Instantaneous Velocity

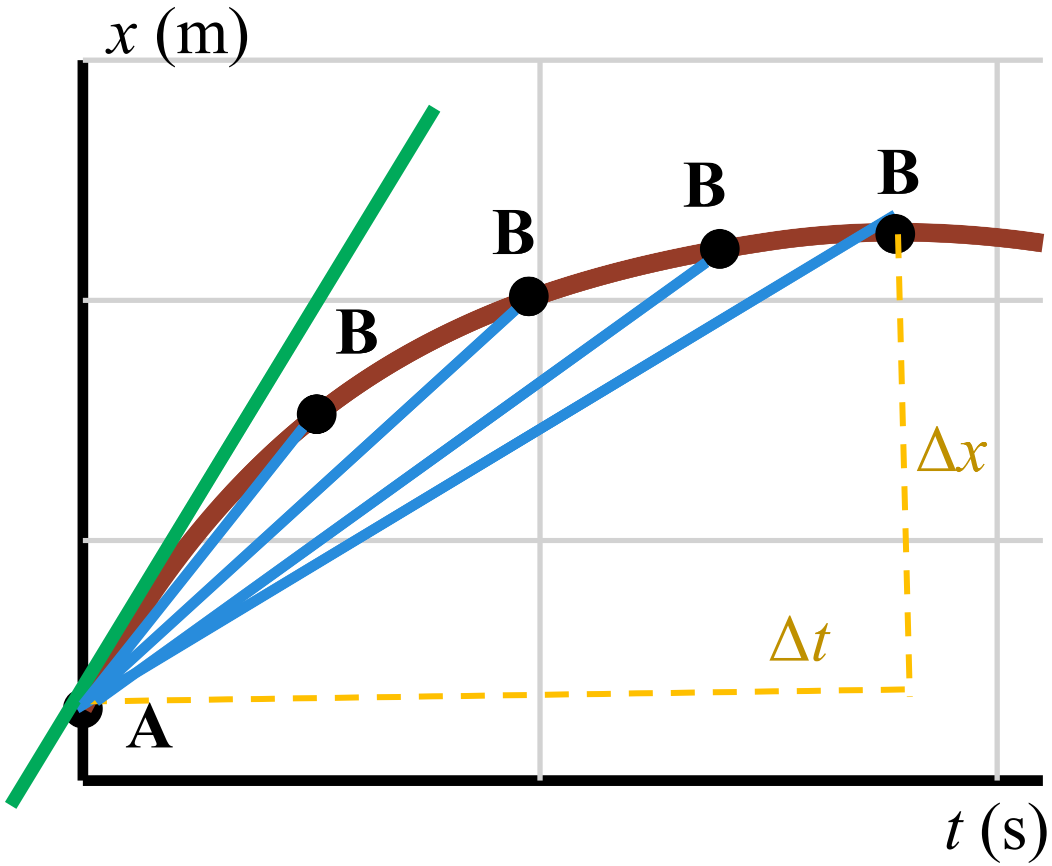
- Instantaneous means "at some given instant". The **instantaneous velocity** indicates what is happening at every point of time: $\boldsymbol{v_x}(t)$

- Connection to average velocity and $x(t)$:

$$\boldsymbol{v_x} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

- Understand instantaneous velocity graphically:

Chords approach the tangent as $\Delta t \to 0$
Slope measure rate of change of position

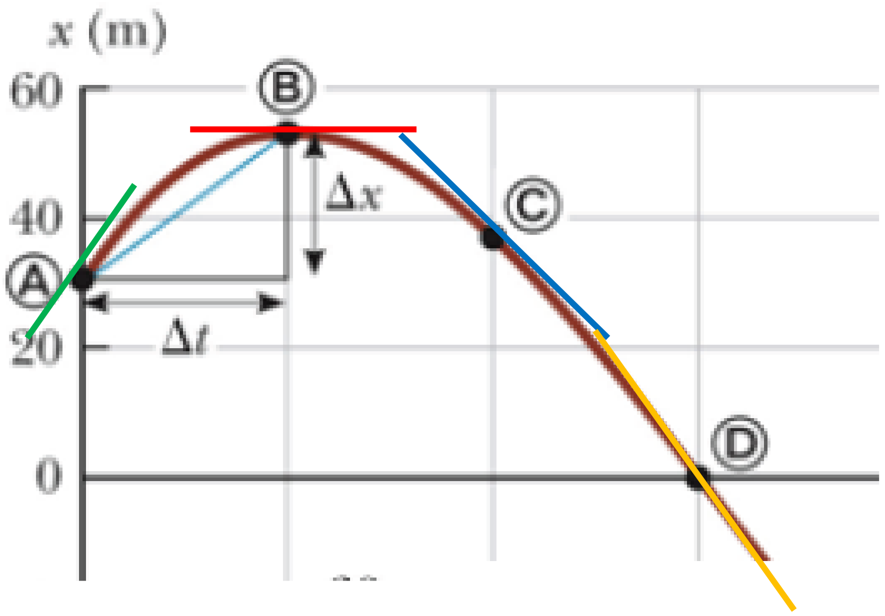


Green line: instantaneous velocity at A.

Instantaneous Velocity

- Instantaneous velocity: $\boldsymbol{v_x} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

- It is a vector quantity.
- SI unit: m/s.
- Instantaneous velocity $\boldsymbol{v_x}(t)$ is a function of time.
- It is the ***slope* of the tangent line** $x(t)$.



