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Cash-Flow Based Dynamic Inventory Management

(Authors' names blinded for peer review)

Firms, such as retailers and wholesalers routinely manage interrelated flows of cash and inventories of goods, and have to make financial and operational decisions simultaneously. In this paper, we model a firm that uses its capital to finance purchases of a single-product inventory, subject to periodic review, lost sales and zero replenishment lead times. The firm earns interest on its cash on hand but pays interest on its debt. Demand is random but not necessarily stationary over periods. The objective is to maximize the expected value of the capital at the end of a finite planning horizon. First, it is shown that the optimal ordering policy is characterized by a sequence of two threshold variables. Then, upper and lower bounds for these threshold values are developed using two myopic ordering policies. Based on these bounds, we provide an efficient algorithm to compute the two threshold values. Subsequently, it is shown that policies of similar structure are optimal when the loan and deposit interest rates are piecewise linear functions and when there is a maximum loan limit. Further managerial insights are provided by numerical studies.

Key words: Inventory-finance decision, threshold variables, myopic policy.

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1. Introduction

In the current competitive environment, firms such as retailers and small to medium scalar wholesalers must make decisions simultaneously on management of interrelated flows of cash and products. For example, most tourist shops in big cities such as New York City, Los Angeles, Chicago, Boston and others are likely to apply capital loan to restock inventory. According to a survey reported by Lindberg and Vaughn (2004), more than one-third of the gift shops, on average, take financial loans from banks or other private lenders, government agencies, venture capital and stock placements. For another example, dried sea food and tonic food retailer stores are commonly seen, almost in every corner, in Hong Kong and some big cities in Japan, and they are selling high-valued products such as shark fin, bird's nest, cordyceps sinensis, Japanese sea cucumber, dried abalone and many others [Conover et al. (1998) and Clarke (2002)]. Those stores are typically facing financial restrictions when making inventory replenishment decisions.

In this study we model and analyze the optimal financial and operational policy of a small or medium-scale firm whose inventory is subject to lost sales, zero replenishment lead times and periodic review over a finite planning horizon. The firm's *treasury* or *inventory-capital profile* consists of inventory level and capital level, where a positive capital level presents the position of cash on hand while a negative capital level means a loan position. In each period, the firm can use its cash on hand or an external short-term loan (if needed) to procure products for inventory. Any cash on hand is deposited in a bank account to earn interest at a given rate, while debt incurs interest at a higher given rate as well. We develop and study a discrete-time model for a single-product, single- and multiple-period inventory system. The objective of the firm is to dynamically optimize the order quantities given the current financial condition in each period via joint operational/financial decisions, so as to maximize the expected value of the firms capital (i.e., total wealth level) at the end of a finite time horizon.

The inventory flow is described as follows. At the beginning of each period, the firm decides on an order quantity and the corresponding replenishment order materializes with zero lead time. During the remainder of the period, no inventory transactions (demand fulfillment or replenishment) take place. Instead, all such transactions are settled at the end of the period. Incoming demand is aggregated over that period, and the total period demand draws down on-hand inventory. However, if the demand exceeds the on-hand inventory, then the excess demand is lost. All the left-over inventory (if any) is carried forward to the next period subject to a holding cost, and at the last period, the remaining inventory (if any) is disposed off either at a salvage value or at a disposal cost.

The cash flow is described as follows. All transactions pertaining to previous period are settled at the beginning of each period, where the firm updates its capital position with the previous period's revenue from sales and interests earned from a deposit or paid if there was a loan. Then, the firm decides the order quantity and pays for replenishment with cash on hand first given deposited cash can be withdrawn without penalty. However, if the cash on hand is insufficient to pay for replenishment, the shortage of capital can be borrowed subject to a higher interest rate. If not all cash on hand is used for replenishment, any unused cash amount is deposited to a bank where it earns interest. Deposited cash may be withdrawn at the beginning of any period without a withdrawal restriction to finance a replenishment order. At the end of each period the resulting cash on hand or debt is carried forward to the next period.

The main contribution of the paper is to establish the optimal ordering policy in terms of the *net worth* of the firm (capital in product units plus the inventory on hand) at the beginning of each period. It is shown that the optimal policy is characterized by a sequence of two threshold critical values α_n and β_n [cf. Theorem 1 for the single period problem and Theorem 3 for the multiple period problem]. The constants α_n and β_n are in general functions of the firms net worth. This optimal policy has the following structure: a) If the net worth is less than α_n, then the firm orders up to α_n, which is referred to as the *over-utilization* case. b) If the net worth is greater than β_n, then the firm orders up to β_n, which is referred to as the *under-utilization* case. c) Otherwise, when the net worth is between α_n and β_n, then the firm orders exactly as many units as it can afford *without* *borrowing*, which is referred to as the *full-utilization* case. Also, for the single period problem, we show that the (α, β) optimal policy yields a positive expected value even with zero values for both initial inventory and capital. For the multiple period problem, we construct two myopic policies which respectively provide upper and lower bounds for the threshold values. Based on these upper and lower bounds, we provide an efficient algorithm to compute the two thresholds, α_n and β_n, recursively for all periods n. It is also shown that policies of similar structure are optimal when the loan and deposit interest rates are piecewise linear functions and when there is a maximum loan limit [cf. Theorem 5 and Theorem 6].

The remainder of this paper is organized as follows. Section 2 reviews the related literature. In Section 3, the single period model is developed and the optimal (α, β) policy is derived, where the threshold values are functions of the demand distribution and the cost parameters of the problem. Section 4 extends the analysis for the dynamic multiple period problem and it derives a two-threshold structure for the optimal policy via a dynamic programming analysis. Section 5 introduces two myopic policies that provide upper and lower bounds for each α_n and β_n. Numerical studies are presented in Section 6. In Section 7, it is pointed out that simple modifications of the study allow the extension of the results to the case of piecewise linear interest rate functions and a maximum loan limit. Section 8 concludes the paper.

2. Literature Review

In the seminal study, Modigliani and Miller (1958) show that, within a perfect capital market, a firm's operational and financial decisions can be made separately. Since then, most literature in inventory management extends the classical newsvendor problem in a variety of perspectives but assuming there is no financial restriction faced by decision makers. Recently, considering the imperfection of the practical capital market, a growing literature has begun to consider the operational decision making under financial constraints. Among those studies, inventories of goods are often treated as special financial instruments [cf. Singhal (1988)]. In such a fashion, portfolios composed of products and regular financial instruments have been studied using finance/investment principles such as *Modern Portfolio Theory* (MPT) and the *Capital Asset Pricing Model* (CAPM). For related literature, we refer the reader to Corbett et al. (1999) and references therein. Further, the relationship between inventories and finance, along with the theoretical and empirical consequences are discussed by Girlich (2003).

The growing literature on the interface between operations management and financial decisions can be categorized into two major streams: single-agent stream and game related multiple-agent stream. Under each stream, the literature can be further classified into two subgroups: single-period and multi-period models. Our study belongs to the single-agent stream with both single-period and multi-period models.

In the single-agent stream, a firm is typical modeled to make operational decisions and financial decisions simultaneously without interacting with other firms. Considering an imperfect market, Xu and Birge (2004) develop models to make production and financing decisions simultaneously in the presence of demand uncertainty. The authors illustrate how a firm's production decisions are affected by the existence of financial constraints. Recently, Birge and Xu (2011) present an extension of a model of Xu and Birge (2004) by assuming that debt and production scale decisions depend on fixed costs necessary to maintain operations, variable costs of production, and volatility in future demand forecasts. Building on their previous work of Xu and Birge (2004), Xu and Birge (2008) review that analysis and consider the effect of different operating conditions on capital structure, including some empirical support of their previous predicted relationship between production margin and market leverage. Those studies focus on single-period problems.

Some studies on single-agent but multi-period problems are done by Hu and Sobel (2005), Chao et al. (2008) and Li et al. (2013). Their multi-period models, except Chao et al. (2008), allow different interest rates on cash on hand and outstanding loans. These papers also demonstrate the importance of the joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing. For example, Hu and Sobel (2005) examine the interdependence of a firm's capital structure and its short-term operating decisions concerning inventories, dividends, and liquidity. To this end, Hu and Sobel (2005) formulate a dynamic model to maximize the expected present value of dividends. Chao et al. (2008) study a multiple period inventory optimization problem faced by a self-financing firm where borrowing is not permitted, demands are independent and identically distributed (i.i.d.) random variables and purchase and sale unit prices are identical across all periods. Li et al. (2013) present and study a dynamic model of managerial decisions in a manufacturing firm in which inventory and financial decisions interact and are coordinated in the presence of demand uncertainty, financial constraints, and default risk. It is shown that the relative financial value of coordination can be made arbitrarily large.

In the second stream of the literature, multi-agent competition between firms and financial institutions has been investigated using game theoretic approaches. This literature includes but is not limited to: Buzacott and Zhang (2004), Dada and Hu (2008), Yasin and Gaur (2010) and Raghavan and Mishra (2011). Most of those literature deals with single-period problem. Buzacott and Zhang (2004) analyze a Stackelberg game between the bank and the retailer in a newsvendor inventory model. Dada and Hu (2008) assume that the interest rate is charged by the bank endogenously and use a game model for the relation between the bank and the inventory controller through which the equilibrium is derived and a non-linear loan schedule is obtained to coordinate the channel.

Yasin and Gaur (2010) study the implications of asset based lending for operational investment, probability of bankruptcy, and capital structure for a borrower firm. Raghavan and Mishra (2011) study a short-term financing problem in a cash-constrained supply chain.

As a part of the second stream literature, multi-agent game theoretic approaches have also been used to model the competition between suppliers and retailers. Such recent studies are Kouvelis and Zhao (2011a) and Kouvelis and Zhao (2011b) and many others. An important area in this literature is the work on the impact of the trade credits provided by suppliers to retailers. Lee and Rhee (2010) study the impact of inventory financing costs on supply chain coordination by considering four coordination mechanisms for investigation: all-unit quantity discount, buy backs, two-part tariff, and revenue-sharing. It is shown that using trade credit in addition to contracts, a supplier can fully coordinate the supply chain and achieve maximum joint profit. Also, Lee and Rhee (2011) model a firm with a supplier that grants trade credit and markdown allowance. Given the supplier's offer, it determines the order quantity and the financing option for the inventory under either trade credit or direct financing from a financial institution. The impact of trade credit is also studied in Yang and Birge (2011) from an operational perspective, in order to investigate the role that trade credit plays in channel coordination and inventory financing. It is shown that when offering trade credit, the supplier balances its impact on operational profit and costs.

