Signal Analysis

Lecture #7
5CT.1-2,4
How to Analyze Different Classes of Signals

• Classes of Signals
  – Periodic vs. Non Periodic
  – Continuous vs. Discrete
  – Bounded vs. Non Bounded
  – Symmetries

• Use Mathematical Transformations
  – such as Fourier Series and Fourier, Laplace, & Z transforms
  – to analyze Signal Properties
    • Frequencies which make up signal: Spectrum
    • Energy Content
  – to analyze and design systems which process these signals
    • Filters
    • etc
Fourier Series

• A method for approximating a signal
• A means to analyze a signal
• Applies to either continuous or discrete signals
• Need to understand/review some background, foundations, and assumptions
**Related Sources Theorem**

- If we know the response to a source, then the response to the derivative/integral of the source is the derivative/integration of the response to the source.
- An intuitive proof:

\[
x(t) \rightarrow y(t) \quad \frac{dx(t)}{dt} \rightarrow \frac{dy(t)}{dt} \quad A(p)y(t) = B(p)x(t) \quad d[A(p)y(t)] = d[B(p)x(t)]
\]

\[
A(p) \frac{dy(t)}{dt} = B(p) \frac{dx(t)}{dt}
\]

\[
\int x(t) dt \rightarrow \int y(t) dt \quad \int A(p)y(t) dt = \int B(p)x(t) dt \quad A(p) \int y(t) dt = B(p)\int x(t) dt
\]
Taylor Series Approximation of a Signal

• From calculus, if we have a single-valued function that is continuous and has continuous derivatives, it can be approximated as

\[
f(t) \approx f_a(t) = f(t_o) + \frac{df(t)}{dt} \bigg|_{t=t_o} (t - t_o) + \frac{d^2 f(t)}{dt^2} \bigg|_{t=t_o} (t - t_o)^2 + \cdots \\
\cdots + \frac{d^{n-1} f(t)}{dt^{n-1}} \bigg|_{t=t_o} (t - t_o)^{n-1} + R_n(t)
\]

• Assuming that \( f(t) \) is the source function, using the related sources theorem, we know the response to a constant source, then we can get the response to any function \( t^n \) by successful integration and then use superposition to get the full response due to \( f_a(t) \)

• \( R_n(t) \) can be considered to be the error between \( f(t) \) and \( f_a(t) \) and gets smaller as more terms are added
An Example

\[ f(t) = \cos \frac{\pi t}{2}; \quad f_a(t) = a_0 + a_1 t + a_2 t^2 \]

- How do we choose the coefficients, the \(a_i\)'s, to get best approximation of \(f(t)\) within the interval \(-1 < t < +1\)?

- Let's choose them that at \(t = -1, 0, +1\),

\[
\begin{align*}
  f_a(-1) &= a_0 - a_1 + a_2 = \cos \frac{-\pi}{2} = 0 \\
  f_a(0) &= a_0 = 1 \\
  f_a(1) &= a_0 + a_1 + a_2 = \cos \frac{\pi}{2} = 0 \\
  a_0 &= 1, a_1 = 0, a_2 = -1 \\
  f_a(t) &= 1 - t^2
\end{align*}
\]
The Error Between $f(t)$ & $f_a(t)$

- Object: Choose the $a_i$'s to minimize the error $\varepsilon(t) = f(t) - f_a(t)$ over the interval of the approximation, but
  - Average error is not a good criterion since we can have large deviations which cancel each other out. Example: $\varepsilon(t) = \sin t$ over the period $0$ to $2\pi$.

