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Production-Inventory Systems with Lost Sales and Compound Poisson Demands

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This paper considers a continuous-review, single-product, production-inventory system with a constant replenishment rate, compound Poisson demands, and lost sales. Two objective functions that represent metrics of operational costs are considered: (1) the sum of the expected discounted inventory holding costs and lost-sales penalties, both over an infinite time horizon, given an initial inventory level; and (2) the long-run time average of the same costs. The goal is to minimize these cost metrics with respect to the replenishment rate. It is, however, not possible to obtain closed-form expressions for the aforementioned cost functions directly in terms of positive replenishment rate (*PRR*). To overcome this difficulty, we construct a bijection from the PRR space to the space of positive roots of *Lundberg's fundamental equation*, to be referred to as the *Lundberg positive root* (*LPR*) space. This transformation allows us to derive closed-form expressions for the aforementioned cost metrics with respect to the LPR variable, in lieu of the PRR variable. We then proceed to solve the optimization problem in the LPR space and, finally, recover the optimal replenishment rate from the optimal LPR variable via the inverse bijection. For the special cases of constant or loss-proportional penalty and exponentially distributed demand sizes, we obtain simpler explicit formulas for the optimal replenishment rate.

Subject classifications: compound Poisson arrivals; integro-differential equation; Laplace transform; Lundberg's fundamental equation; lost sales; production-inventory system; constant replenishment rate.
Area of review: Operations and Supply Chains.

1. Introduction

Production-inventory systems with constant production rates are implemented by a variety of manufacturing firms. Examples can be found in (1) glass manufacturing, where glass furnaces often produce at constant rates (Federal Register 2009); (2) sugar mills, where raw sugar is produced utilizing a constant production rate (Grunow et al. 2007); (3) the electronic computer industry, where displays are manufactured at constant production rates (Display Development News 2000); and (4) the pharmaceutical industry, where cell-free proteins and other products are generally produced at constant production rates (Membrane and Separation Technology News 1997). Additional examples can be found in the carpet manufacturing industry, where the yarning and dyeing processes operate at constant rates over long periods of time. These constant rates are selected by the manufacturer at the production planning stage by taking into account the anticipated demands and its cost structures. At the manufacturing stage, it produces carpet rolls continuously, and specifically, at full capacity for carpet dyeing.

Production-inventory systems with constant production rates are typically deployed when there are high setup times and high setup costs, where frequent modification (e.g., interruption or rate change) of the production line is financially or operationally prohibitive. Thus, for both financial and operational reasons, it is critical to establish the proper production process early in the planning process. The importance of the production rate is self-evident: an overly high production rate results in high holding costs due to excess inventory, whereas a low production rate results in high penalty costs due to frequent stockouts and subsequent lost sales. Thus, it is reasonable to expect that there exists an optimal production rate that balances these two costs. Furthermore, manufactures often employ "full capacity" in production. For example, the refinery industry has an *operable capacity utilization rate* at 92% or even higher.[1] Consequently, the production capacity level corresponding to the production rate has a critical impact on the firm's cost structure, its inventory policies, and its service levels, as well as its management and staff support requirements (cf. Jacobs and Chase 2013). This study sheds light on the optimal production capacity of a firm from a long-term cost minimization perspective.

We study a continuous-review single-product production-inventory system with a constant production/replenishment rate and compound Poisson demands, subject to lost sales. In the sequel, we will use the terms production and replenishment interchangeably. Unsatisfied demand may be partially fulfilled from on-hand inventory (if any) and all excess demand (shortage) is lost; such excess demand will be referred to as the *lost-sales size*. The system incurs two types of costs: a holding cost and a lost-sales cost. The holding cost is incurred as a function of the inventory on hand and assessed at a constant rate per unit on-hand inventory per unit time. The lost-sales cost is a penalty imposed at each loss occurrence and is assumed to be a function of the lost-sales size. The goal of this paper is to derive the optimal replenishment rates that minimize two objective functions that represent metrics of operational costs: (1) the sum of expected discounted inventory holding costs and lost-sales penalties over an infinite time horizon, given an initial inventory level; and (2) the long-run time average of the same costs.

The main objective of this paper is twofold: (1) to provide closed-form expressions for the respective objective functions of the conditional expected discounted costs and of the time-average costs; and (2) to minimize the aforementioned objective functions with respect to the replenishment rate. To this end, we first derive an integro-differential equation for the conditional expected discounted cost function until the first lost-sale occurrence. However, a closed-form formula for that cost function is not available. To overcome this difficulty, we observe that the original optimization problem in terms of the replenishment rate parameter can be reformulated and solved in a tractable form in terms of another variable, and then the requisite optimal replenishment rate can be recovered. More specifically, let the original space of all positive replenishment rates be referred to as the *PRR space*, and define a related space consisting of all positive roots of the so-called *Lundberg's fundamental equation* (see Gerber and Shiu 1998 and Equation (15)), to be referred to as the *Lundberg positive roots* (*LPR*) space. The two spaces, PRR and LPR, will be shown to be related by a bijection (i.e., a one-one and onto mapping); see Equation (17). Indeed, the cost function over the PRR space does not have a closed-form expression, whereas the same cost function over the LPR space does, thereby facilitating its optimization. Finally, having obtained the optimal solution in the LPR space, we shall provide an algorithm to compute the requisite optimal replenishment rate in the PRR space via the inverse bijection (cf. Figure 3). We further obtain explicit solutions for the special cases in which the lost-sales penalty function is either (1) a constant penalty for each lost-sales occurrence, or (2) a loss-proportional penalty. Finally, a numerical study is performed to illustrate the results and demonstrate additional properties of the system.

The methodology employed in this paper gives rise to interesting connections between inventory management and queueing and insurance risk models. In particular, this study is connected to some important aspects of $G/M/1$ queues in equilibrium, such as the joint distribution of the busy period and the idle period (cf. Perry et al. 2005, Adan et al. 2005, and Perry 2011).

In summary, the main analytical contributions of this paper are (1) a closed-form formula for the expected discounted cost function for any initial inventory level, general demand size distributions, and general penalty functions; (2) a characterization of the optimal constant replenishment rate that minimizes the expected discounted cost function for general demand size distributions and general penalty functions; (3) closed-form expressions for the optimal replenishment rate and the attendant costs for the case of exponential demand size, for both constant penalty and loss-proportional penalty functions; (4) a closed-form formula for the long-run time-average cost function for general demand size distributions and general penalty functions; this cost function can also be optimized using the same approach employed for the expected discounted cost function.

The remainder of this paper is organized as follows. Section 2 reviews related literature. Section 3 formulates the production-inventory model under study. Section 4 derives a closed-form expression for the expected discounted cost function and §5 treats its optimization. Section 6 presents a set of numerical studies. Section 7 examines the long-run time-average cost function and its optimization. Section 8 presents ideas on extensions of the model to incorporate variable production cost and service level constraints. Finally, §9 concludes this paper.

2. Literature Review

This section first reviews the literature on continuous-review production-inventory systems, and then compares the production-inventory model with related queueing and insurance risk models.

Most papers on continuous-review inventory systems assume that orders are placed and replenished in batches or lot sizes. One of the well-known ordering policies is the continuous-review (s, S) policy; see Scarf (1960) for a seminal work. In contrast, our study considers a production-inventory system where inventory is replenished continuously at a constant rate, and the goal is to find the optimal replenishment rate. Constant production or replenishment rates are common in continuous-review production-inventory systems. For example, Doshi et al. (1978) consider a production-inventory control model of finite capacity that switches between two possible production rates based on two critical stock levels. The main result of that paper is a formula for the long-run time-average cost as a function of two critical levels of the production rate. De Kok et al. (1984) deal with a production-inventory model subject to a service level constraint, where excess demand is backlogged and the production rate can be dynamically switched between two possible rates. The authors derive a useful approximation for the switch-over level. For the same model, De Kok (1985) considers the corresponding lost-sales case and provides an approximation for the switch-over level. Gavish and Graves (1980) consider a production-inventory system, where the demand process is Poisson and demand size is constant. The authors assume that excess demand is backlogged and the production facility may be set up or shut down. They treat their system as an $M/D/1$ queue and minimize the expected cost per unit time. Graves and Keilson (1981) extend the model of Gavish and Graves (1980) by considering a compound Poisson demand process. The problem is analyzed as a constrained Markov process, using the compensation method, and a closed-form expression is derived for the expected system cost as a function of the policy parameters. For a similar setting, Graves (1982) derives the steady-state distribution of the inventory level using queueing theory. More recently, Perry et al. (2005) study a production-inventory system with a fixed and constant replenishment rate under an M/G (i.e., a compound Poisson) demand process and two "clearing policies" (sporadic and continuous) to avoid high inventory levels. The paper derives explicit results for the associated expected discounted cost functions under both types of clearing policies. We note that although the literature above assumes the replenishment rate to be exogenous and fixed, our paper treats this parameter as a decision variable.

The underlying inventory process studied in this paper can also be ascribed a variety of interpretations, drawn from the contexts of queueing and insurance risk systems. In what follows, we provide a literature review on such connections; interested readers are also referred to Prabhu (1997) for a general treatment of such models under the theme of stochastic models.

The similarity between queuing and inventory models is well recognized in the literature, and a number of papers treat one model from the perspective of the other. From a queueing vantage point, the inventory level can be interpreted as the *attained waiting time* in a $G/M/1$ queue, provided idle periods are removed; see Adan et al (2005), Prabhu (1965) and references therein. An inventory analysis generally includes an explicit cost structure and a solution for optimal policies, whereas researchers in queuing theory have been more interested in the underlying probabilistic structure. However, some papers address inventory problems using queueing theory; two cases in point are Graves (1982) and Perry et al. (2005). Cost optimization has also been considered in queueing models. Such research has been directed toward finding optimal operating policies for a queuing system subject to a given cost/reward structure. Such optimization problems have been considered by Bell (1971), Heyman (1968), Lee and Srinivasan (1989), and Sobel (1969).