Our paper is related to Chao et al. (2008), which consider a single-agent multi-period problem for a self-financing retailer without external loan availability. Chao et al. (2008) show that the optimal, cash flow-dependent, policy in each period, is uniquely determined by a single critical value. Our study differs from Chao et al. (2008) in the following ways: (1) we consider a loan which provides the retailer with flexibility to order a larger quantity; (2) Chao et al. (2008) assume an i.i.d. demand process and time-stationary costs, while we consider non-stationary demand process and time varying (loan and deposit) interest rates. In addition, we introduce two myopic policies which are used to generate an efficient algorithm to compute the threshold values. Our model can be considered as a generalization of Chao et al. (2008) that allows the use of financial loans (with and without a maximum loan limit). After we completed this study, we were informed of the independent work of Gong et al. (2012) which studies a similar model along different lines from ours. Our study differs from Gong et al. (2012) in several aspects including the following. We consider non-stationary demand process and time varying (loan and deposit) interest rates, myopic policies are investigated and utilized in computations, the interest rates can be piecewise linear functions, the case of financing under a maximum loan limit constraint is treated.

3. The Single Period Model

We first introduce necessary notation and assumptions. At the beginning of the period, the "inventory-capital" state of the system can be characterized by a vector (x, y), where x denotes the amount of on-hand inventory (number of product units) and y denotes the amount of product that can be purchased using all the available capital (i.e., y is the capital position measured in "product units"). Note that, $X = c \cdot x$ and $Y = c \cdot y$ represent respectively values of on-hand inventory and available capital position at the beginning of the period. Let D denote the single period random demand. For simplicity, we assume that D is a non-negative continuous random variable with a probability density function $f(\cdot)$ and cumulative distribution $F(\cdot)$. Let p, c, s, denote respectively the selling price, the ordering cost and the salvage price per unit of the product. Note that we allow a negative s in which case s represents a disposal cost per unit, e.g., the unit cost of disposing vehicle tires, etc. Further, let i denote the interest rate for a deposit, and ℓ the interest rate for a loan. The decision variable is the order quantity $q \geq 0$.

To avoid trivialities we assume that $i < \ell$ and it is possible to achieve a positive profit with the aid of a loan, i.e., $(1+\ell)c < p$. This assumption is equivalently written as:

$$\ell < \frac{p}{c} - 1. \tag{1}$$

Note also, that the above assumptions implies $i < \frac{p}{c} - 1$ since $i < \ell$, which says that investing on inventory is preferable to depositing all the available capital Y to the bank. Note that at the beginning of the period it is possible to purchase products with available capital y (when $y = Y/c > 0$) but it is not allowed to convert any of the available on hand inventory x into cash.

Given the initial inventory-capital state is (x, y), if an order of size $q \geq 0$ is placed and the demand during the period is D, then we have the following two cash flows.

1. The cash flow from sales of items (i.e., the realized revenue from inventory) at the end of the period is given by

$$\begin{aligned} R(D,q,x) &= p \cdot \min\{q+x, D\} + s \cdot [q+x-D]^+ \\ &= p \cdot [q+x-(q+x-D)^+] + s \cdot [q+x-D]^+ \\ &= p(q+x) - (p-s) \cdot [q+x-D]^+, \end{aligned} \tag{2}$$

where $[z]^+$ denotes the positive part of real number z, and the second equality holds by $\min\{z,t\} = z - [z-t]^+$.

2. The cash flow from capital at the end of the period can be computed when we consider the following two scenarios:

i) If the order quantity $0 \leq q \leq y$, then the left amount $c \cdot (y-q)$ of cash will be deposited in the bank and it will yield a positive flow of $c \cdot (y-q)(1+i)$ at the end of the period.

ii) Otherwise, if $q > y$ (even if $q = 0 > y$) then a loan amount of $c \cdot (q-y)$ will be incurred during the period and it will result in a negative cash flow of $c \cdot (q-y)(1+\ell)$ at the end of the period.

Consequently, the cash flow from the bank (positive or negative) can be written in general as

$$K(q,y) = c\cdot(y-q)\left[(1+i)\mathbf{1}_{\{q\le y\}} + (1+\ell)\mathbf{1}_{\{q>y\}}\right]. \tag{3}$$

Note that the cash flow from inventory, $R(D,q,x)$ is independent of y, while the cash flow from capital, $K(q,y)$ is independent of the initial on-hand inventory, x and the demand size D. Also, note that the ordering cost, $c\cdot q$, has been accounted for in Eq. (3) while the remaining capital, if any, has been invested in the bank and its value at the end of period is given by $K(q,y)$.

Thus, for any given initial "inventory-capital" state (x,y) and an order quantity q, the expected value of net worth at the end of the period is given by

$$G(q,x,y) = \mathbf{E}_D[\,R(D,q,x)\,] + K(q,y). \tag{4}$$

Substituting Eqs.(2) - (3) into Eq. (4) yields

$$G(q,x,y) = p(q+x) - (p-s)\int_0^{q+x}(q+x-t)f(t)dt + c\cdot(y-q)\left[(1+i)\mathbf{1}_{\{q\le y\}} + (1+\ell)\mathbf{1}_{\{q>y\}}\right]. \tag{5}$$

The following lemma summarizes the important properties of function $G(q,x,y)$.

Lemma 1. The function $G(q,x,y)$ is continuous in q, x and y, and it has the following properties.

i) It is concave in $q\in[0,\infty)$, for all x, y and all $s<p$.

ii) It is increasing and concave in x, for $s\ge 0$.

iii) It is increasing and concave in y, for all $s<p$.

Proof. The continuity follows immediately from Eq. (5). We next prove the concavity via examining the first-order and second-order derivatives. To this end, differentiating Eq. (5) via Leibniz's integral rule yields

$$\frac{\partial}{\partial q}G(q,x,y) = \begin{cases} p - c\cdot(1+i) - (p-s)F(q+x) & \text{if } q<y, \\ p - c\cdot(1+\ell) - (p-s)F(q+x) & \text{if } q>y. \end{cases} \tag{6}$$

Therefore, for $q>y$ or $q<y$

$$\frac{\partial^2}{\partial q^2}G(q,x,y) = -(p-s)f(q+x). \tag{7}$$

Then the concavity in q readily follows since $\frac{\partial^2}{\partial q^2}G(q,x,y)\le 0$ by Eq. (7).

Although the cost function is not differentiable (for the first order and/or the second order) at some specific points (e.g., at $q=y$), we can still consider its derivatives to show its increasing or decreasing properties that allows us to study the optimal solution. Such notational convention will be used throughout the paper.

The increasing property of $G(q,x,y)$ in x and y can be shown by taking the first order derivatives using Eq. (5):

$$\begin{aligned} \frac{\partial}{\partial x}G(q,x,y) &= p\bar{F}(q+x)+sF(q+x)>0, \\ \frac{\partial}{\partial y}G(q,x,y) &= c\cdot\left[(1+i)\mathbf{1}_{\{q<y\}}+(1+\ell)\mathbf{1}_{\{q>y\}}\right]>0. \end{aligned} \tag{8}$$

The joint concavity of $G(q,x,y)$ in x and y can be established by computing the second order derivatives below using again Eq. (5).

$$\begin{aligned} \frac{\partial^2}{\partial x^2}G(q,x,y) &= -(p-s)f(q+x)<0, \\ \frac{\partial^2}{\partial y^2}G(q,x,y) &= 0, \\ \frac{\partial^2}{\partial x\partial y}G(q,x,y) &= 0. \end{aligned}$$

Thus the Hessian matrix is negative semi-definite and the proof is complete. □

Remarks.

1. It is important to point out that $G(q,x,y)$ might not increase in x if $s<0$. In particular, if s represents a disposing cost, i.e., $s<0$, the right side of Eq. (8) might be negative in general, which implies that $G(q,x,y)$ is decreasing for some high values of x.

Further, for the special case with $s<0$, it is of interest to locate the critical value, x' such that $G(q,x,y)$ is decreasing for $x>x'$. To this end, we set Eq. (8) to be zero, which yields

$$(p-s)\cdot F(q+x)=p. \tag{9}$$

Therefore,

$$x'=F^{-1}\left(\frac{p}{p-s}\right)-q, \tag{10}$$

where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$. Eq. (10) shows that a higher disposing cost, $-s$, implies a lower threshold for x' above.

2. Lemma 1 implies that higher values of initial assets, x, y or the net worth $\xi=x+y$, will yield a higher expected revenue $G(q,x,y)$. Further, for any fixed assets (x,y) there is a unique optimal order quantity q^* such that

$$q^*(x,y)=\arg\max_{q\geq 0}G(q,x,y).$$

We next introduce the critical values of α and β as follows:

$$\alpha = F^{-1}(a); \tag{11}$$

$$\beta = F^{-1}(b), \tag{12}$$

where

$$a = \frac{p - c \cdot (1+\ell)}{p-s};$$
$$b = \frac{p - c \cdot (1+i)}{p-s}.$$

It is straightforward to see that $a \leq b$, since $0 \leq i \leq \ell$ by assumption. This implies that $\alpha \leq \beta$, since $F^{-1}(z)$ is increasing in z. The critical value β can been interpreted as the optimal order quantity for the classical firm problem corresponding to the case of sufficiently large Y of our model, in which case no loan is involved, but the unit "price" $c \cdot (1+i)$ has been inflated to reflect the opportunity cost of cash not invested in the bank at an interest rate i. Similarly, α can been interpreted as the optimal order quantity for the classical firm problem corresponding to the case $Y = 0$ of our model, i.e., all units are purchased by a loan at an interest rate ℓ.

Note also that in contrast to the classical firm model, the critical values α and β above, are now functions of the corresponding interest rates and represent opportunity costs that take into account the value of time using the interest factors $1+i$ and $1+\ell$.