- Instead try to minimize the average value of
  \[ E^2 = \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} \varepsilon^2 dt = \frac{1}{t_1 - t_2} \int (f(t) - f_a(t))^2 dt \]
  which is known as the mean squared error.
An Example

\( \varepsilon(t) = f(t) - (a_0 + a_1 t + a_2 t^2) \) over the interval \(-1 < t < +1\)

\[
E^2 = \frac{1}{2} \int_{-1}^{+1} [f(t)]^2 dt - \int_{-1}^{+1} (a_0 + a_1 t + a_2 t^2) f(t) dt + \frac{1}{2} \int_{-1}^{+1} (a_0 + a_1 t + a_2 t^2)^2 dt
\]

To choose the \( a_k \)'s to minimize the mean squared error, we must have:

\[
\frac{\partial E^2}{\partial a_k} = 0, \quad \frac{\partial^2 E^2}{\partial a_k^2} > 0
\]

Since the second partials are positive we will have a minimum. The minimum is \( E^2 = .017 \). But can we do better?
Can we do better?

• Yes, choose more terms, \( f_a(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \)

• Or better yet, choose different approximating functions that are orthogonal in the interval, i.e., choose

\[
f_a(t) = A_0 g_0(t) + A_1 g_1(t) + A_2 g_2(t) + \cdots + A_n g_n(t)
\]

such that

\[
\int_{t_1}^{t_2} g_k(t) g_j(t) \, dt = 0 \quad \text{for } k \neq j
\]

and

\[
\int_{t_1}^{t_2} g_k(t) g_j(t) \, dt = G_k \quad \text{for } k = j
\]
Orthogonal Functions

Using \( f_a(t) = A_0 g_0(t) + A_1 g_1(t) + A_2 g_2(t) + \cdots + A_n g_n(t) \) over the interval \( T \) and choose the \( A_n \)'s to minimize \( E^2 \), we have:

\[
E^2 = \frac{1}{T} \int_T [f(t) - f_a(t)]^2 dt
\]

\[
= \frac{1}{T} \left[ \int_T f(t)^2 dt - 2 \int_T f_a(t) f(t) dt + \int_T f_a(t)^2 dt \right]
\]

\[
\frac{\partial E^2}{\partial A_k} = \frac{1}{T} \frac{\partial}{\partial A_k} \left[ -2 \int_T f_a(t) f(t) dt + \int_T f_a(t)^2 dt \right] = 0
\]

Where \( \frac{\partial E^2}{\partial A_k} \) represents a set \( k+1 \) simultaneous equations

Note: \( \int_T f(t)^2 dt \) is sometimes called the quadratic content or energy associated with \( f(t) \) in interval \( T \)
Coefficients of Orthogonal Functions

It can be shown that the first integral of each set of the \( k+l \) equations is:

\[
\frac{1}{T} \frac{\partial}{\partial A_k} (-2 \int_{T} f_a(t) f(t) \, dt)
\]

\[
= \frac{1}{T} \frac{\partial}{\partial A_k} \left[ -2 \int_{T} (A_0 g_0(t) + A_1 g_1(t) + \cdots + A_n g_n(t)) f(t) \, dt \right]
\]

\[
= \frac{-2}{T} \left[ \int_{T} g_k(t) f(t) \, dt \right]
\]

And applying the orthogonal property to the second integral, we have:

\[
\frac{1}{T} \frac{\partial}{\partial A_k} \int_{T} f_a(t)^2 \, dt = \frac{1}{T} \frac{\partial}{\partial A_k} \left( A_0 g_0(t) + A_1 g_1(t) + \cdots + A_n g_n(t) \right)^2 \, dt
\]

\[
= \frac{1}{T} 2 \int_{T} (A_0 g_0(t) + A_1 g_1(t) + \cdots + A_n g_n(t)) g_k(t) \, dt = A_k \frac{1}{T} 2 \int_{T} g_k(t)^2 \, dt = A_k \frac{2}{T} G_k
\]
Coefficients of Orthogonal Functions

\[
\frac{1}{T} \frac{\partial}{\partial A_k} \left[ -2 \int_{T} f_a(t) f(t) \, dt + \int_{T} f_a(t)^2 \, dt \right]
\]

\[
= \frac{-2}{T} \left[ \int_{T} g_k(t) f(t) \, dt \right] + A_k \frac{1}{T} 2 \int_{T} g_k(t)^2 \, dt
\]

\[
= \frac{-2}{T} \left[ \int_{T} g_k(t) f(t) \, dt \right] + A_k \frac{2}{T} G_k = 0
\]