In the context of classical insurance risk models, the inventory level can be interpreted as a surplus (or capital, or risk reserve) level of an insurance firm, under a constant rate of premium inflows and compound Poisson claim arrivals; see Asmussen (2000) and Gerber and Shiu (1998). Risk theory in general, and ruin probability in particular, are traditionally considered essential topics in the insurance literature. Since the seminal paper by Lundberg (1932), many studies have addressed this topic; cf. Gerber and Shiu (1997, 1998) and Rolski et al. (1999). Two typical questions of interest in classical ruin theory are (a) the deficit at ruin; and (b) the time to ruin. To address those two questions, Gerber and Shiu (1998) have introduced a comprehensive penalty function, the so-called *Gerber-Shiu penalty function*, as a function of surplus immediately prior to ruin and the deficit at ruin; this function has been widely discussed in the recent insurance literature. Additional extensions based on the Gerber-Shiu penalty function include barrier or threshold strategies; see Boxma et al. (2011), Lin et al. (2003), Lin and Pavlova (2006), and references therein. Recently, Boxma et al. (2011) and Löpker and Perry (2010) have further studied insurance risk models (time to ruin, ruin probability, and the total dividend) using methods and results from queueing theory. In most of these studies, it is noted that the inventory process can be interpreted as the content process of a queuing or an insurance risk model. In contrast, the present study differs from the above in terms of its objective function and its conditions for system stability; in particular, our paper treats cost computation and optimization whereas the insurance literature is primarily interested in dividends and risk (e.g., time to ruin and ruin probability), and the queuing literature mainly focuses on quantities such as service levels and workload in the system. Queueing theory also puts emphasis on stability conditions: a stable queue requires the traffic intensity to be strictly less than one; cf. Asmussen (2003) and Prabhu (1997). Stability conditions for an insurance risk model ensure that the average claim is less than the premium rate (i.e., a *positive* security loading), such that the probability of ultimate ruin is less than one; cf. Equation (2.5) in Gerber and Shiu (1998). In our production-inventory context, the condition that the average demand is greater than the replenishment rate (i.e., a *negative* security loading) is necessary for the time-average cost optimization, whereas no such restriction is required for the expected discounted cost analysis.

In this paper, it is not possible to directly solve the integro-differential equation in Equation (12). However, it is possible to solve equations that involve Laplace transforms (cf. Widder 1959), and then invert the transformed functions to obtain the requisite functions. We note that the problem of inverting Laplace transforms is often difficult, so most studies focus on numerical approximations; e.g., Cohen (2007) and Shortle et al. (2004).

In addition to the contributions of analytical results listed in §1, the main methodology contributions of this paper are as follows: (1) we treat the original problem in terms of the LPR variable by taking advantage of *Lundberg's fundamental* equation and a bijection between positive production rates and Lundberg positive roots; and (2) we optimize this cost function in the LPR space and then invert the optimal LPR variable to obtain the requisite optimal replenishment rate in the PRR space using the inverse bijection. To the best of our knowledge, no study in the inventory literature exploits such an optimization technique.

3. Model Formulation

We will use the following notational conventions and terminology. Let $\mathbb{R}$ denote the set of real numbers and $x^+ = \max\{x, 0\}$, for any $x \in \mathbb{R}$. For a random variable X, its probability density function (pdf) is denoted by $f_X(x)$, its cumulative distribution function (cdf) by $F_X(x)$ and its complementary cdf by $\bar{F}_X(x)$. For two real functions $f(x)$ and $g(x)$ on $[0, \infty)$, their convolution function is given by

$$\langle f * g\rangle(u) = \int_0^u f(u-x)g(x)\,dx.$$

The Laplace transform of a function $f(x)$ is defined by

$$\mathscr{L}[f](z) = \tilde{f}(z) = \int_0^\infty e^{-zx} f(x)\,dx, \quad z \geqslant 0.$$

For any nonnegative random variable X, we shall make repeated use of the following relation:

$$\begin{aligned}\tilde{\bar{F}}_X(z) &= \int_0^\infty e^{-zx}\bar{F}_X(x)\,dx = \frac{1}{z}\left[1 + \int_0^\infty e^{-zx}\,d\bar{F}_X(x)\right]\\ &= \frac{1}{z}[1 - \tilde{f}_X(z)], \end{aligned}\tag{1}$$

where the second equality follows from integration by parts. Throughout this paper, we will tacitly assume the existence of a basic probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is the sample space, $\mathscr{F}$ is a σ–field of events, and $\mathbb{P}$ is a probability measure on $\mathscr{F}$. Finally, we assume continuously compounded discounting at rate, $r > 0$.

3.1. Inventory Process

We consider a continuous-review inventory system subject to lost sales. The demand arrival stream constitutes a compound Poisson process with rate λ and arrival times $\{A_i: i \geqslant 0\}$, where $A_0 = 0$ by convention. Thus, the corresponding sequence of interarrival times, $\{T_i: i \geqslant 1\}$, where $T_i = A_i - A_{i-1}$, is exponentially distributed and the sequence is *identically independently distributed* (iid). The corresponding demand sizes form an iid sequence $\{D_i: i \geqslant 1\}$ with a common pdf $f_D(x)$ and common mean demand, $\mathbb{E}[D] < \infty$, where the demand of size D_i arrives at time A_i. Replenishment occurs at a constant (deterministic) rate, $\rho \geqslant 0$. Let $\{I(t): t \geqslant 0\}$ denote the right-continuous inventory process, given by

$$I(t) = I(0) + \rho t - \sum_{i=1}^{N_A(t)} [D_i - L(A_i)],\tag{2}$$

where $N_A(t)$ is the number of demands arriving over $(0, t]$ and

$$L(A_i) = [D_i - I(A_i-)]^+, \quad i = 1, 2, \ldots\tag{3}$$

is the lost-sales size at time A_i. Let $\{\tau_i: i \geqslant 0\}$ be the sequence of loss occurrence times, given by

$$\tau_i = \inf\{A_j > \tau_{i-1}: L(A_j) > 0\},\tag{4}$$

where $\tau_0 = 0$ by convention. Let $\{J_k: k \geqslant 0\}$ be the sequence of random arrival indexes at which a loss occurs, namely, $\tau_k = A_{J_k}$. Figure 1 illustrates a sample path of the inventory process with lost sales over an infinite time horizon.

We note that the inventory process $\{I(t)\}$ of Equation (2) is stable under the condition $\rho < \lambda\mathbb{E}[D]$; cf. Proposition 1.1 in Asmussen (2000). In contrast, it is typically assumed $\rho > \lambda\mathbb{E}[D]$ in queueing theory and classical risk insurance studies. In particular, queueing systems generally assume that the service rate is greater than the arrival rate (cf. Adan et al. 2005 and Asmussen 2003); otherwise the queue length explodes. Classical risk insurance analysis typically assumes that the premium rate is greater than the average claim to ensure a positive drift; cf. Gerber and Shiu (1997, 1998). In our model the stability condition $\rho < \lambda\mathbb{E}[D]$ is only required when studying the time-average cost; it is not imposed for the expected discounted cost, since in this case the objective cost function is always bounded because of discounting even if the inventory process is unstable.

3.2. Cost Functions

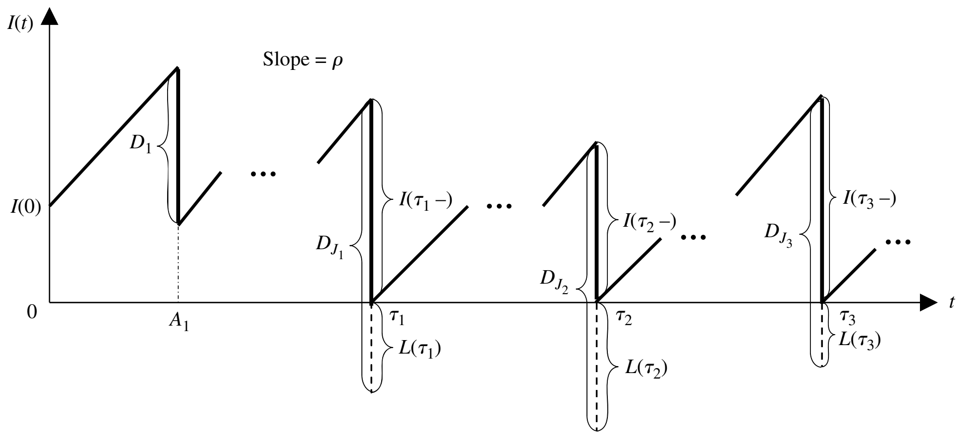
Recall that the production-inventory system under study incurs costs in the form of holding costs and lost-sales penalties. Specifically, a holding cost is incurred at rate h per unit inventory per unit time while there is inventory on hand, and a penalty $w(x)$ is incurred whenever a customer's demand cannot be fully satisfied from on-hand inventory and there is a shortage of size x. The penalty function $w(x)$ is assumed to be nondecreasing in the lost-sales size, x, where $w(0) = 0$. Thus, the total *discounted cost* up until time t is given by

$$C_\rho(t) = h\int_0^t e^{-rz} I(z)\,dz + \sum_{i=1}^{N_A(t)} e^{-rA_i} w(L(A_i)),\tag{5}$$

which is dependent on the initial inventory level $I(0) = u \geqslant 0$. Of particular interest is the *conditional expected discounted cost function* up until and including the first lost-sale occurrence, given by

$$c_\rho(u) = \mathbb{E}[C_\rho(\tau_1) \mid I(0) = u].\tag{6}$$

Figure 1. A sample path of the inventory level process, $\{I(t)\}$.



Furthermore, the *conditional expected discounted cost function* over the interval $(0, t]$ is given by

$$\Phi_\rho(t \mid u) = \mathbb{E}[C_\rho(t) \mid I(0) = u]. \tag{7}$$

It is easy to show that the function $\Phi_\rho(t \mid u)$ is increasing and uniformly bounded in t, for any given u. Hence, it follows that the *conditional expected total discounted cost function*,

$$\Phi_\rho(u) = \lim_{t\to\infty} \Phi_\rho(t \mid u), \tag{8}$$

is well defined. To optimize $\Phi_\rho(u)$ with respect to ρ, we next derive the expected discounted cost function in §4, and then treat its optimization in §5. All proofs omitted from these sections are provided in the appendices (available as supplemental material at http://dx.doi.org/10.1287/opre.2014.1299) .

4. Computation of the Expected Discounted Cost Function

To derive a closed-form formula for the cost function $\Phi_\rho(u)$ of Equation (8), we first establish, in the following theorem, that the expected discounted cost for an arbitrary initial inventory level can be decomposed into two terms: the discounted cost up until the first lost-sale occurrence and the expected discounted cost thereafter.

