Fourier Series for Periodic Functions

Lecture #8
5CT3,4,6,7
Fourier Series for Periodic Functions

- Up to now we have solved the problem of approximating a function $f(t)$ by $f_a(t)$ within an interval $T$.
- However, if $f(t)$ is periodic with period $T$, i.e., $f(t) = f(t+T)$, then the approximation is true for all $t$.
- And if we represent a periodic function in terms of an infinite Fourier series, such that the frequencies are all integral multiples of the frequency $1/T$, where $k=1$ corresponds to the fundamental frequency of the function and the remainder are its harmonics.

$$a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-\frac{j2\pi kt}{T}} dt$$

$$f(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{\frac{j2\pi kt}{T}} + a_k^* e^{-\frac{j2\pi kt}{T}}] = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T}} = a_0 + \sum_{k=1}^{\infty} 2\text{Re}[a_k e^{\frac{j2\pi kt}{T}}]$$

$$f(t) = a_0 + \sum_{k=1}^{\infty} C_k \cos\left(\frac{2\pi kt}{T} + \psi_k\right), \text{ where } 2a_k = C_k e^{j\psi_k} \text{ and } a_0 = C_0$$
Another Form for the Fourier Series

\[ f(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos\left(\frac{2\pi kt}{T} + \psi_k\right) \]

\[ = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi kt}{T}\right) + B_k \sin\left(\frac{2\pi kt}{T}\right) \]

where \( A_k = C_k \cos \psi_k \) and \( B_k = -C_k \sin \psi_k \) and \( A_0 = C_0 \)
Fourier Series Theorem

• Any periodic function $f(t)$ with period $T$ which is integrable ($\int f(t) \, dt < \infty$) can be represented by an infinite Fourier Series.

• If $[f(t)]^2$ is also integrable, then the series converges to the value of $f(t)$ at every point where $f(t)$ is continuous and to the average value at any discontinuity.
Properties of Fourier Series

• Symmetries
  – If \( f(t) \) is even, \( f(t) = f(-t) \), then the Fourier Series contains only cosine terms
  – If \( f(t) \) is odd, \( f(t) = -f(-t) \), then the Fourier Series contains only sine terms
  – If \( f(t) \) has half-wave symmetry, \( f(t) = -f(t+T/2) \), then the Fourier Series will only have odd harmonics
  – If \( f(t) \) has half-wave symmetry and is even, even quarter-wave, then the Fourier Series will only have odd harmonics and cosine terms
  – If \( f(t) \) has half-wave symmetry and is odd, odd quarter-wave, then the Fourier Series will only have odd harmonics and sine terms
Properties of FS Continued

- Superposition holds, if \( f(t) \) and \( g(t) \) have coefficients \( f_k \) and \( g_k \), respectively, then \( Af(t) + Bg(t) \Rightarrow Af_k + Bg_k \)

- Time Shifting:

\[
f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}
\]

\[
f(t-t_1) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T} e^{-j2\pi kt_1/T} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi (k-t_1)/T}
\]

\[
(a_k)_{\text{delayed}} = (a_k)_{\text{original}} e^{-j2\pi k t_1/T}
\]

- Differentiation and Integration

\[
(a_k)_{\text{derivative}} = \frac{j2\pi k}{T} (a_k)_{\text{original}}
\]

\[
(a_k)_{\text{integral}} = \frac{T}{j2\pi k} (a_k)_{\text{original}}
\]
FS Coefficients Calculation Example

\[ f(t) = 0, \ -2 < t < 0 \]
\[ f(t) = E, \ 0 < t < 1 \]
\[ f(t) = 0, \ 1 < t < 2 \]
\[ f(t) = f(t \pm 4) \]

Since \( 2a_k = C_k \angle \psi \), then

\[ f(t) = \frac{E}{4} + \frac{E}{2} \sum_{k=1}^{\infty} \sin \left( \frac{k\pi}{4} \right) \cos \left( \frac{2\pi kt}{4} - \frac{k\pi}{4} \right) \]

\[ a_k = \frac{1}{4} \left[ \int_{-2}^{0} e^{-\frac{j2\pi kt}{4}} dt + \int_{0}^{1} E e^{-\frac{j2\pi kt}{4}} dt + \int_{1}^{2} e^{-\frac{j2\pi kt}{4}} dt \right] \]

\[ = \frac{1}{4} \left[ \frac{E}{-j2\pi k} \frac{e^{-j\pi k}}{4} \right]_0^1 \]

\[ = \frac{E}{-j2\pi k} \left[ e^{-\frac{j\pi k}{4}} - 1 \right] = \frac{E}{\pi k} \left[ \frac{e^{-\frac{j\pi k}{4}}}{4} - \frac{e^{-\frac{j\pi k}{4}}}{2} \right] \]

\[ = \frac{E}{4} \frac{\sin \left( \frac{\pi k}{4} \right)}{\pi k} e^{-\frac{j\pi k}{4}}, \ \ k \neq 0 \]

\[ a_0 = \frac{E}{4} \int_{0}^{1} dt = \frac{E}{4} \]