And, at last we have:

\[
A_k = \frac{\int_{T} g_k(t) f(t) \, dt}{\int_{T} [g_k(t)]^2 \, dt} = \frac{\int_{T} g_k(t) f(t) \, dt}{G_k}
\]
What Functions are Orthogonal

• There is a class of polynomials which form an orthogonal set
• But a better choice are the sinusoidal functions:

\[ f_a(t) = C_0 + \sum_{k=1}^{N} [A_k \cos(\frac{2\pi kt}{T}) + B_k \sin(\frac{2\pi kt}{T})] \]

\[ = C_0 + \sum_{k=1}^{N} C_k \cos(\frac{2\pi kt}{T} + \psi_k) \]

where \( C_k = \sqrt{A_k^2 + B_k^2} \)

\[ \psi_k = \tan^{-1}\left(\frac{-B_k}{A_k}\right) \]
Some Properties of Sinusoids Which Make Things Neater

Recall that $e^{jt} = \cos t + j \sin t$

$$\cos t = \frac{1}{2}(e^{jt} + e^{-jt})$$

$$\sin t = \frac{1}{2j}(e^{jt} - e^{-jt})$$

And for the complex number $s = \alpha + j\omega$, there is its conjugate $s^* = \alpha - j\omega$. Furthermore, $s + s^* = 2\text{Re}[s] = 2\alpha$

Therefore, let's rewrite $f_a(t)$ in terms of complex series of $e^{j\omega t}$ functions and their conjugates.

We now call this the **Fourier Series** of a function within an interval of $T$. 
Fourier Series

\[ f_a(t) = C_0 + \sum_{k=1}^{N} C_k \cos\left(\frac{2\pi kt}{T} + \psi_k\right) \]

\[ = C_0 + C_1 \cos\left(\frac{2\pi 1t}{T} + \psi_1\right) + \cdots + C_k \cos\left(\frac{2\pi kt}{T} + \psi_k\right) + \cdots + C_N \cos\left(\frac{2\pi Nt}{T} + \psi_N\right) \]

Expanding the sum

\[ = C_0 + \frac{C_1}{2} e^{j\left(\frac{2\pi 1t}{T} + \psi_1\right)} + \frac{C_1}{2} e^{-j\left(\frac{2\pi 1t}{T} + \psi_1\right)} + \cdots + \frac{C_k}{2} e^{j\left(\frac{2\pi kt}{T} + \psi_k\right)} + \frac{C_k}{2} e^{-j\left(\frac{2\pi kt}{T} + \psi_k\right)} + \cdots + \frac{C_N}{2} e^{j\left(\frac{2\pi Nt}{T} + \psi_N\right)} + \frac{C_N}{2} e^{-j\left(\frac{2\pi Nt}{T} + \psi_N\right)} \]

Using Euler's formula.

\[ = C_0 + \frac{C_1}{2} e^{j\psi_1} e^{\frac{j2\pi l t}{T}} + \frac{C_1}{2} e^{-j\psi_1} e^{-\frac{j2\pi l t}{T}} + \cdots + \frac{C_k}{2} e^{j\psi_k} e^{\frac{j2\pi k t}{T}} + \frac{C_k}{2} e^{-j\psi_k} e^{-\frac{j2\pi k t}{T}} + \cdots + \frac{C_N}{2} e^{j\psi_N} e^{\frac{j2\pi N l t}{T}} + \frac{C_N}{2} e^{-j\psi_N} e^{-\frac{j2\pi N l t}{T}} \]

Formulation of phasors

Let \( g_k(t) = e^{\frac{j2\pi k t}{T}} \) and then \( g_k(t)^* = e^{-\frac{j2\pi k t}{T}} \) and \( a_k = \frac{C_k}{2} e^{j\psi_k} \) and then \( a_k^* = \frac{C_k}{2} e^{-j\psi_k} \) where \( a_0 = C_0 \)
Fourier Series

\[ f_a(t) = a_0 + a_1 g_1(t) + [a_1 g_1(t)]^* + \cdots + a_k g_k(t) + [a_k g_k(t)]^* + \cdots + a_N g_N(t) + [a_N g_N(t)]^* \]