Theorem 1. *Given any initial inventory $u \geqslant 0$, $\Phi_\rho(u)$ and $c_\rho(u)$ satisfy the following equation,*

$$\Phi_\rho(u) = c_\rho(u) + d_\rho(u)\Phi_\rho(0), \tag{9}$$

where

$$d_\rho(u) = \mathbb{E}[e^{-r\tau_1} \mid I(0) = u]. \tag{10}$$

Proof. The proof follows readily from the strong Markov property of the process $\{I(t)\colon t \geqslant 0\}$. □

In particular, setting $u = 0$ in Equation (9), we obtain

$$\Phi_\rho(0) = \frac{c_\rho(0)}{1 - d_\rho(0)}. \tag{11}$$

The following two subsections study the component functions $c_\rho(u)$ and $d_\rho(u)$ of $\Phi_\rho(u)$.

4.1. The Cost Function $c_\rho(u)$

In this subsection we derive an integro-differential equation for $c_\rho(u)$ in Lemma 1 from which we will later obtain closed-form expressions for $c_\rho(0)$ and $\tilde{c}_\rho(z)$ in Proposition 1.

Lemma 1. *The function $c_\rho(u)$ defined by Equation (6) is continuous, differentiable in $u \geqslant 0$, and satisfies*

$$\rho \frac{\partial}{\partial u} c_\rho(u) - (\lambda + r)c_\rho(u) + \lambda\langle f_D * c_\rho\rangle(u) = -g(u), \tag{12}$$

where

$$g(u) = hu + \lambda \int_u^\infty f_D(x) w(x-u)\, dx, \quad u \geqslant 0. \quad \square \tag{13}$$

To solve Equation (12) for $c_\rho(u)$, we introduce the auxiliary function $\psi(z)$, given by

$$\psi(z) = \lambda \tilde{f}_D(z) + \rho z - \lambda - r, \tag{14}$$

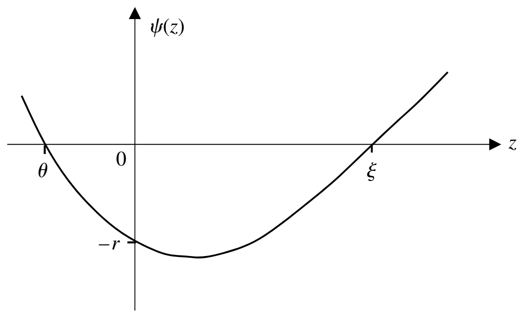
where by convention, $\psi(z) = \infty$ if $\tilde{f}_D(z)$ does not exist. It is of interest to study the roots of the equation $\psi(z) = 0$, that is, the roots of the equation

$$r - \rho z + \lambda[1 - \tilde{f}_D(z)] = 0. \tag{15}$$

Equation (15) is well known in the context of insurance models, where it is referred to as *Lundberg's fundamental equation*; cf. Gerber and Shiu (1998). An important property of the roots of that equation is as follows: for any $r > 0$, the equation $\psi(z) = 0$ has two distinct real roots, ξ and θ, where $\xi > 0$ and $\theta < 0$ (*ibid.*). Figure 2 depicts the structure of the function $\psi(z)$ and its two roots.

We note that either the negative root, θ, or the positive one, ξ, can be employed later to derive the cost function via their one-one and onto relationships with ρ. However, in the sequel, we shall employ ξ (rather than θ) as a decision variable in deriving the cost functions and optimal solutions. There are two reasons for this preference. First, for $\tilde{f}_D(\theta)$ to exist, the negative root θ is constrained to be larger than a certain constant (determined by the demand distribution function),

Figure 2. Illustration of the structure of the function $\psi(z)$ and its two roots.



but such constant is generally difficult to identify. In contrast, the positive root ξ always guarantees the existence of $\tilde{f}_D(\xi)$. Second, the time-average cost function, to be studied in §7, can be derived from the discounted cost function by taking the limit as the discount rate r tends to zero. In this case, θ tends to zero, and ξ remains positive, which can also facilitate the study of the time-average cost case.

Next, setting $z = \xi$ in Equation (15), it follows that the Lundberg positive root, ξ, satisfies

$$\lambda \tilde{f}_D(\xi) + \rho\xi - \lambda - r = 0. \tag{16}$$

Equation (16) motivates the following lemma, which provides the basis for our solution methodology.

Lemma 2. (a) *There is a bijection between ρ and ξ, implicitly given by the equation*

$$\rho = \frac{r}{\xi} + \lambda \tilde{\bar{F}}_D(\xi). \tag{17}$$

(b) *The function $\xi(\rho)$, implicitly defined by Equation* (17), *is strictly decreasing in ρ and satisfies*

(1) $\lim_{\rho\to 0} \xi(\rho) = \infty$ *and* $\lim_{\rho\to 0} \rho\xi(\rho) = \lambda + r$;
(2) $\lim_{\rho\to\infty} \xi(\rho) = 0$ *and* $\lim_{\rho\to\infty} \rho\xi(\rho) = r$. □

The bijection between ρ and ξ, given by Equation (17), allows us to derive a closed-form formula for the attendant cost functions in terms of the LPR variable ξ in lieu of the PRR variable, ρ. Furthermore, the optimization of the cost functions can be performed with respect to ξ, and the corresponding optimal ξ^* can be used to recover the optimal $\rho^* = \rho(\xi^*)$ via the bijection function given by Equation (17). Figure 3 depicts the idea of the solution methodology, which we dub the *bijection solution methodology*.

We next establish expressions for $c_\rho(u)$ by solving Equation (12). To this end, we take the Laplace transform with respect to u on both sides of Equation (12), which yields

$$\rho[z\tilde{c}_\rho(z) - c_\rho(0)] - (\lambda + r)\tilde{c}_\rho(z) + \lambda \tilde{f}_D(z)\tilde{c}_\rho(z) = -\tilde{g}(z), \quad z > 0. \tag{18}$$

Rearranging and simplifying Equation (18), we obtain

$$\psi(z)\cdot \tilde{c}_\rho(z) - \rho c_\rho(0) = -\tilde{g}(z), \quad z > 0, \tag{19}$$

where $\psi(z)$ is given by Equation (14). The following result provides closed-form formulas for $c_\rho(0)$ and $\tilde{c}_\rho(z)$ in terms of the LPR variable ξ.

Proposition 1. *For $\rho > 0$,*

$$c_\rho(0) = \frac{1}{\rho}\tilde{g}(\xi); \tag{20}$$

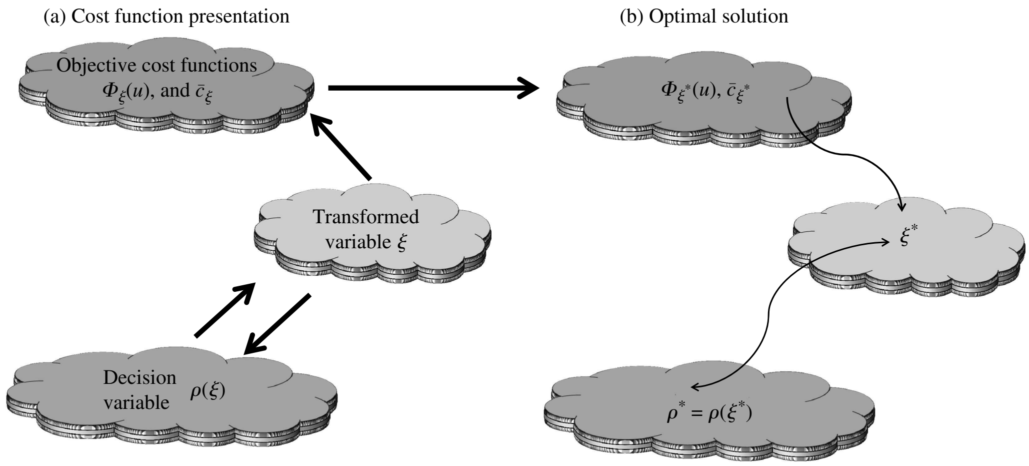
$$\tilde{c}_\rho(z) = \frac{\tilde{g}(\xi) - \tilde{g}(z)}{\psi(z)}, \quad z \neq \xi. \quad \square \tag{21}$$

Next, substituting Equation (17) into Equation (20) yields another expression for $c_\rho(0)$ in terms of the LPR variable ξ, given by

$$c_{\rho(\xi)}(0) = \frac{\xi\tilde{g}(\xi)}{r + \lambda\xi\tilde{\bar{F}}_D(\xi)}. \tag{22}$$

The expression above allows us to optimize $c_{\rho(\xi)}(0)$ with respect to ξ rather than ρ, where the latter is very difficult

Figure 3. The bijection solution methodology over the LPR and PPR spaces.



or even impossible. The optimal ρ^* can then be recovered from the optimal ξ^* via the bijection of Equation (17). The minimization of $\Phi_\rho(u)$ with respect to ρ can be performed in a similar manner.

We mention that for the limiting case of $\rho = 0$, it can be readily shown by Equation (6) that

$$c_0(0) = \frac{\lambda}{\lambda + r}\mathbb{E}[w(D)]. \tag{23}$$

Alternatively, the above result can be obtained by taking limits on both sides of Equation (20), resulting in

$$\lim_{\rho\to 0} c_\rho(0) = \lim_{\rho\to 0}\frac{\xi\tilde{g}(\xi)}{\rho\xi} = \frac{\lim_{\xi\to\infty}\xi\tilde{g}(\xi)}{\lambda + r} = \frac{g(0)}{\lambda + r},$$

where the second equality holds by Lemma 2, part (b) and the third holds by a property of the Laplace transform. The above equation can now be rewritten as Equation (23) by Equation (13).

We note that if there is no holding cost (i.e., $h = 0$), then $c_\rho(0)$ represents the expected discounted value of the deficit at ruin in a classical insurance model. Gerber and Shiu (1998) have given a representation analogous to Equation (20) for this case. If we further specify $w(x)$ to be an exponential function, then Equation (21) can be interpreted in a queueing context as the joint Laplace transform of the busy period and the idle period; cf. Prabhu (1997), Asmussen (2003), and Adan et al. (2005).

4.2. The Function $d_\rho(u)$

In this subsection, we derive a closed-form formula for $d_\rho(0)$ and provide an explicit expression for $\tilde{d}_\rho(z)$. Note that by Equations (5) and (6), $c_\rho(u)$ can be written as

$$c_\rho(u) = \mathbb{E}\left[h\int_0^{\tau_1} e^{-rz} I(z)\,dz + e^{-r\tau_1} w(L(\tau_1))\,\middle|\, I(0) = u\right].$$

The above equation implies that $d_\rho(u)$, given by Equation (10), is a special case of $c_\rho(u)$ when $h = 0$ and $w(x) = 1$. The results for $d_\rho(u)$ contained in the next proposition can be obtained from their counterparts for $c_\rho(u)$.