We next state and prove the following theorem regarding the optimality of the $(\alpha,\, \beta)$ ordering policy.

THEOREM 1. For any given initial inventory-capital state (x, y), the optimal order quantity is

$$q^*(x,y) = \begin{cases} (\beta - x)^+, & \beta \leq x+y; \\ y, & \alpha \leq x+y < \beta; \\ \alpha - x, & x+y < \alpha, \end{cases} \tag{13}$$

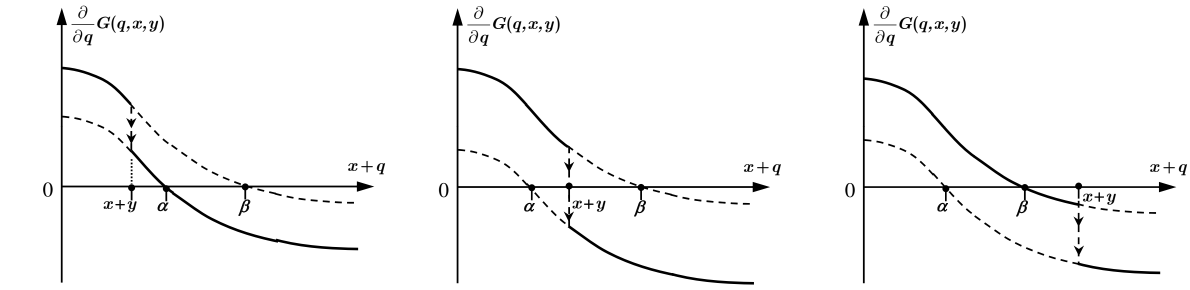
where α and β are given by Eq. (11) and (12), respectively.

Proof. For any given initial state (x, y), Lemma 1 implies that there exists a unique optimal order quantity $q^*(x, y)$ such that the profit function $G(q, x, y)$ is maximized. To prove Eq. (13), we investigate the first order derivative of the expected profit function given by Eq. (6). Figure 1 illustrates its functional structure with respect to three cases for various values of $x+y$.

a) If $x+y < \alpha$, then $G(q, x, y)$ is strictly increasing in q as long as $q+x \leq \alpha$, and decreasing thereafter, while $\partial G(q, x, y)/\partial q = 0$ for $q+x = \alpha$, cf. Figure 1 (a). It follows that in this case the optimal quantity q^* is such that $q^* + x = \alpha$.

b) If $\alpha \leq y < \beta$, then the profit function $G(q, x, y)$ is strictly increasing in q until $q = y$, and decreasing thereafter cf. Figure 1 (b). Then, the optimal quantity is $q^*(x, y) = y$.

c) If $x+y \geq \beta$, then the profit function $G(q, x, y)$ of $x+q$ is strictly increasing until β, and decreasing thereafter, cf. Figure 1 (c). Then, the optimal quantity after ordering is the one such that $q+x$ is close to β as much as it could be. Therefore, the optimal order quantity is $(\beta - x)^+$. This completes the proof. $\square$



(a) $x+y<\alpha$ (b) $\alpha<x+y<\beta$ (c) $x+y>\beta$

Figure 1 Functional Structure for the Derivative of $G(q,x,y)$ with Respective to q

Note that the optimal ordering quantity to a classical newsvendor model [cf. Zipkin (2000) and many others], can be obtained from Theorem 1 as the solution to the extreme case with $i=\ell=0$ whence the optimal order quantity is given by:

$$\alpha=\beta=F^{-1}\left(\frac{p-c}{p-s}\right).$$

We further elucidate the structure of the (α,β) optimal policy below where we discuss the utilization level of the initially available capital Y.

1. **(Over-utilization)** When $x+y<\alpha$, it is optimal to order $q^*=\alpha-x=y+(\alpha-x-y)$. In this case $y=Y/c$ units are bought using all the available fund Y and the remaining $(\alpha-x-y)$ units are bought using a loan of size: $c\cdot(\alpha-x-y)$.

2. **(Full-utilization)** When $\alpha\le x+y<\beta$, it is optimal to order $q^*=y=Y/c$ with all the available fund of Y. In this case, no deposit and no loan is involved.

3. **(Under-utilization)** When $x+y\ge\beta$, it is optimal to order $q^*=(\beta-x)^+$. In particular, if $x<\beta$, it is optimal to order $\beta-x$ using $c\cdot(\beta-x)$ units of the available cash Y, and deposit the remaining cash to earn interest. However, if $x\ge\beta$, then $q^*=0$, i.e., it is optimal not to order any units and deposit all the amount of Y to earn interest.

The above interpretation is illustrated in Figure 2 for the case in which $x=0$, by plotting the optimal order quantity q^* as a function of y. Note that for $y\in(0,\alpha)$ there is over utilization of y ; for $y\in[\alpha,\beta)$ there is full utilization of y and for $y\in[\beta,\infty)$ there is under utilization of y.

We next define the function

$$V(x,y)=\max_{q\ge 0}G(q,x,y). \tag{14}$$

and state and prove the following lemma which will be used in the next section.

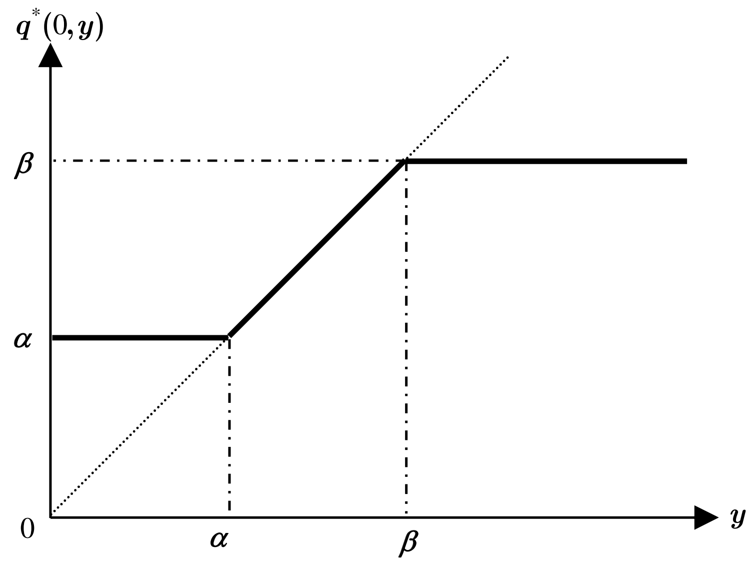


Figure 2 The Optimal Order Quantity when $x = 0$

THEOREM 2. For any initial state (x, y),

i) V(x,y) is given by

$$V(x,y) = \begin{cases} p\cdot x - (p-s)\cdot T(x) + c\cdot y\cdot(1+i), & x > \beta; \\ p\cdot\beta - (p-s)\cdot T(\beta) + c\cdot(x+y-\beta)(1+i), & x \le \beta,\ \beta \le x+y; \\ p\cdot(x+y) - (p-s)\cdot T(x+y), & \alpha \le x+y < \beta; \\ p\cdot\alpha - (p-s)\cdot T(\alpha) + c\cdot(x+y-\alpha)(1+l), & x+y < \alpha, \end{cases} \tag{15}$$

where $T(x) = \int_0^x (x-t)f(t)dt$;

ii) the function $V(x,y)$ is increasing in x and y, and jointly concave in (x,y), for $x, y \ge 0$.

Proof. Part (i) follows from substituting q^* given by Eq. (13) into Eq. (5).

For part (ii) the increasing property of V can be justified straightforwardly. For the concavity of V, note that by Lemma 1, $G(q,x,y)$ is concave in q, x and y. Taking the maximization of G over q and using Proposition A.3.10 of Zipkin (2000), p436, and Eq. (14) we have that the concavity in x and y is preserved and the proof is complete. □

From investment perspective, it is of interest to see the possibility of speculation [cf. Hull (2002)]. The following result shows that the (α,β) policy given in Theorem 1 yields positive value with zero investment. Specifically, when the firm has zero initial inventory asset and capital, i.e., $x = 0$ and $y = 0$, the optimal policy brings a positive expected final asset value.

COROLLARY 1. For $x = 0$ and $y = 0$, the following is true

$$V(0,0) = (p-s)\int_0^{\alpha} tf(t)dt > 0.$$

Proof. The result can be readily proved by setting $x = y = 0$ in Eq. (15). □

Note that arbitrage usually means that it is possible to have a positive profit for any realized demand (i.e., of a risk-free profit at zero cost; cf. Hull (2002)), thus the above speculation possibility does not in general imply an arbitrage. Actually, an arbitrage exists only if the demand is constant.

4. The multi-period problem

In this section, we extend the results of the previous section and consider the finite horizon version of the problem, with $N \geq 2$ periods. As in the single period, at the beginning of a period $n = 1, \ldots, N$, let the "inventory-capital" state of the system be summarized by a vector (x_n, y_n), where x_n denotes the amount of on-hand inventory (number of product units) and y_n denotes the amount of product that can be purchased using all the available capital (i.e., y_n is the capital position measured in "product units"). Note again that, $X_n = c \cdot x_n$ and $Y_n = c \cdot y_n$ represent respectively value of on-hand inventory and capital position at the beginning of period n. Let q_n denote the order quantity the firm uses in the beginning of period $n = 1, \ldots, N$. We assume the lead time of replenishment is zero. Throughout all periods $t = 1, \ldots, N-1$, any unsold units are carried over in inventory to be used in subsequent periods subject to a constant holding cost per unit per period. At the end of the horizon, i.e., period N, all the leftover inventory (if any) will be salvaged (or disposed off) at a constant price (cost) per unit.

Let p_n, c_n, h_n denote the selling price, ordering cost and holding cost per unit in period n, respectively. Let s denote the salvage price (or disposal cost) per unit at the end of period N. Let i_n and ℓ_n, with $i_n \leq \ell_n$, be the interest rates for deposit and loan in period n, respectively.