At what time(s) the instantaneous velocity $\boldsymbol{v_x}(t)$ is zero?

(A), (B), (C), (D)

Example

A car travels in a straight line along a road. The car's distance x from a stop sign is given as a function of time t by the equation

$$x = (1.5 \text{ m/s}^2)t^2 - (0.05 \text{ m/s}^3)t^3$$

Calculate the average velocity of the car for time interval $t = 2.00$ s to $t = 4.00$ s.

Q1: What is the definition of average velocity? $v_{\text{av-}x} = \dfrac{\Delta x}{\Delta t} = \dfrac{20.8 \text{ m} - 5.60 \text{ m}}{2.00 \text{ s}} = +7.60 \text{ m/s}$

Q2: What is Δt? $\Delta t = 4 - 2 = 2$ s

Q3: What is Δx? $\Delta x = x(t = 4 \text{ s}) - x(t = 2 \text{ s}) = 20.8 \text{ m} - 5.6 \text{ m}$

$x(4.00 \text{ s}) = 20.8 \text{ m}$ $x(2.00 \text{ s}) = 5.60 \text{ m},$

Example

A car travels in a straight line along a road. The car's distance x from a stop sign is given as a function of time t by the equation

$$x = (1.5\ \text{m/s}^2)t^2 - (0.05\ \text{m/s}^3)t^3$$

Calculate the instantaneous velocity at $t = 2.00$ s?

Q1: What is the definition of instantaneous velocity?

$$\boldsymbol{v_x} = \frac{dx}{dt} = 2 \times 1.5t - 3 \times 0.05t^2 = 3t - 0.15t^2$$

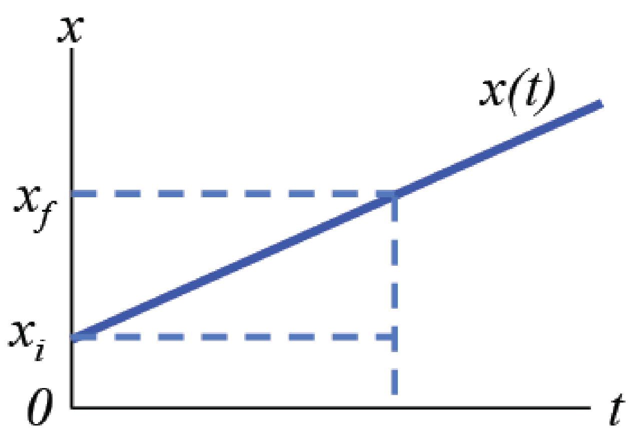
$$\boldsymbol{v_x}(t = 2) = 3 \times 2 - 0.15 \times 2^2 = 5.4$$

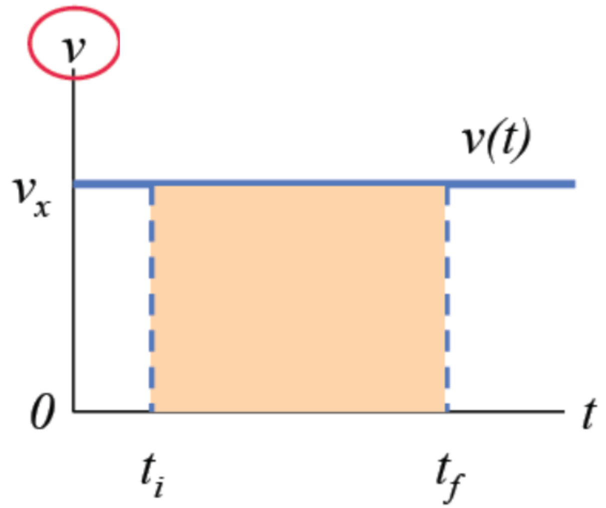
Uniform Velocity

- A special case of velocity is "uniform velocity" or constant velocity.
- What is instantaneous velocity, in this case?

$$\boldsymbol{v_x} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \boldsymbol{v_x}(t) = C$$

- Instantaneous velocities are always the same, all the instantaneous velocities will also equal the average velocity.
- **Q1:** The *x*-*t* graph?
- **Q2:** The *v*-*t* graph?
- **Q3:** What does the area under the graph and above the *t*-axis represent?

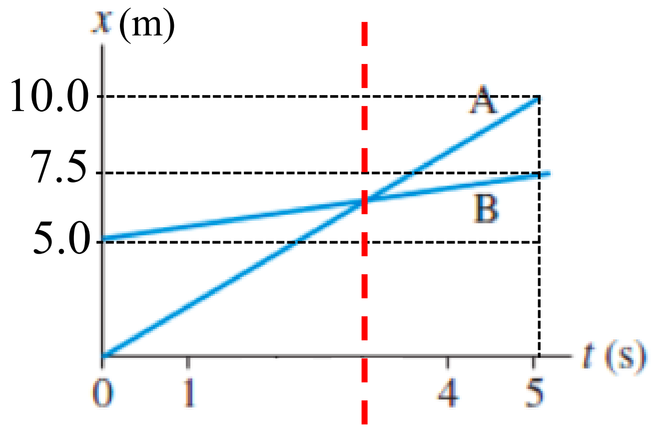




$$\boldsymbol{v_x}\Delta t = \Delta x$$

Example: Constant velocity

The following figure shows the position-time graph for the motion of objects "A" and "B" as they move along the same axis. When "A" catch up with "B", how many meters has "B" traveled?



Q1: What type of motion does A/B have?

Linear *x*-*t* graph, both have **constant velocity**.