Proposition 2. *For $\rho > 0$,*

$$d_\rho(0) = \frac{\lambda}{\rho}\tilde{\tilde{F}}_D(\xi) = 1 - \frac{r}{\rho\xi}, \tag{24}$$

$$\tilde{d}_\rho(z) = \frac{r}{\psi(z)}\left[\frac{1}{z} - \frac{1}{\xi}\right] + \frac{1}{z}, \quad z \neq \xi. \quad \square \tag{25}$$

Note that the definition of $d_\rho(u)$ given by Equation (10) implies its continuity in ρ and

$$d_0(0) = \mathbb{E}[e^{-rA_1}] = \frac{\lambda}{\lambda + r},$$

by virtue of Equation (10), where $\tau_1 = A_1$ when $\rho = 0$. Alternatively, this can be verified by substituting $\lim_{\rho\to 0}\rho\xi(\rho) = \lambda + r$ (cf. Lemma 2) into Equation (24). Note also that $\lim_{\rho\to\infty} d_\rho(0) \to 0$ in view of Equation (10), since $\tau_1 \to \infty$ while $\rho \to \infty$. This can be alternatively verified using the fact that $\lim_{\rho\to\infty}\rho\xi(\rho) = r$ (cf. Lemma 2) and Equation (24).

4.3. The Function $\Phi_\rho(u)$

It appears that it is not possible to derive a closed-form expression for $\Phi_\rho(u)$ as a function of ρ. However, the bijection solution methodology allows us to derive a closed-form expression for $\Phi_\rho(u) = \Phi_{\rho(\xi)}(u)$ as a function of ξ. The main results in this subsection are presented in Theorems 2 and 3. To keep the notation simple, we will use ρ and ξ interchangeably, exploiting the bijection between them. In this fashion, $\Phi_\rho(u)$ and $\Phi_\xi(u)$ denote the same function but given in terms of ρ and ξ, respectively. Similar notational conventions will be adopted in the sequel for other quantities, e.g., $\bar{c}_\rho$ and $\bar{c}_\xi$ for the time-average cost in §7, as well as v_ρ and v_ξ for the production cost in §8.

Theorem 2. *For a zero initial inventory level,*

$$\Phi_\xi(0) = \frac{\rho\xi}{r}c_\rho(0) = \frac{\xi}{r}\tilde{g}(\xi); \tag{26}$$

and for an arbitrary initial inventory level $u \geqslant 0$,

$$\Phi_\xi(u) = c_\xi(u) + \frac{\xi}{r}\tilde{g}(\xi)d_\xi(u), \quad u \geqslant 0; \tag{27}$$

$$\tilde{\Phi}_\xi(z) = \xi\tilde{g}(\xi)\left[\frac{1}{rz} + \frac{1}{z\psi(z)}\right] - \frac{\tilde{g}(z)}{\psi(z)}, \quad z \neq \xi. \quad \square \tag{28}$$

We next obtain a renewal-type representation of $\Phi_\xi(u)$ by inverting Equation (28).

Corollary 1. *For any initial inventory $u \geqslant 0$, $\Phi_\xi(u)$ satisfies the equation,*

$$\Phi_\xi(u) = \Phi_\xi(0) + \langle G_\xi * \eta_\xi\rangle(u), \quad u \geqslant 0, \tag{29}$$

where $\Phi_\xi(0)$ is given by Equation (26), $G_\xi(x)$ is given by

$$G_\xi(x) = \xi\tilde{g}(\xi) - g(x), \tag{30}$$

and $\eta_\xi(u)$ is the inverse Laplace transform of $1/\psi(z)$ at $u \geqslant 0$. $\square$

In view of Corollary 1, $\Phi_\xi(u)$ can be obtained by computing the convolution of $\eta_\xi(u)$ and $G_\xi(x)$. To derive a closed-form expression for $\Phi_\xi(u)$, we introduce the function,

$$V_\rho(z) = \frac{(z - \xi)(z - \theta)}{\psi(z)}. \tag{31}$$

We define $V_\rho(\xi)$ and $V_\rho(\theta)$ to be the limits of $V_\rho(z)$ as z tends to ξ and θ, respectively. Note that by the L'Hôpital rule, $V_\rho(\xi)$ and $V_\rho(\theta)$ can be further simplified as

$$V_\rho(\xi) = \frac{\xi - \theta}{\psi'(\xi)}; \tag{32}$$

$$V_\rho(\theta) = \frac{\theta - \xi}{\psi'(\theta)}, \tag{33}$$

where the derivatives $\psi'(\xi)$ and $\psi'(\theta)$ can be obtained from Equation (14).

The following theorem provides an explicit formula for $\Phi_\xi(u)$ and is a key result of the paper.

THEOREM 3. *For any initial inventory level* $u \geqslant 0$,

$$\Phi_\xi(u) = \frac{\xi\tilde{g}(\xi)}{r} + \frac{V_\rho(\xi)}{\xi-\theta}\left[e^{\xi u}\int_u^\infty g(x)e^{-\xi x}\,dx - \tilde{g}(\xi)\right] + \frac{V_\rho(\theta)}{\xi-\theta}\left[e^{\theta u}\int_0^u g(x)e^{-\theta x}\,dx + \frac{\xi\tilde{g}(\xi)}{\theta}(e^{\theta u}-1)\right], \tag{34}$$

where $V_\rho(\xi)$ *and* $V_\rho(\theta)$ *are given by Equations* (32) *and* (33), *respectively.* □

Theorem 3 shows that the expected discounted cost $\Phi_\xi(u)$ depends on the initial inventory level, u, in a complicated way. We further observe that Equation (34) reduces to Equation (26) when the initial inventory level u is zero.

In the following two subsections, we investigate two special cases of the penalty function: constant lost-sales penalty and loss-proportional penalty.

4.3.1. Constant Lost-Sales Penalty. In this case we have $w(x) = K_0$, for $x > 0$, where $K_0 > 0$ is a constant. Accordingly, Equation (13) becomes

$$g(u) = hu + \lambda K_0 \bar{F}_D(u), \quad u \geqslant 0, \tag{35}$$

and the corresponding Laplace transform is given by

$$\tilde{g}(z) = \frac{h}{z^2} + \lambda K_0 \tilde{\bar{F}}_D(z). \tag{36}$$

Next, setting $z = \xi$ and substituting $\tilde{\bar{F}}_D(\xi)$ from Equation (17) into Equation (36), we have

$$\tilde{g}(\xi) = \frac{h}{\xi^2} + K_0\left(\rho - \frac{r}{\xi}\right). \tag{37}$$

Now substituting Equation (37) into Equation (26) yields

$$\Phi_\xi(0) = \frac{h}{r\xi} + K_0\left(\frac{\rho\xi}{r} - 1\right). \tag{38}$$

Finally, substituting Equations (35) and (37) into Equation (34) yields

$$\Phi_\xi(u) = \Phi_\xi(0) + \frac{V_{\rho(\xi)}(\xi)}{\xi-\theta}\phi_1^c(u,\xi) + \frac{V_{\rho(\xi)}(\theta)}{\xi-\theta}\phi_2^c(u,\theta), \tag{39}$$

where $\Phi_\xi(0)$ is given by Equation (38) and

$$\phi_1^c(u,\xi) = \frac{h}{\xi}u + \lambda K_0 e^{\xi u}\int_u^\infty \bar{F}_D(x)e^{-\xi x}\,dx - \frac{K_0}{\xi}(\rho\xi - r);$$

$$\phi_2^c(u,\theta) = -\frac{h}{\theta}u + \lambda K_0 e^{\theta u}\int_0^u \bar{F}_D(x)e^{-\theta x}\,dx + \frac{1}{\theta}\left(r\Phi_\rho(0) - \frac{h}{\theta}\right)(e^{\theta u}-1).$$

4.3.2. Loss-Proportional Penalty. In this case, we have $w(x) = K_1 x$, for $x \geqslant 0$, where $K_1 > 0$ is a constant. Accordingly, Equation (13) becomes

$$g(u) = hu + \lambda K_1 \int_u^\infty x f_D(x)\,dx, \quad u \geqslant 0, \tag{40}$$

and the corresponding Laplace transform is given by

$$\tilde{g}(z) = \frac{h}{z^2} + \lambda K_1\left[\frac{\mu_D}{z} - \frac{1-\tilde{f}_D(z)}{z^2}\right], \tag{41}$$

where $\mu_D = \mathbb{E}[D]$. Next, setting $z = \xi$ in Equation (41) and using $\tilde{f}_D(\xi)$ as given by Equation (16), we have

$$\tilde{g}(\xi) = \frac{h}{\xi^2} + \lambda K_1\left[\frac{\mu_D}{\xi} - \frac{\rho}{\lambda\xi} + \frac{r}{\lambda\xi^2}\right]. \tag{42}$$

Now substituting Equation (42) into Equation (26) yields

$$\Phi_\xi(0) = \frac{h}{r\xi} + K_1\left[\frac{\lambda\mu_D - \rho}{r} + \frac{1}{\xi}\right]. \tag{43}$$

Finally, substituting Equations (40) and (42) into Equation (34) yields

$$\Phi_\xi(u) = \Phi_\xi(0) + \frac{V_{\rho(\xi)}(\xi)}{\xi-\theta}\phi_1^p(u,\xi) + \frac{V_{\rho(\xi)}(\theta)}{\xi-\theta}\phi_2^p(u,\theta), \tag{44}$$

where $\Phi_\xi(0)$ is given by Equation (43) and

$$\phi_1^p(u,\xi) = \frac{h}{\xi}u + \lambda K_1 e^{\xi u}\int_u^\infty\int_x^\infty z f_D(z)e^{-\xi x}\,dz\,dx + \frac{1}{\xi}\left(\frac{h}{\xi} - r\Phi_\rho(0)\right);$$

$$\phi_2^p(u,\theta) = -\frac{h}{\theta}u + \lambda K_1 e^{\theta u}\int_0^u\int_x^\infty z f_D(z)e^{-\theta x}\,dz\,dx + \frac{1}{\theta}\left(r\Phi_\rho(0) - \frac{h}{\theta}\right)(e^{\theta u}-1).$$