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Example Continued

1 term

10 terms
**Frequency Spectrum of the Pulse Function**

- In the preceding example, the coefficients for each of the cosine terms was proportional to \( \sin(k\pi/4)/(k\pi/4) \).
- We call the function \( Sa(x) = \frac{\sin x}{x} \) the Sampling Function.
- If we plot these coefficients along the frequency axis we have the frequency spectrum of \( f(t) \).
**Frequency Spectrum**

Note that the periodic signal in the time domain exhibits a discrete spectrum (i.e., in the frequency domain)
Another Example

This signal is odd, H/W symmetrical, and mean=0; this means no cosine terms, odd harmonics and mean=0.

\[ a_k = \frac{1}{T} \left[ \int_{-T/2}^{0} -Ve^{\frac{-j2\pi kt}{T}} \, dt + \int_{0}^{T/2} Ve^{\frac{-j2\pi kt}{T}} \, dt \right] \]

\[ = \frac{V}{T} \left[ -\frac{1}{-j2\pi k} \right]^{0}_{-T/2} -j2\pi k + \frac{1}{-j2\pi k} \left[ \frac{T}{2} \right] \]

\[ = \frac{V}{T} \left[ \frac{1}{-j2\pi k} \right]^{j2\pi k0}_{-j2\pi k0} \left[ -e^{\frac{-j2\pi kT}{T}} + e^{\frac{0}{T}} \right] + \left[ e^{\frac{0}{T}} - e^{\frac{-j2\pi kT}{T}} \right] \]

\[ = \frac{V}{-j2\pi k} \left[ -e^{\frac{-j2\pi k0}{T}} + e^{\frac{j2\pi k0}{T}} \right] + \left[ e^{\frac{0}{T}} - e^{\frac{-j2\pi k0}{T}} \right] \]

\[ a_k = \frac{V}{j\pi k} [1 - \cos \pi k] = \begin{cases} 2V \quad \text{for k odd}, \\ 0 \quad \text{for k even.} \end{cases} \]

\[ f(t) = 2 \sum_{k=odd}^{\infty} \frac{2V}{\pi k} \cos \left( \frac{2\pi kt}{T} - \frac{\pi}{2} \right) \]

\[ f(t) = \sum_{k=odd}^{\infty} \frac{4V}{\pi k} \sin \left( \frac{2\pi kt}{T} \right) \]
Fourier Series of an Impulse Train or Sampling Function

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-\frac{j2\pi t}{T}} \, dt \]

\[ = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-\frac{j2\pi t}{T}} \, dt \]

\[ = \frac{1}{T} \text{ for all } k \]

\[ x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{\frac{j2\pi t}{T}} \]
Fourier Series for Discrete Periodic Functions

\[ x[n] = a_0 + \sum_{k=1}^{\infty} a_k e^{\frac{j2\pi kn}{T}} + \sum_{k=1}^{\infty} a_k^* e^{-\frac{j2\pi kn}{T}} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kn}{T}} \]

\[ a_k = \frac{1}{T} \sum_{n=-T/2}^{T/2} x[n] e^{-\frac{j2\pi kn}{T}} \]

Since \( x[n] \) is discrete, this becomes a summation.
Homework

Problem (1)
- Compute the Fourier Series for the periodic functions
  a) $f(t) = 1$ for $0 < t < \pi$, $f(t) = 0$ for $\pi < t < 2\pi$
  b) $f(t) = t$ for $0 < t < 3$

Problem (2)
- Compute the Fourier series of the following Periodic Functions:
  - $f(t) = t$, $2n\pi < t < (2n+1)\pi$ for $n \geq 0$
    - $= 0$, $(2n+1)\pi < t < (2n+2)\pi$ for $n \geq 0$
  - $f(t) = e^{-t/\pi}$, $2n\pi < t < (2n+2)\pi$ for $n \geq 0$ Use Matlab to plot $f(t)$ using $a_k$ for maximum number of components, $N=5,10,100,$ and $1000$. Show your code.

Problem (3)
- Problems: 4.1/3 Find the Fourier series of the following waveforms (choose $t_c = T_o/4$):

![Waveform Diagram]
Homework

- Problem (4)
  - Deduce the Fourier series for the functions shown (hint: deduce the second one using superposition):

- 5CT.7.1