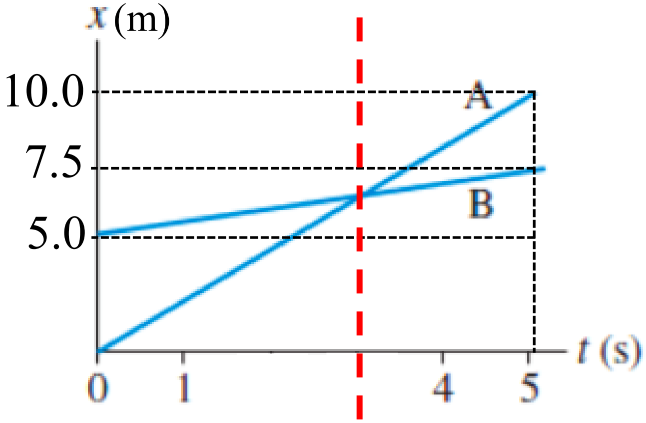
Q2: Which one is faster? A or B?

A, because large slope. "A" starts at the origin, "B" is ahead of A by 5.0 m.

Q3: What is the meaning of "catch up"? Same *x* position? Same time *t*? Same velocity?

At a specific time *t*, A and B has the same *x* position. That is $x_A(t) = x_B(t)$.

Example: Constant velocity



The following figure shows the position-time graph for the motion of objects "A" and "B" as they move along the same axis. When "A" catch up with "B", how many meters has "B" traveled?

Q4: What is the $x(t)$ function for A? $x_A = (2.0 \text{ m/s})t$

Q5: What is the $x(t)$ function for B? $x_B = 5.0 \text{ m} + (0.5 \text{ m/s})t$

$x_A(t) = x_B(t)$ $(2.0 \text{ m/s})t = 5.0 \text{ m} + (0.5 \text{ m/s})t$ $t = 3.33 \text{ s}$

Q6: Which quantity is "how many meters has "B" traveled?"

Displacement. $\Delta x_B = x_B(t = 3.33 \text{ s}) - x_B(t = 0 \text{ s}) = 6.66 \text{ m} - 5.0 \text{ m} = 1.66 \text{ m}$

Acceleration

- Changing position means a velocity is present.
- How about changing velocity (non-uniform)?
- Means an ***acceleration*** is present.
- Acceleration is the rate of change of velocity.
- Vector or scalar?
- Acceleration is a vector quantity.
- Acceleration has both magnitude and direction.
- Acceleration has a dimensions of length/time^2: [m/s^2].

Average/Instantaneous Acceleration

- Similar to the definition of average velocity, we can define average acceleration.

Average *x*-acceleration of a particle in **straight-line motion** during time interval from t_1 to t_2

Change in *x*-component of the particle's velocity

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1}$$

Final *x*-velocity minus initial *x*-velocity

Time interval

Final time minus initial time

- Similar to the definition of instantaneous velocity, we can also define instantaneous acceleration:

The **instantaneous *x*-acceleration** of a particle in **straight-line motion** ...

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

... equals the limit of the particle's average *x*-acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the particle's *x*-velocity.

Example: Average Acceleration of the Tesla

Tesla can accelerate from 0 mph to 60 mph in 2.5 s, becoming the third-fastest-acceleration production car in the world. If so, what is the average acceleration in m/s²?

$$a_x = \frac{\Delta v_x}{\Delta t}$$

$$a_{avg} = \Delta v / \Delta t = (v_f - v_i)/(t_f - t_i) = 60 mph / 2.5 s \quad = 60 \times 0.45 m/s / 2.5 s = 10.8 m/s^2$$

Example: Instantaneous Acceleration

A car travels in a straight line along a road. The car's velocity v_x is given as a function of time t by the equation $v_x = (1.5\ \text{m/s}^3)t^2$. What is the instantaneous acceleration of the car when $t = 4$ s?

Q1: What is definition of instantaneous acceleration?