Finally, let D_n denote the demand of period n. We assume that demands of different periods are independent but could be non-stationary over periods. Let $f_n(\cdot)$, $F_n(\cdot)$ denote respectively the probability density function, the cumulative distribution function of D_n. The system state at the beginning of period n is characterized by (x_n, y_n). The order quantity $q_n = q_n(x_n, y_n)$ is decided at the beginning of period n as a function of (x_n, y_n). It is readily shown that the state (x_n, y_n) process under study is a *Markov decision process* (MDP) with decision variable q_n [cf. Ross (1992)]. Then, the dynamics for the two states of the system are formulated as follows, for $n = 1, 2, ..., N-1$

$$x_{n+1} = [x_n + q_n - D_n]^+; \tag{16}$$

$$y_{n+1} = [R_n(D_n, q_n, x_n) + K_n(D_n, q_n, y_n)]/c_{n+1}, \tag{17}$$

where

$$R_n(D_n, q_n, x_n) = p_n \cdot (x_n + q_n) - (p_n + h_n)\,[x_n + q_n - D_n]^+; \tag{18}$$

$$K_n(D_n, q_n, y_n) = c_n \cdot (y_n - q_n)\left[(1 + i_n)\mathbf{1}_{\{q_n \leq y_n\}} + (1 + \ell_n)\mathbf{1}_{\{q_n > y_n\}}\right]. \tag{19}$$

In particular, at the end of period N, the revenue from inventory is

$$\begin{aligned} R_N(D_N, q_N, x_N) &= p_N \cdot \min\{x_N + q_N, D_N\} - h_N\,[x_N + q_N - D_N]^+ \\ &= p_N[q_N + x_N] - (p_N - s)[q_N + x_N - D_N]^+, \end{aligned} \tag{20}$$

where $h_N = -s$, and the revenue from the bank is

$$K_N(D_N, q_N, y_N) = c_N \cdot (y_N - q_N)\left[(1+i_N)\mathbf{1}_{\{q_N \le y_N\}} + (1+\ell_N)\mathbf{1}_{\{q_N > y_N\}}\right] \tag{21}$$

For a risk-neutral firm, the objective is to maximize the expected value of the total wealth at the end of period N, that is,

$$\max_{q_1, q_2, \cdots, q_N} \mathbf{E}\big[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)\big],$$

where x_N and y_N are sequentially determined by decision variables q_n, $n \le N$. Accordingly, we have the following dynamic programming formulation:

$$V_n(x_n, y_n) = \sup_{q_n \ge 0} \mathbf{E}\left[V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n\right], \qquad n = 1, 2, \cdots, N-1 \tag{22}$$

where the expectation is taken with respect to D_n, and x_{n+1}, y_{n+1} are given by Eqs. (16), (17), respectively. For the final period N, we have:

$$V_N(x_N, y_N) = \sup_{q_N \ge 0} \mathbf{E}\big[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)\big]. \tag{23}$$

Note that for period N, the optimal solution is given by Theorem 1.

In the sequel it is convenient to work with the quantities $p'_n = p_n/c_{n+1}$, $h'_n = h_n/c_{n+1}$ and $c'_n = c_n/c_{n+1}$ and to take $z_n = x_n + q_n$ as the decision variable instead of q_n. Here, z_n refers to the available inventory after replenishment, and it is restricted by $z_n \ge x_n$ for each period n. Accordingly, the dynamic programming (DP) model defined by Eqs. (22)-(23) can be presented as:

$$V_n(x_n, y_n) = \max_{z_n \ge x_n} G_n(z_n, x_n, y_n), \tag{24}$$

where

$$G_n(z_n, x_n, y_n) = \mathbf{E}\left[V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n\right], \tag{25}$$

for $0 \le x_n \le z_n$, and the inventory-capital states are dynamically given by

$$x_{n+1} = [z_n - D_n]^+; \tag{26}$$

$$\begin{aligned} y_{n+1} &= p'_n \cdot z_n - (p'_n + h'_n)\,[z_n - D_n]^+ \\ &\quad + c'_n \cdot (x_n + y_n - z_n)\left[(1+i_n)\mathbf{1}_{\{z_n \le x_n + y_n\}} + (1+\ell_n)\mathbf{1}_{\{z_n > x_n + y_n\}}\right]. \end{aligned} \tag{27}$$

We first present the following result with its proof given in Appendix.

LEMMA 2. For $n = 1, 2, \cdots, N$,

(1) The function $G_n(z_n, x_n, y_n)$ is increasing in x_n and y_n, and it is concave in z_n and (x_n, y_n).

(2) The function $V_n(x_n, y_n)$ is increasing and concave in (x_n, y_n).

We next present and prove the main result of this section.

THEOREM 3. (**The (α_n, β_n) ordering policy**).
For period $n = 1, 2, \cdots, N$ with given state (x_n, y_n) at the beginning of the period, there exist positive constants $\alpha_n = \alpha_n(x_n, y_n)$ and $\beta_n = \beta_n(x_n, y_n)$ with $\alpha_n \le \beta_n$, which define the optimal order quantity as follows:

$$q^*(x_n, y_n) = \begin{cases} (\beta_n - x_n)^+, & x_n + y_n \ge \beta_n; \\ y_n, & \alpha_n \le x_n + y_n < \beta_n; \\ \alpha_n - x_n, & x_n + y_n < \alpha_n. \end{cases} \tag{28}$$

Further, α_n is uniquely identified by

$$\mathbf{E}\left[\left(\frac{\partial V_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n)\frac{\partial V_{n+1}}{\partial y_{n+1}}\right)\mathbf{1}_{\{\alpha_n > D_n\}}\right] = [c'_n(1+\ell_n) - p'_n]\,\mathbf{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\right], \tag{29}$$

and β_n is uniquely identified by

$$\mathbf{E}\left[\left(\frac{\partial V_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n)\frac{\partial V_{n+1}}{\partial y_{n+1}}\right)\mathbf{1}_{\{\beta_n > D_n\}}\right] = [c'_n(1+i_n) - p'_n]\,\mathbf{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\right]. \tag{30}$$

where the expectations are taken with respect to D_n conditionally on the initial state (x_n, y_n).

Theorem 3 establishes that the optimal ordering policy is determined by two threshold variables. More importantly, these two threshold values α_n and β_n can be obtained recursively by solving the implicit equations, Eqs. (29) and (30), respectively.

Remark. The study of Chao et al. (2008) assumes that borrowing is not allowed and thus the firm is firmly limited to order at most y_n units for period n. For this model, it was shown that the optimal policy is determined, in each period, by one-critical value. Our results presented in Theorem 3 contain this study as a special case. This can be seen if we set ℓ_n to be sufficiently large such that a loan is financially prohibited. In this case, α_n becomes zero and β_n can be interpreted as the critical value developed by Chao et al. (2008).

COROLLARY 2. For any period $n < N$ and its initial state (x_n, y_n), the threshold variables of α_n and β_n are only determined by the total worth $\xi_n = x_n + y_n$, i.e., they are of the form: $\alpha_n = \alpha_n(\xi_n)$ and $\beta_n = \beta_n(\xi_n)$. But for the last period N, α_N and β_N are independent of either x_N or y_N.

In view of Corollary 2, one can first compute the α_n and β_n at the beginning of the period based on the total worth ξ_n. The decision on the order quantity q_n can then be made by Eq. (28).

5. Myopic Policies and Threshold Bounds

As shown by Theorem 3, there is a complex computation involved in the calculation of α_n and β_n. In what follows, we study myopic ordering policies that are relatively simple to implement. Such myopic policies optimize a given objective function with respect to any single period and ignore multi-period interactions and cumulative effects. We introduce two types of myopic policies. Specifically, myopic policy (I) assumes the associated cost for the leftover inventory $\hat{s}_n$ is only the holding cost, i.e., $\hat{s}_n = -h_n$. Myopic policy (II) assumes that the leftover inventory cost $\tilde{s}_n$ is not only the holding cost but it also includes its value in the following period, i.e., $\tilde{s}_n = c_{n+1} - h_n$. In the following two subsections, we will show that myopic policy (I) (respectively myopic policy (II)) yields lower bounds, $\hat{\alpha}_n$ and $\hat{\beta}_n$ (respectively upper bounds, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$) for the two threshold values, α_n and β_n.

Before presenting the myopic policies, we present the following lemma that will be applied to derive the upper and lower bounds.

LEMMA 3. For real functions $f(x)$ and $g(x)$,
(a) if both $f(x)$ and $g(x)$ are monotonically increasing or decreasing, then

$$\mathbf{E}[f(X)\cdot g(X)] \geq \mathbf{E}[f(X)]\cdot \mathbf{E}[g(X)],$$

where the expectation is taken with respect to the random variable X.
(b) If $f(x)$ is increasing (decreasing), while $g(x)$ is decreasing (increasing), then

$$\mathbf{E}[f(X)\cdot g(X)] \leq \mathbf{E}[f(X)]\cdot \mathbf{E}[g(X)].$$

5.1. Myopic Policy (I) and Lower Threshold Bounds

Myopic policy (I) is the one period optimal policy obtained when we change the periodic cost structure by assuming that only the holding cost is assessed for any leftover inventory i.e., we assume the following modified "salvage value" cost structure:

$$\hat{s}_n = \begin{cases} -h_n, & n < N, \\ s, & n = N. \end{cases}$$

Let further,

$$\hat{a}_n = \frac{p_n - c_n[1+\ell_n]}{p_n - \hat{s}_n}; \tag{31}$$

$$\hat{b}_n = \frac{p_n - c_n[1+i_n]}{p_n - \hat{s}_n}. \tag{32}$$

and the corresponding critical values are respectively given by

$$\hat{\alpha}_n = F_n^{-1}(\hat{a}_n); \tag{33}$$

$$\hat{\beta}_n = F_n^{-1}(\hat{b}_n). \tag{34}$$

For $n=1,\dots,N$, the order quantity below defines the *myopic policy* (I):

$$\hat{q}_n(x_n,y_n)=\begin{cases}(\hat{\beta}_n-x_n)^+, & x_n+y_n\geq\hat{\beta}_n;\\ y_n, & \hat{\alpha}_n\leq x_n+y_n<\hat{\beta}_n;\\ \hat{\alpha}_n-x_n, & x_n+y_n<\hat{\alpha}_n.\end{cases}$$

The next theorem establishes the lower bound properties of the myopic policy (I).