4.4. Computation of $\Phi_\xi(u)$ for Exponential Demand-Size Distributions

In this subsection, we derive the function $\Phi_\xi(u)$, subject to each penalty function, for the case of exponentially distributed demand sizes with rate $\beta > 0$. Thus,

$$f_D(x) = \beta e^{-\beta x}, \quad x \geqslant 0 \tag{45}$$

and

$$\tilde{f}_D(z) = \frac{\beta}{\beta + z}, \quad z \geqslant 0. \tag{46}$$

Substituting Equation (46) into Equation (14) yields

$$\psi(z) = \frac{\lambda\beta}{\beta+z} + \rho z - \lambda - r = \frac{(z-\theta)(z-\xi)}{V_\rho(z)}, \tag{47}$$

where

$$V_\rho(z) = \frac{z+\beta}{\rho}. \tag{48}$$

Hence, the two real roots of the equation $\psi(z) = 0$ are given by

$$\xi = \frac{\lambda + r - \rho\beta + \sqrt{(\lambda + r - \rho\beta)^2 + 4r\rho\beta}}{2\rho} \geqslant 0, \tag{49}$$

$$\theta = \frac{\lambda + r - \rho\beta - \sqrt{(\lambda + r - \rho\beta)^2 + 4r\rho\beta}}{2\rho} \leqslant 0. \tag{50}$$

4.4.1. Constant Lost-Sales Penalty. Recall that in this case, $w(x) = K_0$, $x > 0$, so Equation (39) can be written as

$$\Phi_\rho(u) = a_0 + a_1 u + a_2 e^{\theta u}, \tag{51}$$

where

$$a_0 = \frac{h}{r}\left(\frac{1}{\xi} + \frac{1}{\beta} + \frac{1}{\theta}\right); \tag{52}$$

$$a_1 = \frac{h}{r}; \tag{53}$$

$$a_2 = \frac{\lambda K_0 \xi}{r(\beta+\xi)} - \frac{h}{r}\left(\frac{1}{\beta} + \frac{1}{\theta}\right). \tag{54}$$

In Equation (51), the initial inventory level, u, appears in both a linear term and an exponential term. Since $\theta < 0$, it follows that when u is relatively small, the exponential term dominates the linear term, whereas for a relatively large u, the opposite is true. A numerical study of $\Phi_\rho(u)$ with exponential demand distribution is presented in §6. Finally, for the special case with $u = 0$, we have

$$\Phi_\xi(0) = a_0 + a_2 = \frac{1}{r}\left(\frac{h}{\xi} + \frac{\lambda K_0 \xi}{\beta + \xi}\right),$$

and a closed-form expression for the optimal ρ^* is provided in Table 1.

4.4.2. Loss-Proportional Penalty. Recall that in this case, $w(x) = K_1 x$, $x > 0$, so Equation (44) can be written as

$$\Phi_\rho(u) = a_0 + a_1 u + a_4 e^{-\beta u} + a_5 e^{\theta u}, \tag{55}$$

where a_0 and a_1 are given by Equations (52) and (53), respectively, and

$$a_4 = \frac{\lambda K_1}{\rho}\left[\frac{1}{(\xi-\theta)(\beta+\theta)} - \frac{1}{\theta(\beta+\xi)}\right] - \frac{h}{r}\left(\frac{1}{\beta} + \frac{1}{\theta}\right);$$

$$a_5 = -\frac{\lambda K_1}{\rho(\beta+\xi)(\beta+\theta)}.$$

In Equation (55) the initial inventory level, u, appears in a linear term and two distinct exponential terms, each with a negative exponent. It follows that when u is relatively small, the exponential terms dominate the linear term, whereas for a relatively large u, the opposite is true. Finally, for the special case $u = 0$, we have

$$\Phi_\rho(0) = a_0 + a_4 + a_5 = \frac{h}{r\xi} + \frac{\lambda K_1}{(\beta+\xi)\rho}\left(\frac{1}{\xi-\theta} - \frac{1}{\theta}\right),$$

and a closed-form expression for the optimal ρ^* is provided in Table 2.

5. Optimization of the Replenishment Rate

In this section, we optimize the expected discounted cost function $\Phi_\rho(u)$ with respect to the replenishment rate, ρ, via an optimization of $\Phi_\xi(u)$ with respect to ξ. We first provide a general structural result in §5.1 for an optimal replenishment rate, ρ^* (admitting the possibility of multiple optimal replenishment rates), and then describe computational simplifications in §5.2 for some selected demand-size distributions.

5.1. Optimal Replenishment Rate

Observe that the cost function $\Phi_\xi(u)$, given by Equation (34), is expressed in terms of the two roots, θ and ξ. In the sequel,

Table 1. Optimal expected discounted costs subject to constant penalty under various demand distributions.

	ξ^*	ρ^*	$\Phi_{\rho*}(0)$
$D = d$ $d > 0$	$\arg\min_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_0 e^{-\xi d}\right\}$	$\frac{r + \lambda[1 - e^{-\xi^* d}]}{\xi^*}$	$\frac{h + K_0\xi^*(\rho^*\xi^* - r)}{r\xi^*}$
$D \sim \mathrm{Exp}(\beta)$ $\beta > 0$	$\begin{cases} \frac{\beta\sqrt{h}}{\sqrt{\lambda\beta K_0} - \sqrt{h}}, & \text{if } \beta\lambda K_0 > h \\ \infty, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{\sqrt{\lambda\beta K_0} - \sqrt{h}}{\beta}\left(\frac{r}{\sqrt{h}} + \frac{\lambda}{\sqrt{\lambda\beta K_0}}\right), & \text{if } \beta\lambda K_0 > h \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} \frac{2\sqrt{\lambda\beta K_0 h} - h}{r\beta}, & \text{if } \beta\lambda K_0 > h \\ \frac{\lambda K_0}{r}, & \text{otherwise} \end{cases}$
$D \sim U(a,b)$ $0 \leqslant a < b$	$\arg\min_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_0 \frac{e^{-a\xi} - e^{-b\xi}}{(b-a)\xi}\right\}$	$\frac{r}{\xi^*} + \frac{\lambda}{\xi^*}\left[1 - \frac{e^{-a\xi^*} - e^{-b\xi^*}}{(b-a)\xi^*}\right]$	$\frac{h + K_0\xi^*(\rho^*\xi^* - r)}{r\xi^*}$
$D \sim \Gamma(\alpha,\beta)$ $\alpha, \beta > 0$	$\arg\min_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_0(1+\xi/\beta)^{-\alpha}\right\}$	$\frac{r}{\xi^*} + \frac{\lambda}{\xi^*}[1 - (1+\xi^*/\beta)^{-\alpha}]$	$\frac{h + K_0\xi^*(\rho^*\xi^* - r)}{r\xi^*}$

Table 2. Optimal expected discounted costs subject to loss-proportional penalty under various demand distributions.

	ξ^*	ρ^*	$\Phi_{\rho*}(0)$
$D=d\;\; d>0$	$\arg\min_{\xi>0}\left\{\frac{h}{\xi}-\lambda K_1\frac{1-e^{-\xi d}}{\xi}\right\}$	$\frac{r+\lambda[1-e^{-\xi^* d}]}{\xi^*}$	$\frac{1}{r}\left[\frac{h}{\xi^*}-\lambda K_1\frac{1-e^{-\xi^* d}}{\xi^*}+\lambda K_1 d\right]$
$D\sim\text{Exp}(\beta)\;\; \beta>0$	$\begin{cases}\frac{\beta\sqrt{h}}{\sqrt{\lambda K_1}-\sqrt{h}}, & \text{if } \lambda K_1>h\\ \infty, & \text{otherwise}\end{cases}$	$\begin{cases}\frac{\sqrt{\lambda K_1}-\sqrt{h}}{\beta}\left[\frac{r}{\sqrt{h}}+\sqrt{\frac{\lambda}{K_1}}\right], & \text{if } \lambda K_1>h\\ 0, & \text{otherwise}\end{cases}$	$\begin{cases}\frac{2\sqrt{\lambda K_1 h}-h}{r\beta}, & \text{if } \lambda K_1>h\\ \frac{\lambda K_1}{r\beta}, & \text{otherwise}\end{cases}$
$D\sim U(a,b)\;\; 0\leqslant a<b$	$\arg\min_{\xi>0}\left\{\frac{h}{\xi}-\frac{\lambda K_1}{\xi}\left[1-\frac{e^{-a\xi}-e^{-b\xi}}{(b-a)\xi}\right]\right\}$	$\frac{r}{\xi^*}+\frac{\lambda}{\xi^*}\left[1-\frac{e^{-a\xi^*}-e^{-b\xi^*}}{(b-a)\xi^*}\right]$	$\frac{1}{r}\left[\frac{h-\lambda K_1}{\xi^*}+\lambda K_1\frac{e^{-a\xi^*}-e^{-b\xi^*}}{(b-a)\xi^{*2}}+\frac{\lambda K_1(b-a)}{2}\right]$
$D\sim\Gamma(\alpha,\beta)\;\; \alpha,\beta>0$	$\arg\min_{\xi>0}\left\{\frac{h}{\xi}-\lambda K_1\frac{1-(1+\xi/\beta)^{-\alpha}}{\xi}\right\}$	$\frac{r}{\xi^*}+\frac{\lambda}{\xi^*}[1-(1+\xi^*/\beta)^{-\alpha}]$	$\frac{1}{r}\left[\frac{h}{\xi^*}-\lambda K_1\frac{1-(1+\xi^*/\beta)^{-\alpha}}{\xi^*}+\lambda K_1\alpha\beta\right]$

we shall express $\Phi_\xi(u)$ in terms of ξ alone by expressing θ in terms of ξ. To this end, we set $z=0$ in Equation (31), and deduce the relation as follows by the fact that $\psi(0)=-r$ in light of Equation (14),

$$\theta=-rV_\rho(0)/\xi. \tag{56}$$

Substituting Equation (56) into Equation (34) then yields

$$\begin{aligned}\Phi_\xi(u)=&\frac{\xi\tilde{g}(\xi)}{r}+\frac{\xi V_\xi(\xi)}{\xi^2+rV_\xi(0)}\left[e^{\xi u}\int_u^\infty g(x)e^{-\xi x}\,dx-\tilde{g}(\xi)\right]\\&+\frac{\xi V_\xi(-rV_\xi(0)/\xi)}{\xi^2+rV_\xi(0)}\left[e^{-rV_\xi(0)u/\xi}\int_0^u g(x)e^{rV_\xi(0)x/\xi}\,dx\right.\\&\left.+\frac{\xi^2\tilde{g}(\xi)}{rV_\xi(0)}(e^{-rV_\xi(0)u/\xi}-1)\right].\end{aligned} \tag{57}$$

The boundedness of $\Phi_\xi(u)$ guarantees the existence of a global minimizing point, $\xi^*=\arg\min_{\xi>0}\{\Phi_\xi(u)\}$. However, the function $\Phi_\xi(u)$ is not convex in general. In fact, it is challenging to prove the uniqueness of the global minimizer, and this still remains an open problem.