$$a_x = \frac{dv_x}{dt} = 2 \times 1.5t = 3.0t$$

$$t = 4.0\ \text{s}$$

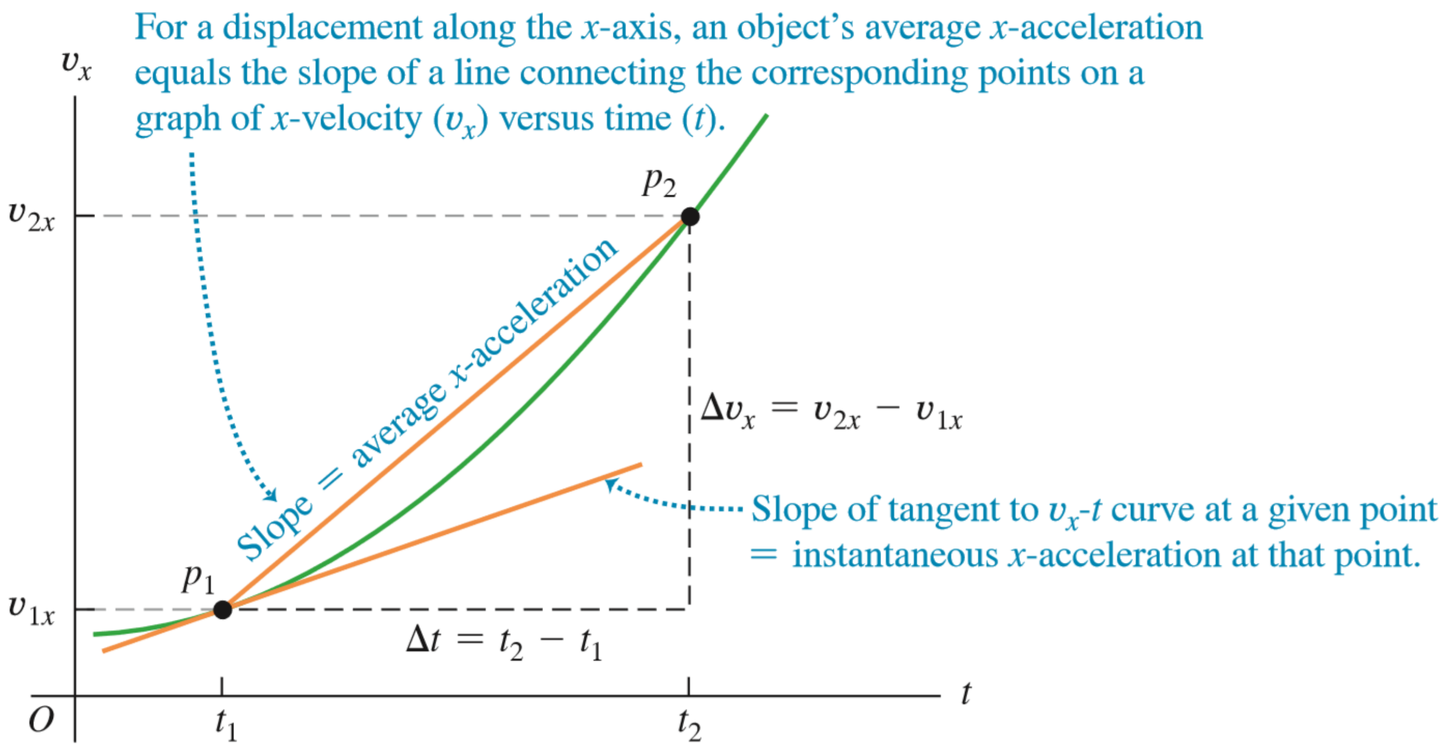
$$a_x = 3.0 \times (4.0\ \text{s}) = 12\ \text{m/s}^2$$

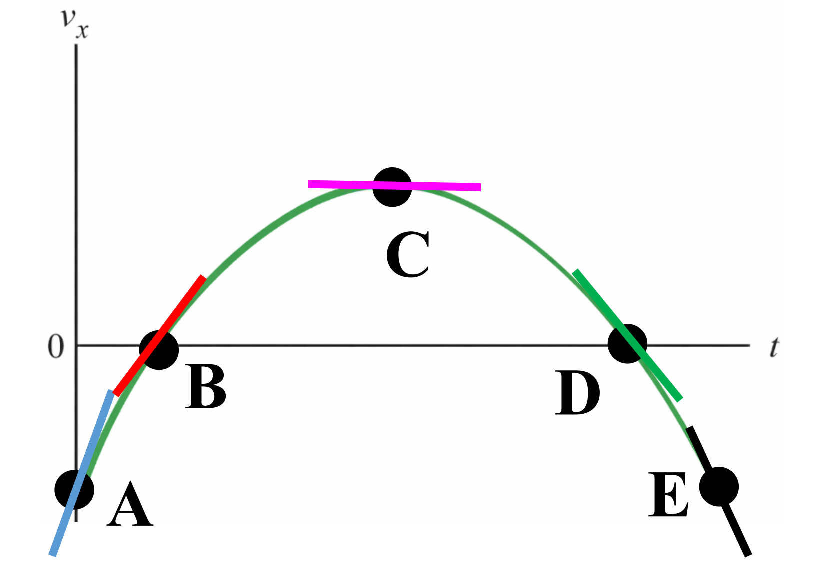
Finding Acceleration on a v_x-t Graph

- Understand the acceleration graphically.