THEOREM 4. The following are true:

i) For the last period N, $\alpha_N=\hat{\alpha}_N$ and $\beta_N=\hat{\beta}_N$.

ii) For any period $n=1,2,\dots N-1$,

$$\alpha_n\geq\hat{\alpha}_n,$$
$$\beta_n\geq\hat{\beta}_n.$$

5.2. Myopic Policy (II) and Upper Bounds

Myopic policy (II) is the one period optimal policy obtained when we change the periodic cost structure by assuming that not only the holding cost is assessed but also the cost in the next period for any leftover inventory i.e., we assume the following modified "salvage value" cost structure:

$$\tilde{s}_n=\begin{cases}c_{n+1}-h_n, & n<N;\\ s, & n=N.\end{cases}\tag{35}$$

One can interpret the new salvage values $\tilde{s}_n$ of Eq. (35) as representing a fictitious income from *inventory liquidation* (or pre-salvage at full current cost) at the beginning of the next period $n+1$, i.e., it corresponds to the situation that the firm can salvage inventory at the price c_{n+1} at the beginning of the period $n+1$. Note that the condition $c_n(1+\ell_n)+h_n\geq c_{n+1}$ is required if inventory liquidation is allowed. Otherwise, the firm will stock up at an infinite level and sell them off at the beginning of period $n+1$. Such speculation is eliminated by the aforementioned condition.
Let further,

$$\tilde{a}_n=\frac{p_n-c_n[1+\ell_n]}{p_n-\tilde{s}_n},\tag{36}$$
$$\tilde{b}_n=\frac{p_n-c_n[1+i_n]}{p_n-\tilde{s}_n}.\tag{37}$$

and the corresponding critical values which are given by

$$\tilde{\alpha}_n=F_n^{-1}(\tilde{a}_n),\tag{38}$$
$$\tilde{\beta}_n=F_n^{-1}(\tilde{b}_n).\tag{39}$$

For $n=1,\dots,N$, the order quantity below defines the *myopic policy* (II):

$$\tilde{q}_n(x_n,y_n)=\begin{cases}(\tilde{\beta}_n-x_n)^+, & x_n+y_n\geq\tilde{\beta}_n;\\ y_n, & \tilde{\alpha}_n\leq x_n+y_n<\tilde{\beta}_n;\\ \tilde{\alpha}_n-x_n, & x_n+y_n<\tilde{\alpha}_n.\end{cases}$$

Let $V_n^L(x_n, y_n)$ denote the optimal expected future value when the inventory liquidation option is available only at the beginning of period $n+1$ (but not the rest of the periods $n+2, \ldots, N$) given the initial state (x_n, y_n) of period n. For notational simplicity, let $\xi_{n+1} = \xi_{n+1}(x_n, y_n, z_n, D_n) = x_{n+1} + y_{n+1}$ represent the total capital and inventory asset value in period $n+1$ when the firm orders $z_n \geq x_n$ in state (x_n, y_n) and the demand is D_n.
Prior to giving the upper bounds of α_n and β_n, we present the following result.

Proposition 1. For any period n and given its initial state (x_n, y_n), function $V_n(x_n - d, y_n + d)$ is increasing in d where $0 \leq d \leq x$.

In view of Proposition 1, V_n^L can be written as

$$V_n^L(x_n, y_n) = \max_{z_n \geq x_n} \mathbf{E}\left[V_{n+1}(0, \xi_{n+1}) \,|\, x_n, y_n\right]. \tag{40}$$

It is straightforward to show $\mathbf{E}\left[V_{n+1}(0, \xi_{n+1}) \,|\, x_n, y_n\right]$ is concave in z_n. Therefore, V_n^L has an optimal policy determined by a sequence of two threshold values α_n^L and β_n^L.

Proposition 2. The following are true: $\alpha_n^L \geq \alpha_n$ and $\beta_n^L \geq \beta_n$, for all n.

We omit a rigorous mathematical proof (by contradiction) of the above proposition and instead we provide the following intuitively clear explanation that holds for both α_n and β_n. Note that inventory liquidation at period $n+1$ provides the firm with more flexibility i.e., the firm can liquidate the initial inventory x_{n+1} into cash so that the firm holds cash $\xi_{n+1} = x_{n+1} + y_{n+1}$ only. Further, note that the firm will chose to stock up to a higher level of inventory when liquidation is allowed. Indeed, if the firm ordered more in period n, all the leftover inventory after satisfying the demand D_n can be salvaged at full cost c_{n+1} at the beginning of the next period $n+1$. In other words, the firm will take the advantage of inventory liquidation to stock a higher level than that corresponding to the case in which liquidation is not allowed in the current period n. The advantage of doing so is twofold: (1) more demand can be satisfied so more revenue can be generated and (2) there is no extra cost while liquidation of the leftover inventory is allowed.

The next result establishes the upper bound properties of the myopic policy (II).

Proposition 3. For any period $n = 1, 2, \ldots, N-1$, if $c_n(1+\ell_n) + h_n \geq c_{n+1}$, then the critical constants of the optimal policy given in Eqs. (29)-(30) and its myopic optimal policy given in Eqs. (38)-(39) satisfy

$$\tilde{\alpha}_n \geq \alpha_n;$$
$$\tilde{\beta}_n \geq \beta_n.$$

For the last period N, $\alpha_N = \tilde{\alpha}_N$ and $\beta_N = \tilde{\beta}_N$.

5.3. An Algorithm to Compute (α_n, β_n)

With the aid of the lower and the upper bounds presented in §5.1 and §5.2, we develop the following heuristic algorithm for a computational-simplification purpose.

Algorithm: For period n, the threshold α_n can be obtained via

$$\alpha_n = \arg\max\left\{\mathbf{E}\left[V_{n+1}(x_{n+1}, y_{n+1}) \,|\, x_n + y_n\right] \;:\; z_n \in (\hat{\alpha}_n, \tilde{\alpha}_n)\right\}, \tag{41}$$

where x_{n+1} is given by Eq. (26) and y_{n+1} is given below following from Eq. (27),

$$y_{n+1} = p_n' \cdot z_n - (p_n' + h_n') \cdot [z_n - D_n]^+ + c_n' \cdot (x_n + y_n - z_n)(1 + \ell_n).$$

For period n, the threshold β_n can be obtained via

$$\beta_n = \arg\max\left\{\mathbf{E}\left[V_{n+1}(x_{n+1}, y_{n+1}) \,|\, x_n + y_n\right] \;:\; z_n \in (\hat{\beta}_n, \tilde{\beta}_n)\right\}, \tag{42}$$

where x_{n+1} is given by Eq. (26) and y_{n+1} is given below by Eq. (27),

$$y_{n+1} = p_n' \cdot z_n - (p_n' + h_n') \, [z_n - D_n]^+ + c_n' \cdot (x_n + y_n - z_n)(1 + i_n).$$

Note that the calculations involved in Eqs. (41) and (42) are optimizations within bounded spaces and we can employ an efficient search procedure based on Eq. (29) for α_n and Eq. (30) for β_n. Those bounds simplify the computational space and thus expedite the calculation process.

6. Numerical Studies

In this section, we provide some numerical studies for the case of Uniform and Exponential demand distributions. Specifically, Subsection 6.1 considers a single period problem, while Subsection 6.2 treats a two-period problem.

6.1. Single Period Model

As shown in Section 3, one major reason for the two distinct threshold variables, α and β, is the two financial rates, i and l. It is of interest to see how sensitive of the variation between the two threshold values with respect to the difference between i and ℓ. In this section, we experiment the single period model with Uniform demand distribution of $D \sim U(0, 100)$ and Exponential demand distribution of $D \sim Exp(50)$. We set the selling price as $p = 50$; cost $c = 20$; salvage cost per unit $s = 10$. We fix the interest rate as $i = 2\%$ and change the loan rate ℓ from 2% to 50%. It shows that the value of β does not change with respect to ℓ. Specifically, $\beta = 74.00$ for the Uniform demand, while $\beta = 67.35$ for the Exponential demand.

Figure 3 depicts the change of α with respect to ℓ for each demand distribution. For both demand distributions, α is decreasing in ℓ. The threshold values, α and β, of Uniform demand are

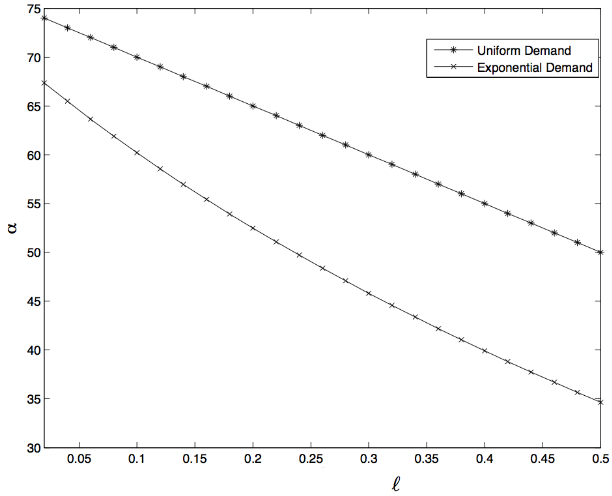


Figure 3 α **of Single Period firm Problem**

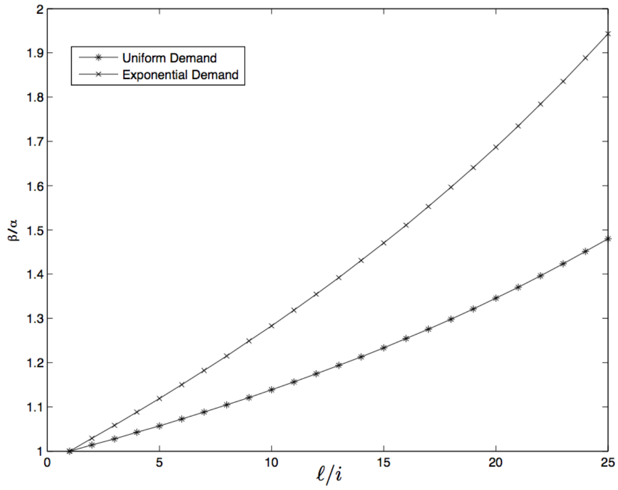


Figure 4 β/α **of Single Period Problem**

larger than those of Exponential demand. This can be explained by the difference between their distributions.

