend on $x(t_i)$ that occur at times after $t_o$, $t_i > t_o$.

- $y(t_o) = \int_{-\infty}^{\infty} h(\tau)x(t_o - \tau)d\tau = \int_{-\infty}^{0} h(\tau)x(t_o - \tau)d\tau + \int_{0}^{\infty} h(\tau)x(t_o - \tau)d\tau$

- For this integral $\int_{0}^{\infty} h(\tau)x(t_o - \tau)d\tau$, $\tau$ is positive, $t_o - \tau < t_o$,

  and for $x(t_i) = x(t_o - \tau)$ we have $t_i < t_o$

- For this integral $\int_{-\infty}^{0} h(\tau)x(t_o - \tau)d\tau$, $\tau$ is negative, $t_o - \tau > t_o$,

  and for $x(t_i) = x(t_o - \tau)$ we have $t_i > t_o$

- Therefore for $y(t)$ to be causal this $\int_{-\infty}^{0} h(\tau)x(t_o - \tau)d\tau$ must be zero.

- For this to happen $h(t) = 0, t < 0.$
Calculating the Unit Impulse Response, \( h(t) \)

Let’s first look at 2 methods:

1. Narrow Pulse approximation
2. Differentiating \( u(t) \)

Better Methods to Come
Narrow Pulse Approximation

ε = width of pulse

1/ε = height of pulse

As ε → 0, the narrow pulse = \( \frac{1}{\varepsilon} [u(t)-u(t-\varepsilon)] \) → δ(t)

So then let’s first look at the response to u(t):

\[
V_s u(t) = i(t) R + L \frac{di(t)}{dt}
\]

\[
i(t) = \frac{V_s}{R} \left( 1 - e^{-\left(\frac{R}{L}\right)t} \right) u(t)
\]

Now we construct the narrow pulse response:

\[
i(t) = \frac{V_s}{R} \frac{1}{\varepsilon} \left\{ (1 - e^{-\left(\frac{R}{L}\right)t}) u(t) - (1 - e^{-\left(\frac{R}{L}\right)(t-\varepsilon)}) u(t-\varepsilon) \right\}
\]
**Narrow Pulse Approximation Continued**

\[ i(t) = \left( \frac{V_s}{R} \right) \left( \frac{1}{\epsilon} \right) \left[ (1 - e^{-(R/L)t})u(t) - (1 - e^{-(R/L)(t-\epsilon)})u(t-\epsilon) \right] \]

OR

\[ i(t) = \begin{cases} 0, & \text{for } t < 0 \\ \left( \frac{V_s}{R} \right) \left( \frac{1}{\epsilon} \right) \left[ (1 - e^{-(R/L)t}) \right], & \text{for } 0 \leq t < \epsilon \\ \left( \frac{V_s}{R} \right) \left( \frac{1}{\epsilon} \right) \left[ (1 - e^{-(R/L)t}) \right] - (1 - e^{-R/L(t-\epsilon)}), & \text{for } t > \epsilon \end{cases} \]

OR

\[ i(t) = \begin{cases} 0, & \text{for } t < 0 \\ \left( \frac{V_s}{R} \right) \left( \frac{1}{\epsilon} \right) \left[ (1 - e^{-(R/L)t}) \right], & \text{for } 0 \leq t < \epsilon \\ \left( \frac{V_s}{R} \right) \left( \frac{1}{\epsilon} \right) \left[ \left( e^{R/L\epsilon} - 1 \right) e^{-(R/L)t} \right], & \text{for } t > \epsilon \end{cases} \]
Taylor Series Approximation for $e^x$

$$e^{ax} = \frac{(ax)^0}{0!} + \frac{(ax)^1}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots$$

For $x \approx 0$, we can drop the higher order terms:

$$e^{ax} \approx \frac{(ax)^0}{0!} + \frac{(ax)^1}{1!} = 1 + ax$$
**Narrow Pulse Approximation Continued**

Applying the approximation for $e^x$, $V_s=1$ (the unit impulse function) and $\alpha = R/L$

\[
i(t) = \begin{cases} 
0, & \text{for } t < 0 \\
(V_s/R)(1/\varepsilon) \left[ (1-e^{-R/Lt}) \right] = (V_s/R)(1/\varepsilon)(R/Lt) = (V_s/R)(\alpha t/\varepsilon), & \text{for } 0 \leq t < \varepsilon \\
(V_s/R)(1/\varepsilon) \left[ (e^{+R/L\varepsilon} - 1)e^{-R/Lt} \right] = (V_s/R)(1/\varepsilon)(\alpha \varepsilon)e^{-\alpha t}, & \text{for } t > \varepsilon
\end{cases}
\]

\[
i(t) = \begin{cases} 
0, & \text{for } t < 0 \\
(1/R)(\alpha t/\varepsilon) = \alpha 1/R, & \text{for } 0 \leq t < \varepsilon \\
(1/R) \alpha e^{-\alpha t}, & \text{for } t > \varepsilon
\end{cases}
\]

\[
h(t) = \lim_{\varepsilon \to 0} i(t) = \frac{\alpha}{R} e^{-\alpha t} u(t)
\]
Differentiating the Unit Step function

