BME 333 Biomedical Signals and Systems
Biomedical Signals and Systems Quiz #2

• Choose 4 out of 5
• Problems 3, 4, & 5 are mandatory
1. a) Sketch the convolution of $t^2u(t)$ with itself
   b) Convolve $tu(t)$ with $u(t)$ and sketch the result.
   "NOTE THE ORDER."
   c) Sketch the convolution of $e^{-2t}u(t)$ with itself.

2. Compute the Fourier Transform for the following functions without integrating
   a) $f(t) = e^{-t}u(t)$
   b) $f(t) = \cos(t)u(t)$
   c) $f(t) = e^{-t}\cos(t)u(t)$

3. Compute the Discrete Fourier Transform for the following function and sketch the FT for $N = 10$
   a) $x[n] = 10$ for $n = 0$
   b) $x[n] = 10$ for $n = 1, 2, \ldots, N-1$
   c) Describe why the results for both a) and b) makes sense.
   What is the relationship of these two problems?

\[ \sum_{0}^{N-1} a^n = \frac{1 - a^N}{1 - a} \]
4. The following periodic signal is passed through an ideal bandpass filter with cutoff frequencies of 22.5 kHz and 27.5 kHz.

\[
x(t) = \begin{cases} 
1 & 0 \leq t < 20\mu\text{sec} \\
0 & 20\mu\text{sec} \leq t < 40\mu\text{sec}
\end{cases}
\]

a) Find and sketch the spectrum of both the input and output signals.

b) What happens if the bandpass filter’s cutoff frequencies are changed to 95 kHz and 105 kHz? Find and sketch the spectrum of both the input and output signals.

c) What happens if low pass filter’s with cutoff frequency of 200 kHz is used? Find and sketch the spectrum of both the input and output signals.

d) What happens if the band elimination filter with cutoff frequencies 210 kHz and 230 kHz is used? Find and sketch the spectrum of both the input and output signals.
5. Apply the signal to the input of the circuit and calculate and plot the spectrum of the output signal. What happens if the carrier frequency is changed to 10kHz?

\[ v_{in}(t) = (1 + m \cos t) \cos 100t \]
1. a) Sketch the convolution of $t^2u(t)$ with itself.

\[ C = \int_{-\infty}^{\tau} \tau^2 u(\tau) (t-\tau)^2 u(t-\tau) d\tau \]

**Case #1**

For $t < 0$

\[ C = \int_{-\infty}^{t} \tau^2 u(\tau) (t-\tau)^2 u(t-\tau) d\tau = 0 \]

**Case #2**

For $t \geq 0$

\[ C = \int_{0}^{t} \tau^2(u(t-\tau)) (t-\tau)^2 u(t-\tau) d\tau \]

\[ C = \int_{0}^{t} \tau^2 (t-\tau)^2 d\tau = \int_{0}^{t} \tau^2 (t^2 - 2t\tau + \tau^2) d\tau = \int_{0}^{t} (\tau^2 t^2 - 2t\tau^3 + \tau^4) d\tau = \]

\[ = \left( \frac{\tau^3 t^2}{3} - \frac{2t\tau^4}{4} + \frac{\tau^5}{5} \right) \bigg|_{0}^{t} = \frac{t^5}{3} - \frac{t^5}{2} + \frac{t^5}{5} = \frac{t^5}{30}; \text{ for } t > 0 \]

\[ C = \frac{t^5}{30} u(t) \]
b) Convolve $tu(t)$ with $u(t)$ and sketch the result. *NOTE THE ORDER THIS IS THE WRONG ORDER*

Case #1

\[
C = \int_{-\infty}^{t} \tau u(\tau)u(t-\tau) d\tau = 0 \text{ for } t < 0
\]

Case #2

\[
C = \int_{0}^{t} \tau u(\tau)u(t-\tau) d\tau \text{ for } t > 0
\]

\[
C = \int_{0}^{t} \tau d\tau = \frac{\tau^2}{2} \bigg|_{0}^{t} = \frac{t^2}{2} \text{ for } t > 0
\]

\[
C = \frac{t^2}{2} u(t)
\]
b) Convolve \( tu(t) \) with \( u(t) \) and sketch the result. *NOTE THE ORDER
THIS IS THE CORRECT ORDER*

\[
(t - \tau)u(t - \tau) \quad u(\tau) \quad (t - \tau)u(t - \tau) \quad u(\tau)
\]