Figure 4 depicts the change of β/α with respect to ℓ. This numerical study shows that the difference between α and β, measured by β/α is not significantly sensitive to the difference between i and ℓ, measured by ℓ/i. Specifically, while $\ell/i = 25$, $\beta/\alpha = 1.48$ for the Uniform demand, and $\beta/\alpha = 1.94$ for the Exponential demand.

6.2. Two-Period Model

In this experiment, we consider a two-period problem and apply the algorithm presented in §5.3 to calculate the optimal solutions for each period. We assume i.i.d. Uninform demand distributions, $D \sim U(0, 200)$, for each period and set the selling price as $p = 50$; cost $c = 35$; salvage cost per unit

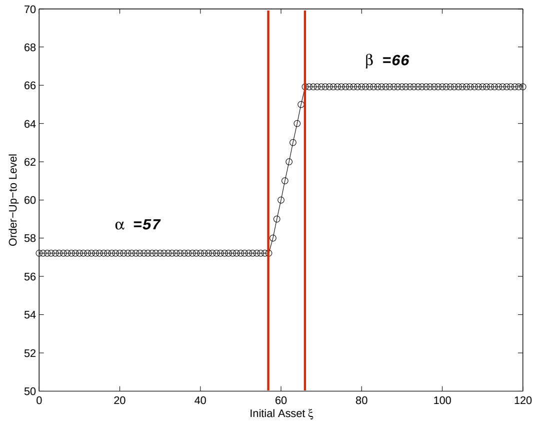


Figure 5 **Optimal Threshold Values in Period 2 versus Initial Asset**

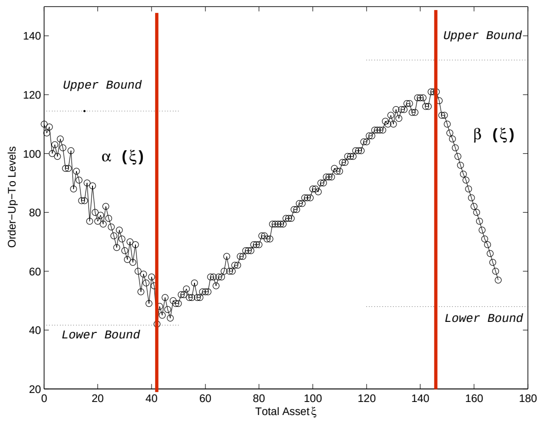


Figure 6 **Optimal Threshold Values in Period 1 versus Initial Total Worth**

$s = 10$ and holding cost $h = 5$. In the first experiment, we fix the interest rate as $i = 5\%$ and the loan rate $\ell = 10\%$, while in the second one, we fix the interest rate as $i = 5\%$ and vary the loan rate ℓ from 5% to 20%.

6.2.1. Optimal threshold values. The first numerical study shows the sensitivities of the optimal order quantity in each period with respect to the total initial asset ξ_n, where $n = 1$ indicates the beginning period while $n = 2$ indicates the ending period. Figure 5 depicts the optimal order-up-to levels (i.e., the two threshold values α and β) for the ending period. In this case, $\alpha = 57.2125$ and $\beta = 65.9188$. Note that the structure of the optimal order quantity obtained in this numerical study repeats Figure 2 presented via analysis.

Figure 6 depicts the threshold variables for the beginning period. Here, the zigzag shape is caused by the rounding calculations to approximate y_n by $[Y_n/c]$ for each period, which explains the

Table 1 **Sensitivity of the Loan Rate**

ℓ	q_1^*	q_2^*	$V_1(0,0)$	$V_2(0,0)$	V_1/V_2
5%	111	66	1097.13	432.30	2.54
10%	110	57	935.08	324.88	2.88
15%	108	49	775.81	232.77	3.33
20%	105	40	624.45	156.00	4.00

non-monotonicity observation of $\alpha(\xi)$ in ξ. For $\alpha(\xi)$, its lower bound of $\hat{\alpha} = 41.61$, while its upper bound is $\tilde{\alpha} = 114.43$. For $\beta(\xi)$, its lower bound is $\hat{\beta} = 47.94$, while its upper bound is $\tilde{\beta} = 131.84$. Furthermore, for initial net worth $\xi \in (42, 145)$, the firm would order ξ with all available capital.

6.2.2. Sensitivity of the optimal solution to the loan rate. In this numerical study, we illustrate the sensitivity of the optimal solution with respect to the loan rate. As shown in Corollary 1, the optimal order policy has positive value even for zero initial inventory and zero initial capital. In this study we stay with the initial states $x_n = 0$ and $y_n = 0$ in each period $n = 1, 2$. It is of our interest to see the value of optimal ordering policy through the time horizon. Table 1 exhibits the optimal values at each period for $\ell \in [5\%, 20\%]$, for $i = 5\%$. First, the optimal order quantity q_2^* in the ending period is relatively sensitive to the loan rate ℓ, and it is decreasing in ℓ; while the optimal order quantity q_1^* in the beginning period is not sensitive to ℓ. This can be explained by the additional variability of the longer horizon associated with q_1^*. Second, the expected total ending wealths, V_1 and V_2 are both decreasing in ℓ. Importantly, the (α_n, β_n)-policy yields a positive value by generating a positive expected ending wealth (e.g., $V_n > 0$) even if the initial asset and capital are both zero. In addition, these calculations also show a positive value of time since V_1 is always larger than V_2. The calculations also demonstrate that the value of time at a small loan rate (keeping everything else the same) is smaller than that at a large loan rate. For example, $V_1/V_2 = 2.54$ for $\ell = 5\%$, while $V_1/V_2 = 4$ for $\ell = 20\%$.

7. Model Extensions

In this section, we study two extensions of the model. First we consider the case of piecewise linear (loan and deposit) interest rate functions and second the case where there is a maximum loan limit.

7.1. Piecewise Type of Loan and Deposit Functions

Up to this section, the loan function was assumed to be a linear function: $L(x) = (1+\ell) \cdot x$ with a constant loan rate ℓ. However, $L(x)$ can have a more complex form in practice. In this section we investigate the often occurring case in which $L(x)$ is a piecewise linear function, i.e., it has the form:

$$L(x) = (1 + \ell^{(m)}) \cdot x, \qquad x \in (x^{(m-1)}, x^{(m)}],$$

where $x^{(m-1)} < x^{(m)}$, $x^{(0)} = 0$ and $\ell^{(m)} < \ell^{(m+1)}$ for $m = 1, 2, 3, ...$

Similarly, we consider the deposit interest function to be a piecewise linear function of the form:

$$M(y) = (1 + i^{(k)}) \cdot y, \qquad y \in (y^{(k-1)}, y^{(k)}],$$

where $y^{(k-1)} < y^{(k)}$, $y^{(0)} = 0$ and $i^{(k)} \le i^{(k+1)}$ for $k = 1, 2, 3, ...$

Without loss of generality, we assume that the loan interest rates are always greater than the deposit interest rates, that is $\bar{i} < \ell^{(1)}$ where $\bar{i} = \sup_k \{i^{(k)}\}$.
Prior to characterizing the optimal ordering policy, we next introduce the critical values of $\alpha^{(m)}$ and $\beta^{(k)}$ for $m, k = 1, 2, 3, ...$ as follows:

$$\alpha^{(m)} = F^{-1}(a^{(m)}); \tag{43}$$

$$\beta^{(k)} = F^{-1}(b^{(k)}), \tag{44}$$

where

$$a^{(m)} = \frac{p - c \cdot (1 + \ell^{(m)})}{p - s};$$

$$b^{(k)} = \frac{p - c \cdot (1 + i^{(k)})}{p - s}.$$

It is straightforward to see that $\beta^{(1)} \ge \cdots \beta^{(k)} \ge \beta^{(k+1)} \cdots \ge \bar{\beta} > \alpha^{(1)} \ge \cdots \alpha^{(m)} \ge \alpha^{(m+1)} \cdots \ge 0$ where $\bar{\beta} = F^{-}(\frac{p-c\cdot(1+\bar{i})}{p-s})$.

We present the structure of the optimal ordering policy in Theorem 5 below. The proof follows by a straightforward modification of the proof of Theorem 1 and thus it is omitted for simplicity.

THEOREM 5. For any given initial inventory-capital state (x, y), the optimal order quantity is

$$q^*(x, y) = \begin{cases} \beta^{(k)} - x, & \beta^{(k+1)} \le x + y \le \beta^{(k)}; \\ \cdots, & \cdots \\ y, & \alpha^{(1)} \le x + y < \bar{\beta}; \\ \cdots, & \cdots \\ \alpha^{(m)} - x, & \alpha^{(m+1)} \le x + y < \alpha^{(m)}, \end{cases}$$

where $\{\alpha^{(m)}\}$ and $\{\beta^{(k)}\}$ are given by Eq. (43) and (44), respectively.

For the multi-period problem, the optimal order quantity for each period shares a similar structure to that of single period problem, given by Theorem 5. However, the threshold values pertaining to each period $n < N$ depend on the period's initial state values x_n and y_n.

7.2. Financing under a Maximum Loan Limit Constraint

In practice, the outstanding loan amount is often restricted to be less than or equal to a maximum limit. Let $L_n > 0$ denote the maximum loan limit for period n. In this case, we have the following structural results for the optimal ordering policy. The proof follows by straightforward modifications of the ones for Theorem 1 and Theorem 3 and thus it is omitted for simplicity.

THEOREM 6. (**The optimal ordering policy under a maximum loan limit**).
For period $n = 1, 2, \cdots, N$ with given state (x_n, y_n) at the beginning of the period, if there is a loan limit L_n, then there exist positive constants $\alpha_n^L = \alpha_n^L(x_n, y_n)$ and $\beta_n^L = \beta_n^L(x_n, y_n)$ with $\alpha_n^L \leq \beta_n^L$, which define the optimal order quantity as follows:

$$q^*(x_n, y_n) = \begin{cases} (\beta_n^L - x_n)^+, & x_n + y_n \geq \beta_n^L; \\ y_n, & \alpha_n^L \leq x_n + y_n < \beta_n^L; \\ \alpha_n^L - x_n, & \alpha_n^L - L'_n \leq x_n + y_n < \alpha_n^L; \\ L'_n, & x_n + y_n < \alpha_n^L - L'_n. \end{cases}$$

where $L'_n = L_n / c_n$.