In light of Theorem 3, a minimizer, ξ^*, can be computed in several ways. A straightforward, but relatively time-consuming, method is global search. However, when $\Phi_\xi(u)$ is convex, the availability of the derivative $(\partial/\partial\xi)\Phi_\xi(u)$ allows us to apply the relatively fast Newton's method. The above discussion can be summarized as follows.

Corollary 2. *Given $I(0)=u$, the optimal replenishment rates for $\Phi_\rho(u)$ are given by*

$$\rho^*=\frac{r+\lambda[1-\tilde{f}_D(\xi^*)]}{\xi^*}, \tag{58}$$

where $\xi^*=\arg\min_{\xi>0}\{\Phi_\xi(u)\}$ and $\Phi_\xi(u)$ is given by Equation (57). □

5.2. Optimal Replenishment Rate Under Delayed Replenishment

Suppose the system operates under *delayed replenishment*, that is, replenishment starts only after the first lost-sale occurrence. For example, suppose the system has an initial setup period during which replenishment is unavailable (e.g., a production facility that requires a setup time to gear up for production). Accordingly, minimizing the corresponding expected discounted cost, $\hat{\Phi}_\rho(u)$, over an infinite time horizon can be written as

$$\hat{\Phi}_\rho(u)=c_0(u)+d_0(u)\Phi_\rho(0). \tag{59}$$

From Equation (59), it is readily seen that minimizing $\hat{\Phi}_\rho(u)$ with respect to ρ is equivalent to minimizing $\Phi_\rho(0)$ with respect to ρ, since only the second term is a function of ρ. In the following two subsections, we treat the optimization of $\Phi_\rho(0)$ for the special cases of constant lost-sales penalty and loss-proportional penalty.

5.2.1. Constant Lost-Sales Penalty. Recall that in this case, $w(x)=K_0$, $x>0$, where $K_0>0$ is a constant, and $\Phi_\xi(0)$ is given by Equation (38). In view of Equation (17), Equation (38) can be rewritten as

$$\Phi_\xi(0)=\frac{h+\lambda\xi K_0[1-\tilde{f}_D(\xi)]}{r\xi}. \tag{60}$$

By Equation (60), the optimal ξ^* is given by

$$\xi^*=\arg\min_{\xi>0}\left\{\frac{h}{\xi}-\lambda K_0\tilde{f}_D(\xi)\right\}. \tag{61}$$

Table 1 exhibits the optimal ξ^*, $\rho^*$and $\Phi_{\rho*}(0)$ with closed-form formulas, when available, for selected demand distributions; detailed derivations are given in Appendix B.

In the table above and elsewhere, the arg min operation corresponds to a search for the optimal ξ^*, whenever a closed-form formula for it is either unavailable or not readily available. In particular, for an exponential demand distribution, the optimal solution is available in closed form, and the condition $\beta\lambda K_0 > h$ ensures a positive optimal replenishment rate; otherwise, it is optimal to have zero replenishment and bear the repeated penalty costs (a degenerate case).

5.2.2. Loss-Proportional Penalty. Recall that in this case, $w(x) = K_1 x$, for $x > 0$, where $K_1 > 0$ is constant, and $\Phi_\xi(0)$ is given by Equation (43). In view of Equation (17), Equation (43) can be rewritten as

$$\Phi_\xi(0) = \frac{1}{r}\left[\frac{h}{\xi} - \lambda K_1 \frac{1-\tilde{f}_D(\xi)}{\xi}\right] + \frac{\lambda K_1 \mu_D}{r}, \qquad (62)$$

where $\mu_D = \mathbb{E}[D]$. Consequently, by Equation (62), the optimal ξ^* is given by

$$\xi^* = \arg\min_{\xi>0}\left\{\frac{h}{\xi} - \lambda K_1 \frac{1-\tilde{f}_D(\xi)}{\xi}\right\}. \qquad (63)$$

Table 2 exhibits the optimal ξ^*, $\rho^*$and $\Phi_{\rho*}(0)$ with closed-form formulas, when available, for selected demand distributions; detailed derivations are given in Appendix B.

Again, for an exponential demand distribution, the optimal solution is available in closed form, and the condition $\lambda K_1 > h$ ensures a positive optimal replenishment rate; otherwise, it is optimal to have zero replenishment and bear the repeated penalty costs (a degenerate case).

6. Numerical Study

This section contains two numerical studies of production-inventory systems with selected demand-size distributions, subject to constant lost-sales penalty. Both studies were conducted with the following common parameters: $\lambda = 1$, $h = 1$, $K_0 = 100$, and $r = 0.1$. Recall that only the exponential demand-size distribution gives rise to a closed-form optimal solution; in all other cases, optimal solutions were obtained by a simple search.

6.1. Optimal Numerical Solutions for Zero Initial Inventories

In this study we compute and compare the numerical values of $\Phi_{\rho^*}(0)$ for increasing mean demand sizes, and under the following demand-size distributions: constant, exponential, uniform and Gamma. Table 3 displays the optimal ρ^* and ξ^* as functions of the mean demand, $\mathbb{E}[D] = 1/\beta$, for the four aforementioned demand-size distributions.

From Table 3, it can be seen that the respective ρ^* and the corresponding $\Phi_{\rho^*}(0)$ increase in this order of distributions: exponential, uniform, gamma, and constant. Note that as the average demand increases, ρ^* and $\Phi_{\rho^*}(0)$ increase as expected. Furthermore, for each selected demand-size distribution, we observe that $\rho^* > \lambda\mathbb{E}[D]$ for $\mathbb{E}[D] < 7$ (case 1), whereas $\rho^* < \lambda\mathbb{E}[D]$ for $\mathbb{E}[D] > 15$ (case 2). One possible explanation for these observations can be derived by examining the optimal production attendant to a demand rate, noting that discounting implies that the objective function is driven by the behavior of the system in an initial interval (starting at 0). Thus, in case 1, the optimal production rate would be driven above the demand rate, because otherwise, the inventory level would stay low, thereby incurring excessive penalty costs. Conversely, in case 2, the optimal production rate would be driven below the demand rate, because otherwise, the inventory level would stay high, thereby incurring excessive holding costs.

The above observation can be explained analytically for the case of exponential demand, $D \sim \text{Exp}(\beta)$, with the aid of the explicit solution given in Table 1. In particular, assuming that $\beta\lambda K_0 > h$ holds, the optimal production rate is given in closed form by $\rho^* = ((\sqrt{\lambda\beta K_0} - \sqrt{h})/\beta)(r/\sqrt{h} + \lambda/\sqrt{\lambda\beta K_0})$, whence the difference $\rho^* - \lambda\mathbb{E}[D]$ is given by

$$\rho^* - \frac{\lambda}{\beta} = \sqrt{\frac{\lambda}{\beta}}\left(\frac{r\sqrt{K_0}}{\sqrt{h}} - \frac{\sqrt{h}}{\beta\sqrt{K_0}}\right) - \frac{r}{\beta}. \qquad (64)$$

Thus, for sufficiently large β, i.e., sufficiently small $\mathbb{E}[D]$, the right-hand side of Equation (64) becomes positive,

Table 3. Optimal $\Phi_{\rho^*}(0)$ for selected demand-size distributions.

	$D = 1/\beta$		$D \sim \text{Exp}(\beta)$		$D \sim U(0, 2/\beta)$		$D \sim \Gamma(4, 1/(4\beta))$	
$\lambda E[D]$	ρ^*	$\Phi_{\rho^*}(0)$	ρ^*	$\Phi_{\rho^*}(0)$	ρ^*	$\Phi_{\rho^*}(0)$	ρ^*	$\Phi_{\rho^*}(0)$
0.05	0.27	44.47	0.27	44.22	0.27	44.39	0.27	44.41
0.30	0.82	108.03	0.80	106.54	0.82	107.53	0.82	107.66
1.30	2.30	221.40	2.16	215.04	2.25	219.19	2.26	219.79
3.30	4.63	346.27	4.20	330.32	4.47	340.58	4.53	342.18
5.30	6.66	432.79	5.87	407.43	6.37	423.54	6.47	426.23
6.30	7.64	468.98	6.61	439.00	7.22	457.96	7.35	461.20
7.30	8.59	501.97	7.28	467.37	8.08	489.12	8.24	492.95
8.30	9.47	532.35	7.94	493.19	8.87	517.68	9.06	522.11
9.30	10.36	560.62	8.60	516.92	9.67	544.11	9.88	549.15
10.00	10.95	579.30	9.02	532.46	10.19	561.50	10.43	566.99
15.00	14.86	693.46	11.58	624.60	13.41	666.27	13.98	675.10
20.00	18.37	784.52	13.61	694.43	16.28	747.54	16.88	760.16
25.00	21.23	860.50	15.00	750.00	18.21	813.34	19.60	830.15
30.00	23.87	925.43	16.14	795.45	19.81	867.66	21.50	889.22

Table 4. Optimal quantities for selected demand-size distributions with respect to their coefficient of variation.

	$D\sim U(a,b)$					$D\sim\Gamma(\alpha,\beta)$				
c_v	a	b	ξ^*	ρ^*	$\Phi_{\rho^*}(0)$	α	β	ξ^*	ρ^*	$\Phi_{\rho^*}(0)$
$1/\sqrt{3}$	0	20	0.041	10.165	561.497	3	0.300	0.041	10.242	562.971
1/2	1.340	18.660	0.040	10.363	566.176	4	0.400	0.040	10.404	566.983
1/3	4.226	15.774	0.039	10.676	573.615	9	0.900	0.039	10.691	573.768
1/4	5.670	14.330	0.039	10.783	576.124	16	1.600	0.039	10.785	576.171

implying $\rho^* > \lambda\mathbb{E}[D]$. Conversely, for sufficiently small but positive β, i.e., sufficiently large $\mathbb{E}[D]$, the right-hand side of Equation (64) becomes negative, implying $\rho^* < \lambda\mathbb{E}[D]$. Furthermore, by Equation (64), the cut-off point for $\rho^* = \lambda\mathbb{E}[D]$ is identified by $\lambda = r^2\beta(r\beta\sqrt{K_0/h} - \sqrt{h/K_0})^{-2}$. In this numerical study with the selected parameters and $\lambda = 1$, it shows that the cut-off mean demand is $\mathbb{E}[D] = 13.7$. That is, $\rho^* > \lambda\mathbb{E}[D]$ for $\mathbb{E}[D] < 13.7$, whereas $\rho^* < \lambda\mathbb{E}[D]$ for $\mathbb{E}[D] > 13.7$, which explains our observations.