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

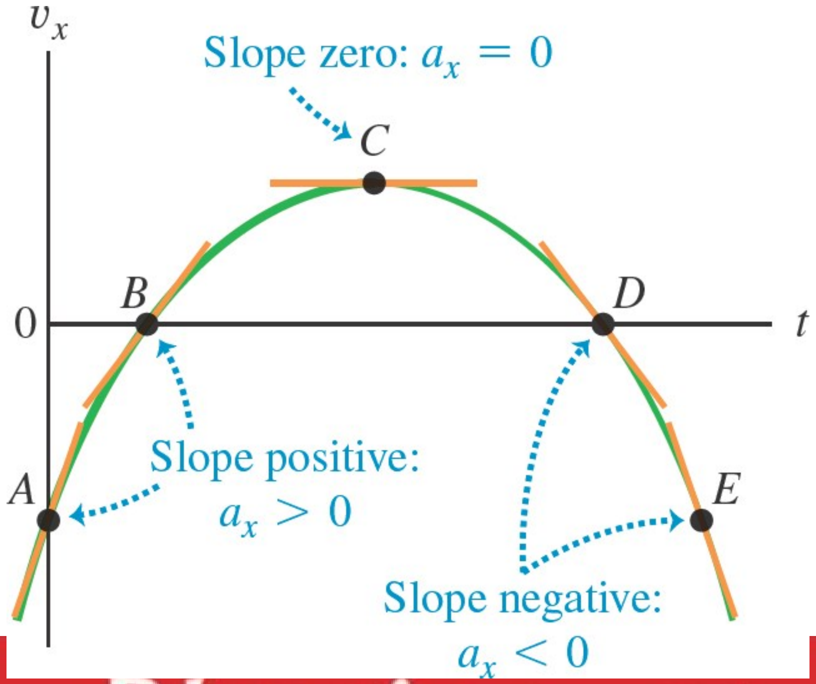


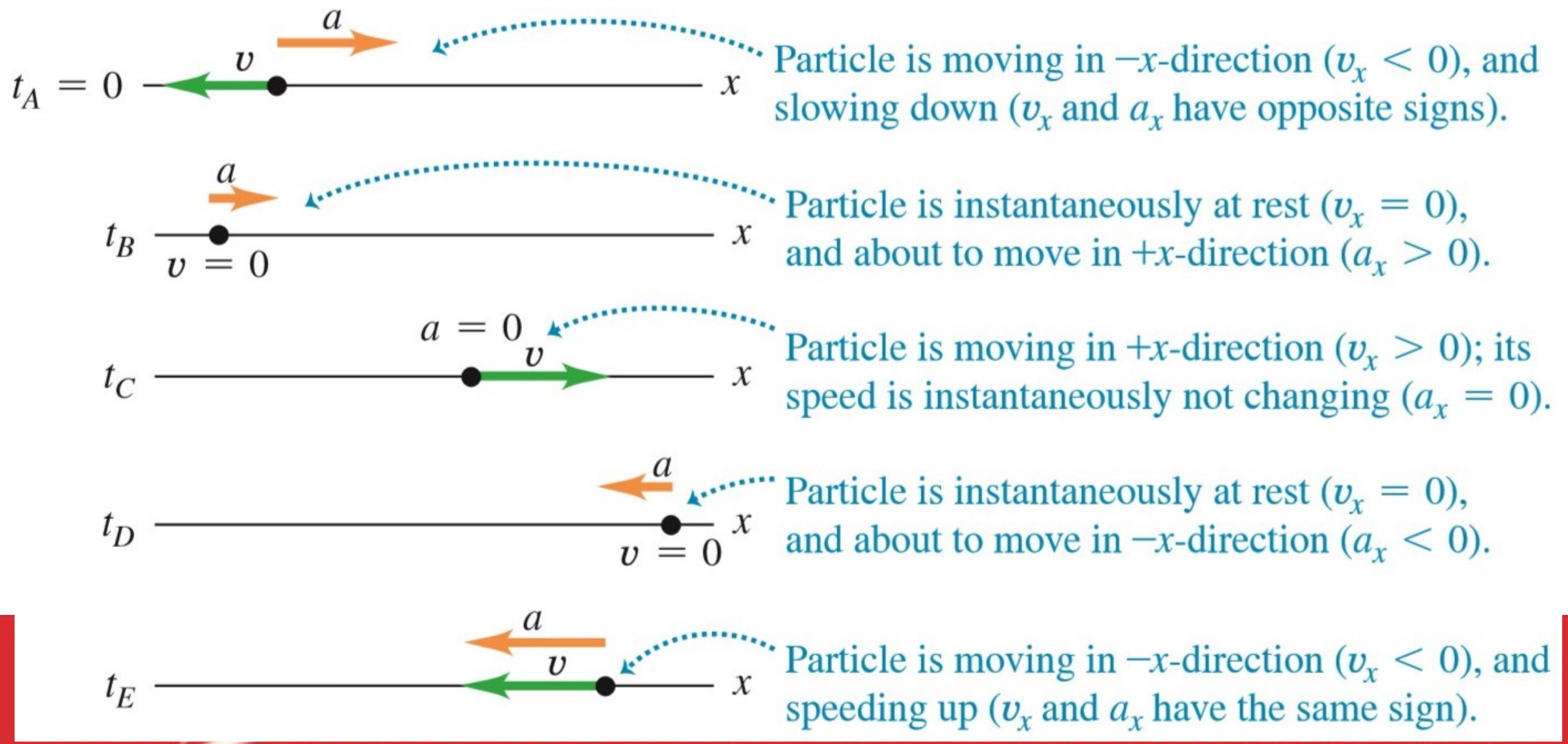


Velocity and acceleration both are vectors. What is the relationship for their directions?