Case #1

\[
C = \int_{\infty}^{t} (t - \tau)u(\tau)u(t - \tau)d\tau = 0 \quad \text{for} \ t < 0
\]

Case #2

\[
C = \int_{0}^{t} (t - \tau)u(\tau)u(t - \tau)d\tau \quad \text{for} \ t > 0
\]

\[
C = \int_{0}^{t} (t - \tau)d\tau = -\frac{(t - \tau)^2}{2} \bigg|_{0}^{t} = \frac{t^2}{2}; \quad \text{for} \ t > 0
\]

\[
C = \frac{t^2}{2}u(t)
\]
Biomedical Signals and Systems
Quiz #2

1. b) Sketch the convolution of $e^{-t}u(t)$ with itself.

$$C = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau = 0 \text{ for } t < 0$$

**Case #1**

$$C = \int_{0}^{t} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau \text{ for } t > 0$$

$$= \int_{0}^{t} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau = \int_{0}^{t} e^{-2\tau} e^{-2(t-\tau)}d\tau$$

$$= \int_{0}^{t} e^{-2\tau} e^{-2t+2\tau}d\tau = \int_{0}^{t} e^{-2\tau-2t+2\tau}d\tau = \int_{0}^{t} e^{-2t}d\tau = e^{-2t} \int_{0}^{t} 1d\tau$$

$$= e^{-2t}t; \text{ for } t > 0$$

$$= e^{-2t}tu(t)$$

BME 333 Biomedical Signals
and Systems - J.Schesser
2. Compute the Fourier Transform for the following functions without integrating

a) \( f(t) = e^{t}u(t) \)

b) \( f(t) = \cos(t)u(t) \)

c) \( f(t) = e^{-t}\cos(t)u(t) \)

a) \( \mathcal{F}[u(t)] = \frac{1}{j\omega} \)

using the frequency displacement property

\( \mathcal{F}[e^{-\alpha t}u(t)] = \frac{1}{\alpha + j\omega}; \mathcal{F}[e^{-t}u(t)] = \frac{1}{1 + j\omega} \)

b) \( \mathcal{F}[\cos(t)u(t)] = \frac{1}{2} \left[ \mathcal{F}[e^{jt}u(t)] + \mathcal{F}[e^{-jt}u(t)] \right] \)

using the frequency displacement property

\[ \begin{align*}
&= \frac{1}{2} \left[ \frac{1}{1 - j + j\omega} + \frac{1}{j + j\omega} \right] \\
&= \frac{1}{2} \left[ \frac{j + j\omega + -j + j\omega}{(1 - j + j\omega)(j + j\omega)} \right] = \frac{j\omega}{(1 - \omega^2)}
\end{align*} \]

c) \( \mathcal{F}[e^{-t}\cos(t)u(t)] = \frac{1}{2} \left[ \mathcal{F}[e^{(-1+j)t}u(t)] + \mathcal{F}[e^{(-1-j)t}u(t)] \right] \)

using the frequency displacement property

\[ \begin{align*}
&= \frac{1}{2} \left[ \frac{1}{1 - j + j\omega} + \frac{1}{1 + 1j + j\omega} \right] \\
&= \frac{1}{2} \left[ \frac{1 + j + j\omega + 1 - 1j + j\omega}{(1 - j + j\omega)(1 + j + j\omega)} \right] \\
&= \frac{1 + j\omega}{(2 - \omega^2 + j2\omega)}
\end{align*} \]
3. Compute the Discrete Fourier Transform for the following function and sketch the FT for $N = 10$

a) $x[n] = 10$ for $n = 0$
   $= 0$ for $n = 1, 2, \ldots, N-1$

b) $x[n] = 10$ for $n = 0, 1, 2, \ldots, N-1$

c) Describe why the results for both a) and b) makes sense. What is the relationship of these two problems?