Recasting in terms of general orthogonal functions.

\[ a_0 + \sum_{k=1}^{N} a_k g_k(t) + [a_k g_k(t)]^* \quad \text{Simplifying the sum.} \]

where \( g_k(t) = e^{\frac{j2\pi kt}{T}} \), \( g_k(t)^* = e^{-\frac{j2\pi kt}{T}} \), \( a_k = \frac{C_k}{2} e^{j\psi_k} \), \( a_k^* = \frac{C_k}{2} e^{-j\psi_k} \), \( a_0 = C_0 \)

and \( a_k = \frac{1}{T} \int_{t_1}^{t_1+T} f(t)g_k(t)^* \, dt \)

\[ = \frac{1}{T} \int_{t_1}^{t_1+T} f(t)e^{-\frac{j2\pi kt}{T}} \, dt \]

\[ f_a(t) = a_0 + \sum_{k=1}^{N} [a_k e^{\frac{j2\pi kt}{T}} + a_k^* e^{-\frac{j2\pi kt}{T}}] = \sum_{k=-N}^{N} a_k e^{\frac{j2\pi kt}{T}} = C_0 + \sum_{k=1}^{N} 2 \text{Re}[a_k e^{\frac{j2\pi kt}{T}}] \]

Note that since the magnitude of the \( a_k \) coefficients are 1/2 the value of the \( C_k \) coefficients, 2 real part is required.

\[ f_a(t) = a_0 + \sum_{k=1}^{N} C_k \cos\left(\frac{j2\pi kt}{T} + \psi_k\right), \quad \text{where} \quad 2a_k = C_k e^{j\psi_k} \quad \text{and} \quad a_0 = C_0 \]
Homework

• Fourier Series
  – Problem (3)

• Compute the Fourier Series for the function using 3 terms in the series:

\[ f(t) = 1 \text{ for } 0 < t < \pi \text{ and } f(t) = 0 \text{ for } \pi < t < 2\pi \]

\[
a_k = \frac{1}{2\pi} \int_{0}^{2\pi} f(t) e^{-jkt} dt = \frac{1}{2\pi} \int_{0}^{\pi} e^{-jkt} dt = \left( \frac{1}{2\pi} \right) \left( \frac{1}{-jk} \right) e^{-jkt} \bigg|_{0}^{\pi} = \frac{1}{-2\pi k} (e^{-j\pi} - 1)
\]

\[
= \frac{1}{-2\pi k} e^{-j\pi k/2} (e^{-j\pi k/2} - e^{+j\pi k/2})
\]

\[
= \frac{\sin k\pi}{\pi k}; \text{ for } k \neq 0
\]

\[
a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} f(t) dt = \frac{1}{2\pi} \int_{0}^{\pi} 1 dt = \frac{1}{2}
\]

\[
f(t) = \frac{1}{2} + 2 \sum_{k=1}^{N} \frac{\sin \frac{k\pi}{2}}{\pi k} \cos(kt - k\frac{\pi}{2})
\]
Homework

• Mean Squared Error
  – Problem (1)
    • For our example in class, prove that $E^2 = 0.017$ for $f(t) = \cos(\pi t/2)$
  – Problem (2)
    • It is desired to approximate $f(t) = \sin(t)$ in the interval $0 < t < \pi/2$ by the straight line $f_a(t) = mt + b$. Determine the values of $m$ and $b$ for a least mean square error approximation and calculate the corresponding MSE.

• Fourier Series
  – Problem (3)
    • Compute the Fourier Series for the function using 3 terms in the series:
      \[
      f(t) = 1 \quad \text{for} \quad 0 < t < \pi, \quad f(t) = 0 \quad \text{for} \quad \pi < t < 2\pi
      \]
  – Problem (4)
    • Compute the Fourier Series for the function using 4 terms in the series:
      \[
      f(t) = t \quad \text{for} \quad 0 < t < 3
      \]

• 5CT.1.1, 5CT.1.2