Note that, for period n, it is allowed that $\alpha_n^L < L'_n$. In this case, the maximum loan limit is not binding since the optimal order quantity $q_n^* \leq \alpha_n^L < L'_n$.

8. Conclusions

In this paper, we studied the optimal inventory-finance policy for a single-item inventory system within an environment that allows interest earning for deposit and capital loans. We showed that the optimal ordering policy, called (α_n, β_n)-policy, for each period is characterized by two threshold variables. In addition, we provided two myopic policies which give a lower bound and a upper bound of the threshold variables. With the two bounds, we developed an algorithm to compute the two threshold values α_n and β_n.

There are various possible directions of research to follow up with our current study, some of them are: a) To include a fixed ordering cost and/or a fixed financial transaction cost. b) Our study assumes risk neutral decision making. It is an interesting direction to analyze the risk performance pertaining to the system, e.g., bankruptcy probabilities [cf. Babich et al. (2012), Gong et al. (2012) and Li et al. (2013)]. In addition, it is not difficult to see many other related topics and generalizations of this work for future research.

9. Appendix

Proof of Lemma 2. We prove the result by induction. In particular, in each iteration, we will prove properties (1) and (2) by recursively repeating two steps: deducing the property of G_n from the property of V_{n+1} and obtaining the property of V_n from the property of G_n. Throughout the proof, for a matrix or a vector w, we denote its transpose by w^T. The *Hessian Matrix* (if it exists) of a function $G = G(x, y)$ will be denoted by $\mathbf{H}^G(x, y)$. For example, the Hessian Matrix of $V_n(x_n, y_n)$ is denoted by

$$\mathbf{H}^{V_n}(x_n, y_n) = \begin{bmatrix} \frac{\partial^2 V_n}{\partial x_n \partial x_n} & \frac{\partial^2 V_n}{\partial x_n \partial y_n} \\ \frac{\partial^2 V_n}{\partial y_n \partial x_n} & \frac{\partial^2 V_n}{\partial y_n \partial y_n} \end{bmatrix}. \quad (45)$$

1. For V_N, we have a one period problem. In this case, the result for function $G_N(z_N, x_N, y_N)$ is obtained by Lemma 1 with $z_n = x_n + q_n$ and the result for $V_N(x_N, y_N)$ is given by Theorem 2 of the single period problem.

2. For $n = 1, 2, \cdots, N-1$, we prove the results recursively using the following two steps:

Step 1. We show that $G_n(z_n, x_n, y_n)$ is increasing in y_n and concave in z_n and (x_n, y_n) if $V_{n+1}(x_{n+1}, y_{n+1})$ is increasing in y_{n+1} and concave (x_{n+1}, y_{n+1}).

We first compute the partial derivatives that will be used in the sequel for any given z_n. From Eq. (26) we have:

$$\frac{\partial x_{n+1}}{\partial z_n} = \frac{\partial x_{n+1}}{\partial x_n} = \mathbf{1}_{\{z_n > D_n\}}, \tag{46}$$

$$\frac{\partial x_{n+1}}{\partial y_n} = 0. \tag{47}$$

Similarly, from Eq. (27) we obtain:

$$\frac{\partial y_{n+1}}{\partial z_n} = p_n' \mathbf{1}_{\{z_n < D_n\}} - h_n' \mathbf{1}_{\{z_n > D_n\}} - c_n' \left[(1 + i_n) \mathbf{1}_{\{z_n < x_n + y_n\}} + (1 + \ell_n) \mathbf{1}_{\{z_n > x_n + y_n\}} \right], \tag{48}$$

and

$$\frac{\partial y_{n+1}}{\partial x_n} = p_n' \mathbf{1}_{\{z_n < D_n\}} - h_n' \mathbf{1}_{\{z_n > D_n\}}, \tag{49}$$

$$\frac{\partial y_{n+1}}{\partial y_n} = c_n' \left[(1 + i_n) \mathbf{1}_{\{z_n < x_n + y_n\}} + (1 + \ell_n) \mathbf{1}_{\{z_n > x_n + y_n\}} \right]. \tag{50}$$

From Eqs. (46)-(50), it readily follows that the second order derivatives of x_{n+1} and y_{n+1} with respect to z_n, x_n and y_n are all zero.

In the sequel we interchange differentiation and integration in several places, this is justified by the *Lebesgue's Dominated Convergence* Theorem [cf. Bartle (1995)].

The increasing property of function $G_n(z_n, x_n, y_n)$ in y_n can be established by taking the first order derivative of Eq. (5) with respect to y_n. Then,

$$\begin{aligned}\frac{\partial}{\partial y_n} G_n(z_n, x_n, y_n) &= \mathbf{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial y_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n} \right] \\ &= \mathbf{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n} \right] \geq 0,\end{aligned}$$

where the second equality holds since $\partial x_{n+1} / \partial y_n = 0$, by Eq. (47), and the inequality holds by Eq. (50) and the induction hypothesis that V_{n+1} is increasing in y_{n+1}.

To prove the concavity of $G_n(z_n, x_n, y_n)$ in z_n, we next show that $\partial^2 G_n(z_n, x_n, y_n) / \partial z_n^2 \leq 0$. To this end we compute the first and second order derivatives as follows:

$$\frac{\partial}{\partial z_n} G_n(z_n, x_n, y_n) = \mathbf{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial z_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial z_n} \right] \tag{51}$$

and

$$\frac{\partial^2}{\partial z_n^2} G_n(z_n, x_n, y_n) = \mathbf{E}\left[\left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right] \cdot \mathbf{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right]^T\right], \tag{52}$$

where

$$\mathbf{H}^{V_{n+1}} = \begin{bmatrix} \frac{\partial^2 V_{n+1}}{\partial x_{n+1} \partial x_{n+1}} & \frac{\partial^2 V_{n+1}}{\partial x_{n+1} \partial y_{n+1}} \\ \frac{\partial^2 V_n}{\partial y_{n+1} \partial x_{n+1}} & \frac{\partial^2 V_{n+1}}{\partial y_{n+1} \partial y_{n+1}} \end{bmatrix}$$

is the Hessian matrix of $V_{n+1}(x_{n+1}, y_{n+1})$. Now the induction hypothesis regarding V_{n+1}, implies that $\mathbf{H}^{V_{n+1}}$ is negative semi-definite, i.e., $w\mathbf{H}^{V_{n+1}}w^T \le 0$ for any 1 by 2 vector w, thus the result follows using Eq. (52).

To prove the concavity of $G_n(z_n, x_n, y_n)$ in (x_n, y_n), we compute its Hessian matrix and show that it is negative semi-definite. To this end we compute the first and second order partial derivatives of V_{n+1} with respect to x_n and y_n for any given z_n, as follows:

$$\frac{\partial G_n}{\partial x_n} = \mathbf{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial x_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial x_n}\right], \tag{53}$$

$$\frac{\partial G_n}{\partial y_n} = \mathbf{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial y_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n}\right], \tag{54}$$

and

$$\frac{\partial^2 G_n}{\partial x_n^2} = \mathbf{E}\left[J_{n+1}^{(1)}\right], \tag{55}$$

$$\frac{\partial^2 G_n}{\partial x_n \partial y_n} = \mathbf{E}\left[J_{n+1}^{(2)}\right], \tag{56}$$

$$\frac{\partial^2 G_n}{\partial y_n^2} = \mathbf{E}\left[J_{n+1}^{(3)}\right], \tag{57}$$

where, by Eqs. (46) - (50), the terms involved with the second order derivatives of x_{n+1} and y_{n+1} with respect to x_n and y_n have vanished and where for notational convenience we have defined:

$$J_{n+1}^{(1)} = \left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right] \cdot \mathbf{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right]^T,$$

$$J_{n+1}^{(2)} = \left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right] \cdot \mathbf{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]^T,$$

$$J_{n+1}^{(3)} = \left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right] \cdot \mathbf{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]^T.$$

Thus, the Hessian matrix of G_n in terms of (x_n, y_n) is:

$$\mathbf{H}^{G_n}(x_n, y_n) = \begin{bmatrix} \frac{\partial^2 G_n}{\partial x_n \partial x_n} & \frac{\partial^2 G_n}{\partial x_n \partial y_n} \\ \frac{\partial^2 G_n}{\partial y_n \partial x_n} & \frac{\partial^2 G_n}{\partial y_n \partial y_n} \end{bmatrix}, \tag{58}$$

with its elements given by Eqs. (55) - (57). To prove it is negative semi-definite, we consider the quadratic function below for any real z and t,

$$\begin{aligned}[z,t]\cdot\mathbf{H}^{G_n}\cdot[z,t]^T &= \frac{\partial^2 G_n}{\partial x_n \partial x_n} z^2 + 2\frac{\partial^2 G_n}{\partial x_n \partial y_n}\cdot z\cdot t + \frac{\partial^2 G_n}{\partial y_n \partial y_n}\cdot t^2 \\ &= \mathbf{E}\left[J_{n+1}^{(1)} z^2 + 2J_{n+1}^{(2)} zt + J_{n+1}^{(3)} t^2\right]. \end{aligned} \tag{59}$$

If we define the 1×2 vector $w = w(n,z,t)$ as follows:

$$w = z\cdot\left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right] + t\cdot\left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right], \tag{60}$$

then Eq. (59) can be further written as

$$[z,t]\cdot\mathbf{H}^{G_n}\cdot[z,t]^T = \mathbf{E}\left[w\cdot\mathbf{H}^{V_{n+1}}\cdot w^T\right]. \tag{61}$$

Since by the induction hypothesis $\mathbf{H}^{V_{n+1}}$ is negative semi-definite, we have

$$w\cdot\mathbf{H}^{V_{n+1}}\cdot w^T \leq 0$$

and this implies that the right side of Eq. (61) is non-positive. Thus, the proof for Step 1 is complete.

Step 2. We show that $V_n(x_n,y_n)$ is concave in (x_n,y_n) if $G_n(z_n,x_n,y_n)$ is concave in z_n and (x_n,y_n).