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}$$

$$x(n) = \sum_{n=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}}$$

**NOTE:**

$$\sum_{0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

**a)** $x(n) = 10$ for $n = 0$

$= 0$ for $n = 1, 2, \ldots N$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}$$

$$= x(0)e^{-j\frac{2\pi 0}{N}} + x(1)e^{-j\frac{2\pi 1}{N}} + \cdots + x(N)e^{-j\frac{2\pi N}{N}}$$

$$= 10$$
3. Compute the Discrete Fourier Transform for the following function and sketch the FT for \( N = 10 \)

b) \( x[n] = 10 \) for \( n = 0, 1, 2, \ldots, N-1 \)

c) Describe why the results for both a) and b) makes sense. What is the relationship of these two problems?

\[
\begin{align*}
 b) x[n] &= 10 \\
 X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n/N} \\
 &= \sum_{n=0}^{N-1} 10 e^{-j2\pi n/N} = 10 \sum_{n=0}^{N-1} (e^{-j2\pi /N})^n \\
 &= 10 \frac{1 - e^{-j(k2\pi /N)N}}{1 - e^{-j(k2\pi /N)}} \\
 &= 10 \frac{1 - e^{-jk2\pi}}{1 - e^{-j(k2\pi /N)}} = 0; k \neq 0 \\
\end{align*}
\]

For \( k = 0 \), use L'Hopital's Rule

\[
\begin{align*}
\lim_{k \to 0} 10 \frac{d}{dk} \frac{1 - e^{-jk2\pi}}{1 - e^{-j(k2\pi /N)}} &= \lim_{k \to 0} 10 \frac{-j2\pi e^{-jk2\pi}}{-j2\pi N e^{-j(k2\pi /N)}} = 10 \frac{-j2\pi /N}{-j2\pi /N} = 10 N = 100 \\
X(k) &= 10 N = 100; k = 0 \\
X(k) &= 0; k \neq 0
\end{align*}
\]
3. Compute the Discrete Fourier Transform for the following function and sketch the FT for \( N = 10 \)

\[
\text{c) Describe why the results for both a) and b) makes sense. What is the relationship of these two problems?}
\]

\[
\text{c) They are duals of each other.}
\]

Problem a) is an impulse in the time domain which yields a constant in the frequency domain (needs all frequencies).

Problem b) is a constant in the time domain which yields an impulse in the frequency domain (single frequency at DC).
Biomedical Signals and Systems
Quiz 2

4. The following periodic signal is passed through an idea bandpass filter with cutoff frequencies of

\[ x(t) = \begin{cases} 1 & 0 \leq t < 20 \mu\text{sec} \\ 0 & 20 \mu\text{sec} \leq t < 40 \mu\text{sec} \end{cases} \]

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jko_0 t} \]

\[ a_k = \frac{1}{T_o} \int_{-T_o/2}^{0} x(t)e^{-jko_0 t} dt = \frac{1}{T_o} \int_{0}^{T_o/2} e^{-jko_0 t} dt \]

\[ = \frac{1}{T_o} e^{-jko_0} \bigg|_{0}^{T_o/2} = \frac{1}{j2k\pi} \left[ e^{-j2\pi kT_o/2} - e^{-j2\pi k0/2} \right] \]

\[ = \frac{1}{j2k\pi} \left[ 1 - e^{-jk\pi} \right] \]

Recall that \( e^{-jk\pi} = \cos k\pi - j \sin k\pi = 1 \) for even values of \( k \)

\[ = -1 \] for odd values of \( k \)

\[ a_0 = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) dt = \frac{1}{T_o} \int_{0}^{T_o/2} 1 dt = \frac{1}{2} \]
4. The following periodic signal is passed through an idea bandpass filter with cutoff frequencies of

\[
\begin{align*}
    x(t) &= \begin{cases} 
        1 & 0 \leq t < 20\mu\text{sec} \\
        0 & 20\mu\text{sec} \leq t < 40\mu\text{sec}
    \end{cases}
\end{align*}
\]

a) Since \( T_o = 40\mu\text{sec} \) then \( f_o = 25\text{kHz} \), then output signal will have components of 25 kHz, 75 kHz, 125 kHz, etc. (odd values of \( k \)). Since the bandpass filter’s cutoff frequencies are 22.5 kHz and 27.5 kHz, only the spectral component for \( k=1 \) (25 kHz) passes.
The following periodic signal is passed through an idea bandpass filter with cutoff frequencies of