The response due to a Unit Step function is \( i(t) = \frac{V_s}{R} \left( 1-e^{-\alpha t} \right) u(t) \)

and since

\[
\delta(t) = \frac{du(t)}{dt}, \text{ then } h(t) = \frac{di(t)}{dt}
\]

\[
h(t) = \frac{1}{R} \left[ \delta(t) + \alpha e^{-\alpha t} u(t) - e^{-\alpha t} \delta(t) \right]
\]

\[
= \frac{1}{R} \left[ (1-e^{-\alpha t}) \delta(t) + \alpha e^{-\alpha t} u(t) \right]
\]

\[
= \frac{1}{R} \alpha e^{-\alpha t} u(t)
\]
Convolution for Discrete Systems

- For an LTI system, let’s define $h[n]$ as the system response to a unit impulse source, $\delta[n]$.
- $\delta[n]=1$, $n=0$ and 0 for $n \neq 0$
- We have:
  \[ x[n] = \sum x[m] \delta[n-m] \]
  \[ y[n] = \sum x[m] h[n-m] \]
- In addition the same convolution properties hold:
  - Commutative  \[ f_1[n] \otimes f_2[n] = f_2[n] \otimes f_1[n] \]
  - Associative  \[ f_1[n] \otimes \{f_2[n] \otimes f_3[n]\} = \{f_1[n] \otimes f_2[n]\} \otimes f_3[n] \]
  - Distributive  \[ f_1[n] \otimes \{f_2[n] + f_3[n]\} = \{f_1[n] \otimes f_2[n]\} + \{f_1[n] \otimes f_3[t]\} \]
Stability of Systems
Continuous Systems

• If a system is stable, i.e., Bounded Input, Bounded Output (BIBO), the following must be true:

\[ x(t) < \infty, \text{ then } y(t) < \infty \]

\[ y(t) = \int h(\tau)x(t - \tau)d\tau < \int h(\tau)x_{\text{max}} d\tau = x_{\text{max}} \int h(\tau)d\tau < \infty \]

OR

\[ \int h(\tau)d\tau < \infty \]

• However, this is not always the case.
  – Positive Feedback causes instability
What is needed for BIBO

• For a continuous time system, the poles of \( H(p) \) must lie within the left hand complex plane such that \( \text{Re } s_i < 0 \) where \( s_i \) are the poles of \( H(p) \). This will assure that the free response will be damped and not grow exponentially. **THIS IS WHY WE STUDIED SOLUTIONS OF LINEAR ODE IN TERMS OF SOURCE-FREE AND SOURCE COMPONENTS.**

• **THIS IMPLIES THAT \( H(p) \) AND THE IMPULSE RESPONSE, \( h(t) \), MAY BE RELATED.**
Stability of Systems

Discrete Systems

• If a system is stable, i.e., Bounded Input, Bounded Output (BIBO), the following must be true:

\[ x[n] < \infty, \text{ then } y[n] < \infty \]

\[
y[n] = \sum h[m]x[n - m] = x_{\text{max}} \sum h[m] < \infty
\]

OR

\[ \sum h[m] < \infty \]

• However, this is not always the case.
  – Positive Feedback causes instability
What is needed for BIBO

For a discrete time system, the eigenvalues of \( h[n] \) must lie within the unit circle such that \( z_i < 1 \) where \( z_i \) are the eigenvalues of \( h[n] = \sum_i A_i z_i^n \).

This will assure that free response will not diverge and \( \sum_n h[n] = \sum_i \sum_n A_i z_i^n \to \infty \). Using the formula for the partial sums of a geometric series, where \( N \) is the number of roots of the Characteristic Equation

\[
\lim_{L \to \infty} \sum_{n=0}^{L-1} h[n] = \lim_{L \to \infty} \sum_{n=0}^{L-1} \sum_{i=1}^{N} z_i^n = \lim_{L \to \infty} \sum_{n=0}^{L-1} \sum_{i=1}^{N} z_i^n = \lim_{L \to \infty} \sum_{i=1}^{N} \frac{1 - z_i^L}{1 - z_i} \to \sum_{i=1}^{N} \frac{1}{1 - z_i}; \text{ provided } |z_i| < 1
\]

If \( |z_i| \geq 1 \),

\[
\lim_{L \to \infty} \sum_{n=0}^{L-1} \sum_{i=1}^{N} z_i^n = \lim_{L \to \infty} \sum_{i=1}^{N} \left| \frac{1 - z_i^L}{1 - z_i} \right| \to \infty
\]

\( z_i = |z_i| \angle \text{angle}(z_i) \)

\( z_i^L = |z_i|^L \angle (L \times \text{angle}(z_i)) \)

\[
\lim_{L \to \infty} z_i^L = \lim_{L \to \infty} \{ |z_i|^L \angle (L \times \text{angle}(z_i)) \} = \lim_{L \to \infty} \{ |z_i|^L \} \subseteq \{ -\pi < L \times \text{angle}(z_i) \leq \pi \}
\]

\[
\lim_{L \to \infty} \{ |z_i|^L \} = 0; z_i < 1 \text{ and } \lim_{L \to \infty} \{ |z_i|^L \} \to \infty; z_i \geq 1
\]
Homework

- Convolution  Verify your all your results of these convolution problems using Matlab and its conv function.
  - Problem (1)
    - Assume that a system response is given by the following:
      
      ![h(t) diagram]

      Sketch the response to a) \( u(t) \), b) \( u(t) - u(t-a) \) for \( a=0.5 \), \( a=1 \), and \( a=5 \), and c) evaluate \( e^{-t} u(t) \) at \( t=1 \) and \( t=2 \)
  - Problem (2)
    - Assume that a system response is given by the following:
      
      ![h(t) diagram]

      Evaluate the response to \( te^{-t} u(t) \) at \( t=1 \) and \( t=3 \)
Homework

- Stability
  - Determine the stability of the following systems with poles in the complex plane, describe the form of the transient response:

```
Imaginary axis  Imaginary axis  Imaginary axis
Real axis       Real axis       Real axis
```

- 2CT.3.1, 2CT.3.2