In the next numerical study, we use the same parameters as before, but fix $\mathbb{E}[D] = 10$ and vary the value of the coefficient of variation c_v (ratio of standard deviation to mean) of the random demand. For each selected value of c_v, we chose the parameters of uniform and gamma distributions for D so as to keep the corresponding values of c_v the same. Table 4 displays several such parameter values and the corresponding ρ^*, ξ^* and $\Phi_{\rho^*}(0)$ for selected c_v ranging between $1/\sqrt{3}$ to $1/4$.

From Table 4, it can be seen that the respective ρ^* and the corresponding $\Phi_{\rho^*}(0)$ increase in c_v. For each case, it is shown $\rho^* > \lambda\mathbb{E}[D] = 10$. Note that although the variation in ξ^*, ρ^* and $\Phi_{\rho^*}(0)$ is not significant compared with the change in c_v, it reveals to what extent the optimal rates depend on more than the first two moments of the demand distribution. Furthermore, observe that when the demand distribution is $\Gamma(\alpha,\beta)$, we have larger ρ^* and $\Phi_{\rho^*}(0)$ than their counterparts for demand distribution $U(a,b)$. This phenomenon can be explained by the longer tail of the $\Gamma(\alpha,\beta)$ distribution (cf. De Kok 1987).

6.2. Optimal Numerical Solutions for Arbitrary Initial Inventory Levels

In this study we compute and compare the numerical values of ξ^*, ρ^*, and $\Phi_{\rho^*}(u)$ for selected demand-size distributions (constant, exponential, and uniform) with increasing initial inventory levels and for low and high average demands. Tables 5 and 6 display ρ^*, ξ^*, and $\Phi_{\rho^*}(u)$ for sample low and high demands as functions of the initial inventory level, $I(0) = u$.

Tables 5 and 6 reveal similar behavior patterns of ρ^* and $\Phi_{\rho^*}(u)$, as functions of $I(0) = u$. For each demand-size distribution in each table, ρ^* decreases as $I(0) = u$ increases, whereas the corresponding $\Phi_{\rho^*}(u)$ first decreases and then increases in u. Also, for any given initial inventory level, $\Phi_{\rho^*}(u)$ increases as the average demand, $\lambda\mathbb{E}[D]$, increases. Moreover, for each demand-size distribution, the optimal initial inventory level $u^* = \arg\min_{u\geqslant 0}\{\Phi_{\rho^*}(u)\}$ increases in the average demand. For example, $u^* = 10$ in Table 5 and $u^* \in [25, 35]$ in Table 6 are cases in point. In other words, a larger demand size is more beneficial when the initial inventory level is high. This is intuitive since higher demand is more likely to deplete the inventory quickly, which reduces the holding cost incurred because of a high initial inventory level. We also observe that in each of these tables, ρ^* decreases in the demand-size distribution in this order: constant, uniform, and exponential; this, however, does not generally hold for $\Phi_{\rho^*}(u)$.

7. Time-Average Cost and Optimization

The long-run time-average (undiscounted) cost can be treated similarly to its discounted counterpart. In this case, we need

Table 5. Optimal quantities for selected demand-size distributions under a low demand with $\lambda\mathbb{E}[D] = 1/\beta = 2$.

	$D = 1/\beta$			$D\sim\text{Exp}(\beta)$			$D\sim U(0,2/\beta)$		
$I(0)=u$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$
0	0.076	3.169	272.590	0.082	2.936	262.840	0.078	3.087	269.170
5	0.130	2.530	157.450	0.113	2.515	193.450	0.113	2.614	172.160
10	0.194	2.173	150.260	0.155	2.171	181.490	0.177	2.165	163.060
15	0.301	1.835	175.720	0.208	1.893	191.810	0.215	1.996	177.840
20	0.372	1.679	197.900	0.273	1.660	212.640	0.367	1.569	204.800
25	0.513	1.445	229.660	0.357	1.447	239.160	0.387	1.528	232.330
30	0.547	1.399	262.250	0.466	1.250	269.040	0.469	1.383	264.410
35	0.717	1.202	295.100	0.610	1.065	301.060	0.674	1.119	297.570
40	1.160	0.864	330.040	0.812	0.885	334.520	1.065	0.816	329.030
45	1.714	0.623	364.120	1.109	0.712	368.970	1.439	0.644	362.650
50	7.598	0.145	392.380	1.591	0.541	404.150	3.055	0.333	396.290

Table 6. Optimal quantities for selected demand-size distributions under a high demand with $\lambda\mathbb{E}[D]=1/\beta=20$.

	$D=1/\beta$			$D\sim\text{Exp}(\beta)$			$D\sim U(0,2/\beta)$		
$I(0)=u$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$	ξ^*	ρ^*	$\Phi_{\rho^*}(u)$
0	0.030	18.299	784.500	0.041	13.519	694.430	0.034	16.187	747.520
5	0.031	18.251	774.650	0.041	13.356	684.430	0.034	16.187	736.510
10	0.032	17.943	759.150	0.043	13.147	676.910	0.034	16.187	726.210
15	0.032	17.943	736.390	0.044	12.928	671.680	0.037	15.709	716.850
20	0.032	17.943	705.290	0.045	12.685	668.580	0.037	15.709	709.080
25	0.035	17.324	705.480	0.047	12.437	667.440	0.039	15.203	703.600
30	0.036	16.994	707.410	0.049	12.171	668.130	0.039	15.203	700.800
35	0.036	16.994	705.130	0.051	11.889	670.500	0.042	14.642	698.260
40	0.040	16.303	706.100	0.053	11.609	674.430	0.042	14.642	698.590
45	0.040	16.303	709.320	0.055	11.317	679.790	0.045	14.073	701.890
50	0.042	15.929	715.630	0.058	11.007	686.490	0.045	14.073	707.640

to assume the stability condition, $\rho<\lambda\mathbb{E}[D]$ (or equivalently $\mathbb{E}[\tau_1]<\infty$); otherwise the long-run time-average cost is infinite. In the sequel, we derive the time-average cost directly from the results for the discounted cost by taking limits as $r\downarrow 0$ and using the renewal reward theorem (cf. Ross 1996).

For $r=0$, the *Lundberg's fundamental equation* of Equation (15) becomes

$$\lambda\tilde{f}_D(z)+\rho z-\lambda=0. \tag{65}$$

Under the stability condition $\rho<\lambda\mathbb{E}[D]$, it follows that Equation (65) has two real roots: $\theta_0=0$ and $\xi_0>0$. Next, by Equations (1) and (65), one has

$$\rho=\lambda\tilde{\tilde{F}}_D(\xi_0), \tag{66}$$

which implies that ρ and ξ_0 are connected by a bijection.

In view of Equation (66), the stability condition $\rho<\lambda\mathbb{E}[D]$ can be written as $\tilde{\tilde{F}}_D(\xi_0)<\mathbb{E}[D]$. Since $\tilde{\tilde{F}}_D(z)$ is monotonically decreasing in z and $\tilde{\tilde{F}}_D(z)=\mathbb{E}[D]$ at $z=0$, the stability condition $\rho<\lambda\mathbb{E}[D]$ in the PRR space, can be equivalently expressed as $\xi_0>0$ in the LPR space.

Under the stability condition $\xi_0>0$ in the LPR space (i.e., $\rho<\lambda\mathbb{E}[D]$ in the PRR space), the inventory process over time intervals of the form $(\tau_i,\tau_{i+1}]$ is a renewal process, and the corresponding cost process can be regarded as a renewal reward process, with finite expectations of interrenewal times and cycle rewards. Consequently, by Theorem 3.6.1 in Ross (1996), the long-run time-average cost is independent of the initial inventory level, and can be represented by

$$\bar{c}_\rho=\frac{c_\rho(0)}{\mathbb{E}[\tau_1\mid I(0)=0]}, \tag{67}$$

where $c_\rho(0)$ is given by Equation (6) with $r=0$ and $u=0$. Next, we use ξ_0 as the decision variable to derive $\bar{c}_\rho$ in closed form and analyze its optimal solution. Following our notational fashion, we let $\bar{c}_\rho$ and $\bar{c}_{\xi_0}$ denote the time-average cost function of Equation (67) in terms of ρ and ξ_0, respectively. To derive the time average cost, we use the fact that $d_\rho(u)$, defined by Equation (10), can be interpreted as the moment generating function of τ_1 at $-r$; cf. Karr (1993). Consequently, by Equation (24), the expected time to the first shortage conditioned on the initial inventory level can be written as

$$\mathbb{E}[\tau_1\mid I(0)=0]=\frac{1}{\rho\xi_0}=\frac{1}{\lambda-\lambda\tilde{f}_D(\xi_0)}. \tag{68}$$

Note also that Equation (68) can be interpreted as the expected value of the time to ruin in the classical insurance model (cf. Gerber and Shiu 1998), conditioned on a zero initial surplus level. The following theorem provides a closed-form expression for the time-average cost.

Theorem 4. *Under the stability condition $\rho<\lambda\mathbb{E}[D]$, the time-average cost is given by*

$$\bar{c}_{\xi_0}=\xi_0\tilde{g}(\xi_0), \tag{69}$$

where $\xi_0>0$. □

We mention that De Kok (1987) studies a corresponding production-inventory system, but with two switchable production rates, and provides an approximation for the time average of inventory holding and switching cost (cf. Equation (2.10) therein). Actually, our production-inventory model can be treated as the aforementioned model, provided the two production rates as the equal and there is no switching cost. In this case, the approximated carrying cost in De Kok (1987) is exactly equivalent to the time-average holding cost in Equation (69). However, the approximation proposed by De Kok (1987) only accounts for the holding cost but ignores the lost-sale penalty component.

In view of Theorem 4, minimizing $\bar{c}_\rho$ with respect to ρ is equivalent to minimizing $\bar{c}_{\xi_0}$ with respect to the positive variable ξ_0. To this end, we first optimize $\bar{c}_{\xi_0}=\xi_0\tilde{g}(\xi_0)$ in the LPR space to find the optimal ξ_0^*, and then compute the corresponding optimal ρ^* in the PRR space. The following corollary provides a general structural result for the optimal replenishment rate, ρ^*.