\[ x(t) = \begin{cases} 
1 & 0 \leq t < 20 \mu\text{sec} \\
0 & 20 \mu\text{sec} \leq t < 40 \mu\text{sec} 
\end{cases} \]

b) Since the bandpass filter’s cutoff frequencies are 95 kHz and 105 kHz, NO spectral component passes.
4. The following periodic signal is passed through an ideal bandpass filter with cutoff frequencies of

\[ x(t) = \begin{cases} 
1 & 0 \leq t < 20\mu\text{sec} \\
0 & 20\mu\text{sec} \leq t < 40\mu\text{sec} 
\end{cases} \]

c) Since the low filter’s cutoff frequency is 200 kHz, the spectral component for \( k = 0 \) (DC), 1 (25kHz), 3 (75kHz), and 5 (125kHz) 7 (175kHz) pass.
Final Answers

4. The following periodic signal is passed through an ideal bandpass filter with cutoff frequencies of

\[ x(t) = \begin{cases} 
1 & 0 \leq t < 20 \mu\text{sec} \\
0 & 20 \mu\text{sec} \leq t < 40 \mu\text{sec} 
\end{cases} \]

d) Since the band elimination filter’s cutoff frequencies are 210 kHz and 230 kHz, only the spectral component for \( k=9 \) (225 kHz) is blocked.
5. Calculate the Transfer Function $V_{out}(j\omega)/V_{in}(j\omega)$ for these circuits and sketch the Bode plots:

$$v_{in}(t) = (1 + m_0 \cos t) \cos 100t$$

$$v_{in}(t) = \cos 100t + \frac{m_0}{2} \cos 101t + \frac{m_0}{2} \cos 99t$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{(1 - \omega^2 LC) + j\omega CR} = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + (\omega CR)^2}} \angle - \tan^{-1}\left(\frac{\omega CR}{1 - \omega^2 LC}\right) \omega_o = \sqrt{\frac{1}{LC}} = 10^2; RC = 10^{-3}$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} \bigg|_{\omega=0} = 1 \angle 0$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} \bigg|_{\omega=\frac{1}{\sqrt{LC}}} = \frac{1}{(1 - 1) + j100 \times 10^{-3}} = 10 \angle -\frac{\pi}{2}$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} \bigg|_{\omega \to \infty} = \frac{1}{-\omega^2 LC} = 0 \angle \pi$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} \bigg|_{\omega=99} \approx 10 \angle -\frac{\pi}{2}$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} \bigg|_{\omega=100} \approx 10 \angle -\frac{\pi}{2}$$

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} \bigg|_{\omega=101} \approx 10 \angle -\frac{\pi}{2}$$

$$v_{out}(t) \approx 10 \cos(100t - \frac{\pi}{2}) + \frac{10m_0}{2} \cos(101t - \frac{\pi}{2}) + \frac{10m_0}{2} \cos(99t - \frac{\pi}{2})$$
5. Calculate the Transfer Function $V_{out}(j\omega)/V_{in}(j\omega)$ for these circuits and sketch the Bode plots:

\[ v_{in}(t) = (1 + m_{0} \cos t) \cos 2\pi 10,000 t \]
\[ v_{in}(t) = \cos 62831.9 t + \frac{m_{0}}{2} \cos 62832.9 t + \frac{m_{0}}{2} \cos 62830.9 t \]

\[
\left. \frac{V_{out}(j\omega)}{V_{in}(j\omega)} \right|_{\omega = 62830.9} \approx 2.5 \times 10^{-6} \angle -\pi \\
\left. \frac{V_{out}(j\omega)}{V_{in}(j\omega)} \right|_{\omega = 62831.9} \approx 2.5 \times 10^{-6} \angle -\pi \\
\left. \frac{V_{out}(j\omega)}{V_{in}(j\omega)} \right|_{\omega = 62832.9} \approx 2.5 \times 10^{-6} \angle -\pi \\
\]

\[ v_{out}(t) \approx 2.5 \times 10^{-6} [\cos(62831.9 t - \pi) + \frac{m_{0}}{2} \cos(62832.9 t - \pi) + \frac{m_{0}}{2} \cos(62830.9 t - \pi)] \]

None of the signal gets through the filter.