Speed up and Slow down

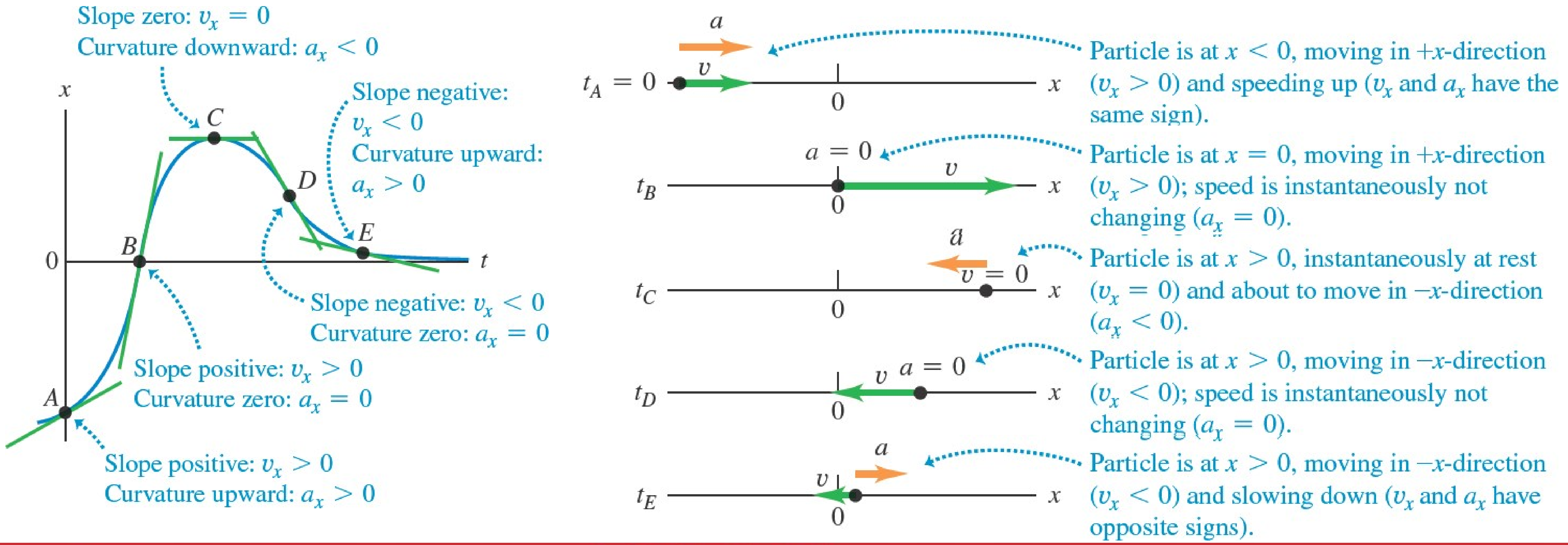
- When the sign of the velocity and the acceleration are the same (either positive or negative), then the speed is increasing (speed up).
- When the sign of the velocity and the acceleration are in the opposite directions, the speed is decreasing (slow down).





Velocity and acceleration in *x*-*t* graph

- Understand the velocity and acceleration in *x*-*t* graph?



Example

The position of an object moving along the *x*-axis is given by

$$x = (6.0\ \text{m/s})t - (3.0\ \text{m/s}^2)t^2 + (1.0\ \text{m/s}^3)t^3$$

What is the object doing at $t = 1.0$ s?

A. It is moving and speeding up.
B. It is moving and slowing down.
C. It is moving, and its acceleration is zero.
D. It is momentarily at rest.

Q1: Which quantity do we need to find to figure out if it is moving?

Instantaneous velocity. $\boldsymbol{v_x = \frac{dx}{dt}}$

Q2: How to determine if it is speeding up or slowing down?

Instantaneous acceleration $a_x = \frac{dv_x}{dt}$

Example

The position of an object moving along the x-axis is given by

$$x = (6.0\ \text{m/s})t - (3.0\ \text{m/s}^2)t^2 + (1.0\ \text{m/s}^3)t^3$$

What is the object doing at $t = 1.0$ s?

A. It is moving and speeding up.

B. It is moving and slowing down.

C. It is moving, and its acceleration is zero. ✓

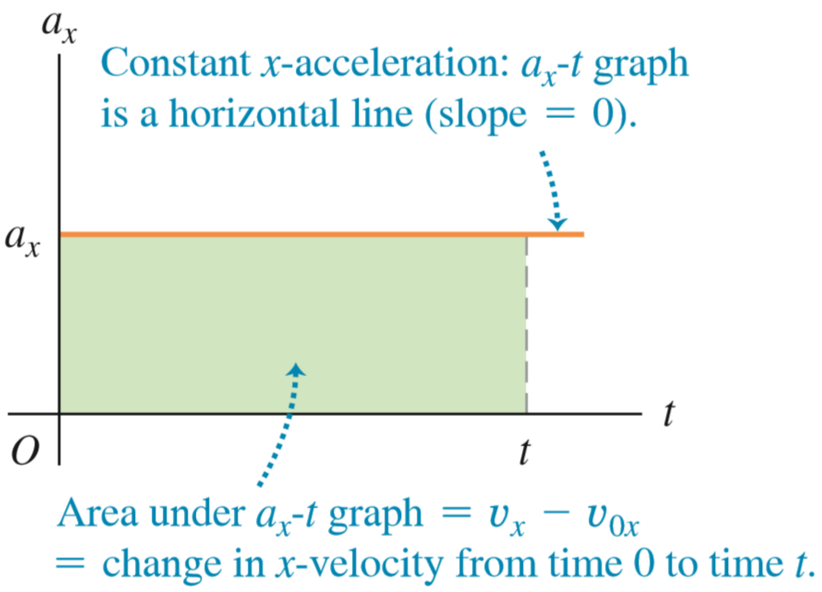
D. It is momentarily at rest.

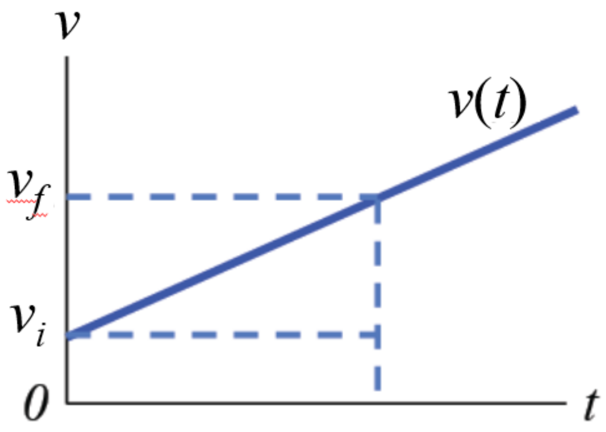
Instantaneous velocity. $\boldsymbol{v} = \frac{dx}{dt}$ $\quad v = (6.0) - (6.0)t + (3.0)t^2 \quad v(t = 1\ \text{s}) = 3\ \text{m/s}$