Since $G_n(z_n,x_n,y_n)$ is concave in z_n and (x_n,y_n), then $V_n(x_n,y_n) = \max_{z_n\geq x_n} G_n(z_n,x_n,y_n)$ is concave in x_n, y_n by the fact that concavity is reserved under maximization [cf. Proposition A.3.10 in Zipkin (2000), p436].

Thus the induction proof is complete. □

Proof of Theorem 3. Given state (x_n,y_n) at the beginning of period $n=1,2,\cdots,N$, we consider the equation:

$$\frac{\partial}{\partial z_n} G_n(z_n,x_n,y_n) = 0, \tag{62}$$

where $\partial G_n(z_n,x_n,y_n)/\partial z_n$ is given by Eq. (51). Substituting Eqs. (46) and (48) into Eq. (51) we consider the following cases:

(1) for $z_n \leq x_n + y_n$,

$$\frac{\partial G_n}{\partial z_n} = \mathbf{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\mathbf{1}_{\{z_n>D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{z_n<D_n\}} - h_n'\mathbf{1}_{\{z_n>D_n\}} - c_n'(1+i_n)\right)\Big|x_n,y_n\right]; \tag{63}$$

(2) for $z_n > x_n + y_n$,

$$\frac{\partial G_n}{\partial z_n} = \mathbf{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\mathbf{1}_{\{z_n>D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{z_n<D_n\}} - h_n'\mathbf{1}_{\{z_n>D_n\}} - c_n'(1+\ell_n)\right)\Big|x_n,y_n\right], \tag{64}$$

where for each case above, random variables x_{n+1} and y_{n+1} within the expectations are given by Eqs. (26) and (27), respectively.

The results follow readily by setting the right sides of Eqs. (63) and (64) equal to zero and simple simplifications. Note that $\partial G_n(z_n, x_n, y_n)/\partial z_n$ is monotonically decreasing in z_n due to its concavity shown in part (1) of Lemma 2, therefore, there are unique solutions to each of these equations. □

Proof Corollary 2. For period N, the independence of x_N or y_N is obvious since this is a single period. For period $n < N$, let us revisit Eqs. (63) and (64). Note that x_{n+1} is independent of (x_n, y_n) by Eq. (26) while y_{n+1} is dependent of $x_n + y_n$ by Eq. (27). Therefore, α_n and β_n implicitly given by Eqs. (63) and (64) are dependent of $\xi_n = x_n + y_n$ only, and thus completes the proof. □

Proof of Lemma 3. To prove (a), we only give the proof for the case that $f(x)$ and $g(x)$ are increasing. The same argument can be applied for the case of decreasing $f(x)$ and $g(x)$.
Let X' be another random variable which is i.i.d. of X. Since $f(x)$ and $g(x)$ are increasing, we always have

$$[f(X) - f(X')][g(X) - g(X')] \geq 0.$$

Taking expectations with respect to X and X' yields

$$\begin{aligned}
&\mathbf{E}\big[[f(X) - f(X')][g(X) - g(X')]\big] \\
&= \mathbf{E}[f(X)g(X) + f(X')g(X') - f(X')g(X) - f(X)g(X')] \\
&= \mathbf{E}[f(X)g(X)] + \mathbf{E}[f(X')g(X')] - \mathbf{E}[f(X')]\mathbf{E}[g(X)] - \mathbf{E}[f(X)]\mathbf{E}[g(X')] \\
&= 2\mathbf{E}[f(X)g(X)] - 2\mathbf{E}[f(X)]\mathbf{E}[g(X)] \geq 0.
\end{aligned}$$

The result of part (a) readily follows from the above.
In a similar vein, we can prove part (b) via changing the direction of the inequality above. □

Proof of Theorem 4. We only prove the result for α_n. The same argument can be applied to prove the result for β_n.
In view of Eq. (63), α_n is uniquely given as the solution to the equation below,

$$\mathbf{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{\alpha_n < D_n\}} - h_n'\mathbf{1}_{\{\alpha_n > D_n\}} - c_n'(1+\ell_n)\right)\right] = -\mathbf{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\mathbf{1}_{\{\alpha_n > D_n\}}\right]. \tag{65}$$

Since $\frac{\partial V_{n+1}}{\partial x_{n+1}} \geq 0$ by Lemma 2 part (2), the equation above is negative, which implies

$$\mathbf{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{\alpha_n < D_n\}} - h_n'\mathbf{1}_{\{\alpha_n > D_n\}} - c_n'(1+\ell_n)\right)\right] \leq 0. \tag{66}$$

Further note that for any realization of demand $D_n = d > 0$, the two terms of the left hand side of Eq. (66):

$$\frac{\partial V_{n+1}(x_{n+1}(d), y_{n+1}(d))}{\partial y_{n+1}(d)}$$

and

$$p_n' \mathbf{1}_{\{\alpha_n < d\}} - h_n' \mathbf{1}_{\{\alpha_n > d\}} - c_n'(1+\ell_n)$$

are both increasing in d. Specifically, the first term is increasing by the concavity of V_{n+1} [cf. Lemma 2, part (2)] and Eq. (27). Then, by Lemma 3 and Eq. (66), one has,

$$\mathbf{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\right] \mathbf{E}\left[p_n' \mathbf{1}_{\{\alpha_n < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n > D_n\}} - c_n'(1+\ell_n)\right] \le 0 \tag{67}$$

Since $\frac{\partial V_{n+1}}{\partial y_{n+1}} \ge 0$ by Lemma 2 part (2), the above inequality implies

$$\mathbf{E}\left[p_n \mathbf{1}_{\{\alpha_n < D_n\}} - h_n \mathbf{1}_{\{\alpha_n > D_n\}} - c_n(1+\ell_n)\right] \le 0,$$

which, after simple algebra, is equivalent to

$$p_n - c_n \cdot (1+\ell_n) - (p_n - s_n) F_n(\alpha_n) \le 0.$$

The above further simplifies to

$$F(\alpha_n) \ge \frac{p_n - c_n \cdot (1+\ell_n)}{p_n - s_n}.$$

By Eqs. (31) and (33), the right hand side in the above inequality is $F_n(\hat{\alpha}_n)$. Thus, we have $F_n(\alpha_n) \ge F_n(\hat{\alpha}_n)$, which completes the proof for $\alpha_n \ge \hat{\alpha}_n$ by the increasing property of $F_n(\cdot)$. □

Proof of Proposition 1. It is sufficient to prove that for an arbitrarily small value of $d > 0$, $V_n(x_n, y_n) \le V_n(x_n - d, y_n + d)$. To this end, consider the initial state to be $(x_n - d, y_n + d)$. In this case, the firm can always purchase d units without any additional cost to reset the initial state to be (x_n, y_n). This means $V_n(x_n, y_n) \le V_n(x_n - d, y_n + d)$, and thus completes the proof. □

Proof of Proposition 3. For period N, the result readily follows from the optimal solution of single period model. We only prove for $\tilde{\alpha}_n \ge \alpha_n$ as a similar argument (with replacing ℓ_n with i_n) can be applied to prove $\tilde{\beta}_n \ge \beta_n$.

By Proposition 2, we have $\alpha_n \le \alpha_n^L$ and α_n^L is determined by taking derivative of Eq. (40) and setting it equal to zero, that is

$$\mathbf{E}\left[\frac{\partial V_{n+1}(0, \xi_{n+1})}{\partial \xi_{n+1}} \left(\mathbf{1}_{\{\alpha_n^L > D_n\}} + p_n' \mathbf{1}_{\{\alpha_n^L < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n^L > D_n\}} - c_n'(1+\ell_n)\right)\right] = 0. \tag{68}$$

For any realization of the demand $D_n = d > 0$ the term

$$\frac{\partial V_{n+1}(0, \xi_{n+1}(d))}{\partial \xi_{n+1}(d)}$$

is decreasing in d by the concavity of V_{n+1} [cf. Lemma 2 part (2)] and the fact that ξ_{n+1} is increasing in d by Eqs. (26)-(27).

In addition the term

$$\begin{aligned}&\mathbf{1}_{\{\alpha_n > d\}} + p_n' \mathbf{1}_{\{\alpha_n < d\}} - h_n' \mathbf{1}_{\{\alpha_n > d\}} - c_n'(1+\ell_n)\\ &= p_n' - (p_n' + h_n' - 1)\mathbf{1}_{\{\alpha_n > d\}} - c_n'(1+\ell_n),\end{aligned}$$

is increasing in d.

By Eq. (68) and Lemma 3, one has

$$\mathbf{E}\left[\frac{\partial V_{n+1}(0,\xi_{n+1})}{\partial \xi_{n+1}}\right] \cdot \mathbf{E}\left[\mathbf{1}_{\{\alpha_n^L > D_n\}} + p_n' \mathbf{1}_{\{\alpha_n^L < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n^L > D_n\}} - c_n'(1+\ell_n)\right] \geq 0.$$

Since $\partial V_{n+1}(0,\xi_{n+1})/\partial \xi_{n+1} \geq 0$ by Lemma 2 part (2), the above inequality implies

$$\mathbf{E}\left[\mathbf{1}_{\{\alpha_n^L > D_n\}} + p_n' \mathbf{1}_{\{\alpha_n^L < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n^L > D_n\}} - c_n'(1+\ell_n)\right] \geq 0, \tag{69}$$

which, after simple algebra, is equivalent to

$$p_n - c_n \cdot (1+\ell_n) - (p_n + h_n - c_{n+1}) F_n(\alpha_n^L) \geq 0.$$

The above further simplifies to

$$F(\alpha_n^L) \leq \frac{p_n - c_n \cdot (1+\ell_n)}{p_n + h_n - c_{n+1}}.$$

Note that the right hand side of the above is less than 1 since $c_n(1+\ell_n) + h_n \geq c_{n+1}$ by assumption. Next, by Eqs. (36) and (38), the right hand side in the above inequality is $F_n(\hat{\alpha}_n)$. Thus, we have $F_n(\tilde{\alpha}_n) \geq F_n(\alpha_n^L)$, which means $\tilde{\alpha}_n \geq \alpha_n^L$. Thus, the proof for $\tilde{\alpha}_n \geq \alpha_n$ is complete, since $\alpha_n^L \geq \alpha_n$ by Proposition 2. □

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