COROLLARY 3. *The optimal replenishment rate for the time-average cost $\bar{c}_\rho$ under the stability condition $\rho < \lambda\mathbb{E}[D]$ is given by*

$$\rho^* = \lambda\tilde{\tilde{F}}_D(\xi_0^*), \tag{70}$$

where

$$\xi_0^* = \arg\min_{\xi_0>0}\{\xi_0\tilde{g}(\xi_0)\}. \quad \square \tag{71}$$

8. Further Extensions

The research presented in this paper can be extended in several directions. First, the methodology can be extended to include in the objective function a variable production cost modeled as a nonnegative and increasing function $a(\rho)$ of the replenishment rate. In this case, the expected discounted production cost is

$$v_\rho = \int_0^\infty a(\rho)e^{-rt}\,dt = \frac{a(\rho)}{r}. \tag{72}$$

By Equation (17), we can rewrite v_ρ in Equation (72) as

$$v_\xi = \frac{a(r/\xi + \lambda\tilde{\tilde{F}}_D(\xi))}{r}, \tag{73}$$

where v_ρ and v_ξ denote the same cost function, but of ρ and ξ, respectively. Finally, we can express the total expected discounted cost function as $\Phi_\xi(u) + v_\xi$, where $\Phi_\xi(u)$ is given by Equation (57) and v_ξ by Equation (73). This closed form of the objective function allows one to compute the optimal ξ^* directly, from which the optimal replenishment rate ρ^* can be recovered via Equation (58).

For the case of time-average cost, adding the production cost $a(\rho)$ to Equation (69) yields the total cost function representation

$$a(\rho) + \xi_0\tilde{g}(\xi_0) = a(\lambda\tilde{\tilde{F}}_D(\xi_0)) + \xi_0\tilde{g}(\xi_0), \tag{74}$$

by virtue of Equation (66). The above closed-form expression allows one to compute the optimal ξ_0^* directly. The requisite optimal replenishment rate ρ^* can then be obtained from Equation (70).

Second, we point out that the results of this paper can be applied to cost optimization (discounted or time average) subject to a given service-level constraint, e.g., a *fill rate* π, defined as the percentage of demand arrivals that are immediately satisfied in full from inventory on hand. Let the *lost-sales rate* be denoted by $\bar{\pi} = 1 - \pi$. Then, $\bar{\pi} = \lim_{t\to\infty} N_B(t)/N_A(t)$, where $N_A(t)$ and $N_B(t)$ denote the number of demand arrivals and lost-sale occurrences, respectively, in the interval $(0, t]$. The lost-sales rate, $\bar{\pi}$, can be alternatively represented as (cf. Ross 1996, Theorem 3.4.4)

$$\bar{\pi} = \frac{\mathbb{E}[T_1]}{\mathbb{E}[\tau_1 \mid I(0) = 0]}. \tag{75}$$

Substituting $\mathbb{E}[T_1] = 1/\lambda$ and Equation (68) into Equation (75) yields

$$\bar{\pi} = \frac{\rho\xi_0}{\lambda} = 1 - \tilde{f}_D(\xi_0). \tag{76}$$

Consequently, we have the following representation for the fill rate

$$\pi = \tilde{f}_D(\xi_0). \tag{77}$$

For optimization problems with objective functions of expected discounted cost or long-run time-average cost, constrained by a given minimal *fill rate*, $0 < \pi' \leqslant 1$, one can apply Equation (77) to compute the critical value ξ' such that $\tilde{f}_D(\xi') = \pi'$. It follows that the cost optimization problem (e.g., the time average cost studied in §7) with a constrained fill rate, π', can be solved by a search in the LPR space, restricted to the interval $0 < \xi_0 \leqslant \xi'$, in lieu of the original search space, $\xi_0 > 0$.

9. Conclusions and Future Research

This paper investigated a continuous-review single-product production-inventory system with a constant replenishment rate, compound Poisson demands, and lost sales. Two objective functions that represent metrics of operational costs were investigated: (1) the sum of the expected discounted inventory holding costs and the lost-sales penalties, over an infinite time horizon, given an initial inventory level; and (2) the long-run time-average of the same costs. A bijection between the PRR space and LPR space was established to facilitate optimization. For any initial inventory level, a closed-form expression was derived for the expected discounted cost, given an initial inventory level, in terms of an LPR variable. The resultant cost function was then readily optimized in the LPR space, and the requisite optimal value of the replenishment rate was recovered via the aforementioned bijection. In addition, the time-average cost was also derived in closed form under a stability condition, and an optimization methodology similar to the one used for the expected discounted cost was applied to optimize the requisite time-average cost.

Additional work in this area may include the following. First, for the general model (with general cost functions and general demand distributions), one might admit multiple optimal replenishment rates, though it is likely that a single optimal replenishment rate is unique under fairly general conditions. The type of conditions necessary to ensure uniqueness is a future research topic. Second, one might introduce inventory capacity constraints (e.g., base stock level), such that replenishment is suspended or shut down when the inventory level reaches or is at capacity. Third, it is of interest to investigate similar production-inventory systems with discrete replenishment, that is, where replenishment orders are triggered by demand arrivals that drop the inventory level below some prescribed base stock level.

Finally, regarding the discrete-time version of these problems, we note that the integro-differential equation obtained in Lemma 1 is no longer valid, since its derivation is based on time continuity. Therefore, a different approach that utilizes Markov chain and/or renewal theory might be employed to treat the corresponding discrete-time models.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2014.1299.

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Endnotes

1. According to a recent publication of the Independent Statistics and Analysis, U.S. Energy Information Administration.

References

Adan I, Boxma O, Perry D (2005) The $G/M/1$ queue revisited. *Math. Methods Oper. Res.* 62(3):437–452.

Asmussen S (2000) *Ruin Probabilities*, Advanced Series on Statistical Science and Applied Probability (World Scientific, Singapore).

Asmussen S (2003) *Applied Probability and Queues*, 2nd ed. (Springer, New York).

Bell CE (1971) Characterization and computation of optimal policies for operating an $M/G/1$ queuing system with removable server. *Oper. Res.* 19(1):208–218.

Biotech/Biomedical: Cell-Free Protein Production (1997) *Membrane and Separation Technology News* (1 April).

Boxma OJ, Löpker A, Perry D (2011) Threshold strategies for risk processes and their relation to queueing theory. *J. Appl. Probab.* 48A(Special volume):29–38.

Churchill RV (1971) *Operational Mathematics* (McGraw-Hill).

Cohen AM (2007) *Numerical Methods for Laplace Transform Inversion* (Springer, New York).

De Kok AG (1985) Approximations for a lost-sales production/inventory control model with service level constraints. *Management Sci.* 31(6): 729–773.

De Kok AG (1987) Approximations for operating characteristics in a production-inventory model with variable production rate. *Eur. J. Oper. Res.* 29(3):286–297.

De Kok AG, Tijms HC, Van Der Duyn Schouten FA (1984) Approximations for the single product production-inventory problem with compound Poisson demand and service level constraints. *Adv. Appl. Probab.* 16:378–401.

Display Development News (2000) PixTech Hits Production Milestone, November 1.

Doshi BT, Van der Duyn Schouten FA, Talman AJJ (1978) A production inventory control model with a mixture of back-orders and lost-sales. *Management Sci.* 24(10):1078–1087.

Federal Register (2009) Mandatory Reporting of Greenhouse Gases, part III of IX, April 10, vol. 74, no. 68.

Gavish B, Graves SC (1980) A one-product production/inventory problem under continuous review policy. *Oper. Res.* 28(5):1228–1236.

Gerber HU, Shiu ESW (1998) On the time value of ruin. *North Amer. Actuarial J.* 2:48–78.

Gerber HU, Shiu SW (1997) The joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at Ruin. *Insurance: Math. Econom.* 21:129–137.

Graves SC (1982) The application of queueing theory to continuous perishable inventory systems. *Management Sci.* 28(4):400–406.

Graves SC, Keilson J (1981) The compensation method applied to a one-product production inventory model. *Math. Oper. Res.* 6(2): 246–262.

Grunow M, Günther HO, Westinner R (2007) Supply optimization for the production of raw sugar. *Internat. J. Production Econom.* 110:224–239.

Heyman DP (1968) Optimal operating policies for $M/G/1$ queuing systems. *Oper. Res.* 16(2):362–382.

Jacobs FR, Chase RB (2013). *Operations and Supply Management*, 14th ed. (McGraw-Hill, New York).

Lee H, Srinivasan MM (1989) Control policies for the $M^x/G/1$ queueing system. *Management Sci.* 35(6):708–721.

Lin XS, Pavlova KP (2006) The compound Poisson risk model with a threshold dividend strategy. *Insurance: Math. Econom.* 38(1): 57–80.

Lin XS, Willmot GE, Drekic S (2003) The classical risk model with a constant dividend barrier: Analysis of the Gerber-Shiu discounted penalty function. *Insurance: Math. Econom.* 33:551–566.

Löpker A, Perry D (2010) The idle period of the finite $G/M/1$ queue with an interpretation in risk theory. *Queueing Systems* 64:395–407.

Lundberg F (1932) Some supplementary researches on collective risk theory. *Scandinavian Actuarial J.* 1932(3):137–158.

Perry D (2011) The $G/M/1$ Queue. Cochran JJ, ed., *Wiley Encyclopedia of Operations Research and Management Science*. (John Wiley & Sons, Hoboken, NJ).

Perry D, Stadje W, Zacks S (2005) Sporadic and continuous clearing policies for a production/inventory system under an M/G demand process. *Math. Oper. Res.* 30(2):354–368.

Prabhu NU (1965) *Queues and Inventories: A Study of Their Basic Stochastic Processes* (John Wiley & Sons, New York).

Prabhu NU (1997) *Stochastic Storage Processes: Queues, Insurance Risk, Dams and Data Communication*, 2nd ed. (Springer, New York).

Ross SM (1996) *Stochastic Processes*, 2nd ed. (John Wiley & Sons, New York).

Saff EB, Snider AD (1993) *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering* (Prentice Hall, Upper Saddle River, NJ).

Scarf H (1960) The optimality of (s, S) policies for the dynamic inventory problem. Arrow K, Karlin S, Suppes P, eds. *Proc. 1st Stanford Sympos. Math. Methods Soc. Sci.* (Stanford University Press, Stanford, CA), 196–202.

Shortle JF, Brill PH, Fischer MJ, Gross D, Masi DMB (2004) An algorithm to compute the waiting time distribution for the $M/G/1$ queue. *INFORMS J. Comput.* 16(2):152–161.

Sobel MJ (1969) Optimal average-cost policy for a queue with start-up and shut-down costs. *Oper. Res.* 17(1):145–162.

Widder DV (1959) *The Laplace Transform* (Princeton University Press, Princeton, NJ).

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