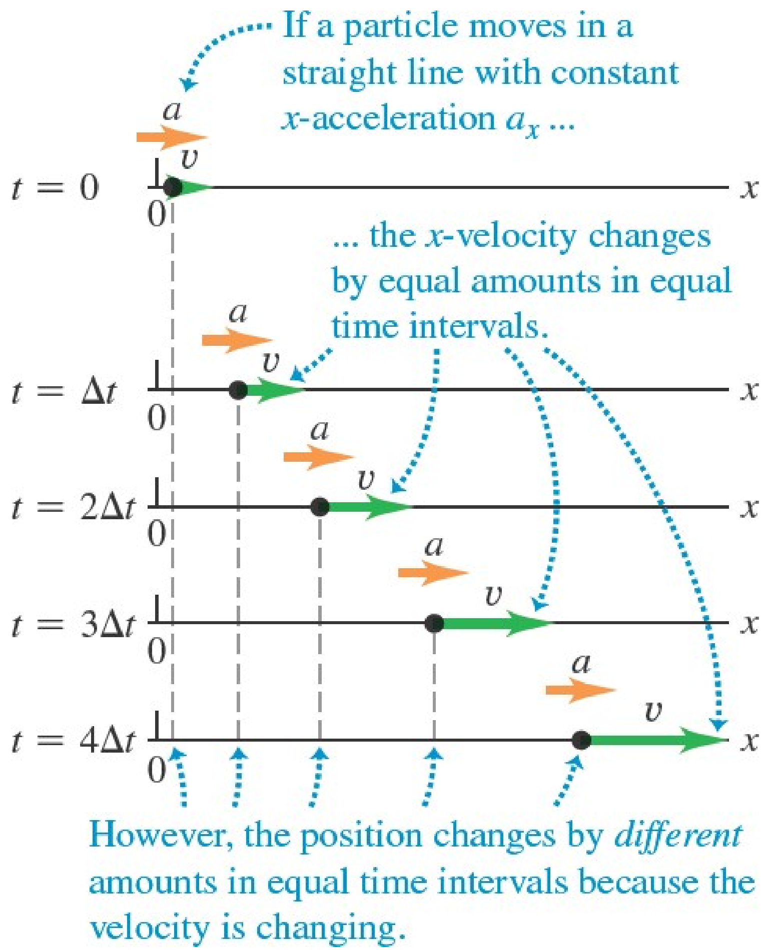
Instantaneous acceleration $\quad a_x = \frac{dv_x}{dt} \quad a = -(6.0) + (6.0)t \quad a(t = 1\ \text{s}) = 0$

Uniform Acceleration

- When the instantaneous accelerations are always the same, the acceleration will be uniform. The instantaneous acceleration will be equal to the average acceleration.

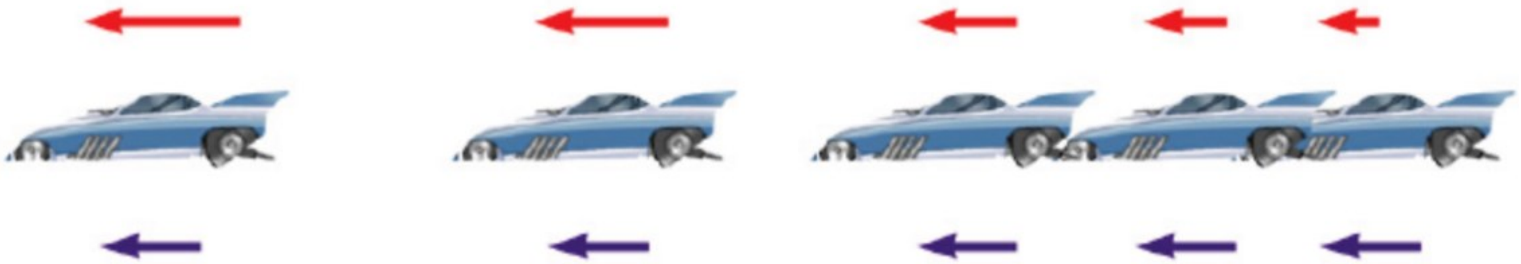


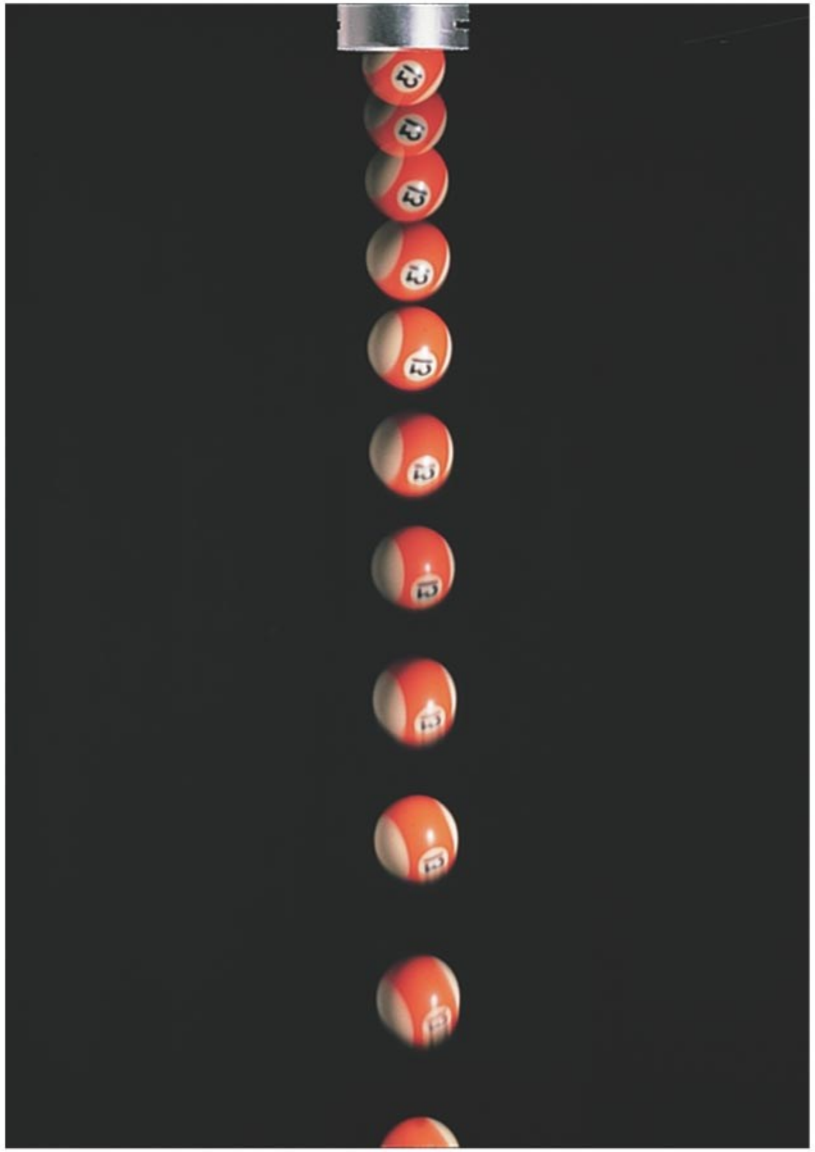




Freely falling bodies

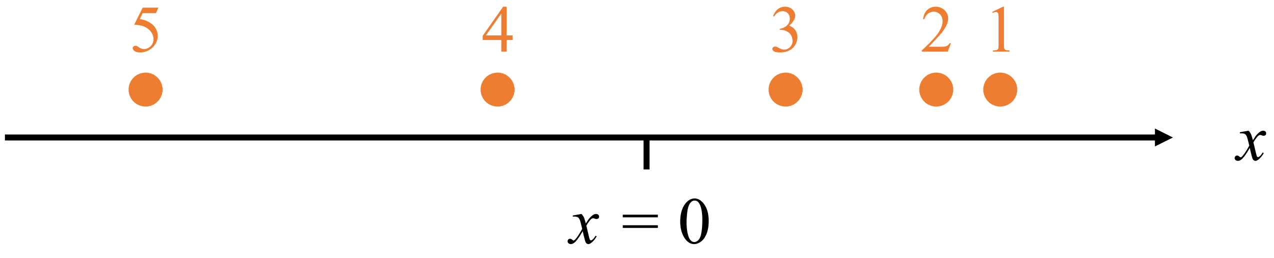
- In the figure, a strobe light flashes with equal time intervals between flashes.
- The velocity change is the same in each time interval, so the acceleration is constant.



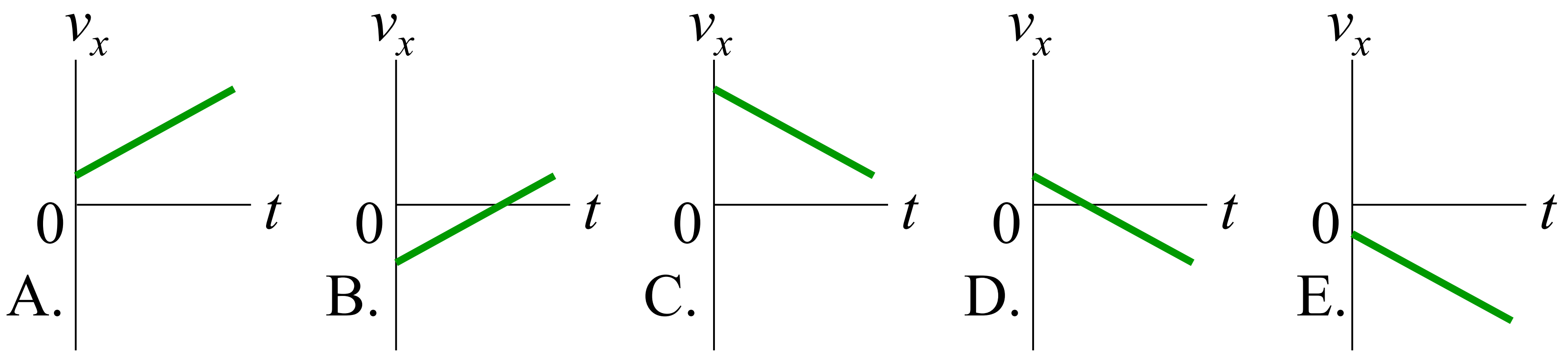


Example:

This is a motion diagram of an object moving along the x-direction with constant acceleration. The dots 1, 2, 3, … show the position of the object at equal time intervals Δt.



Which of the following v_x-t graphs best matches the motion shown in the motion diagram?



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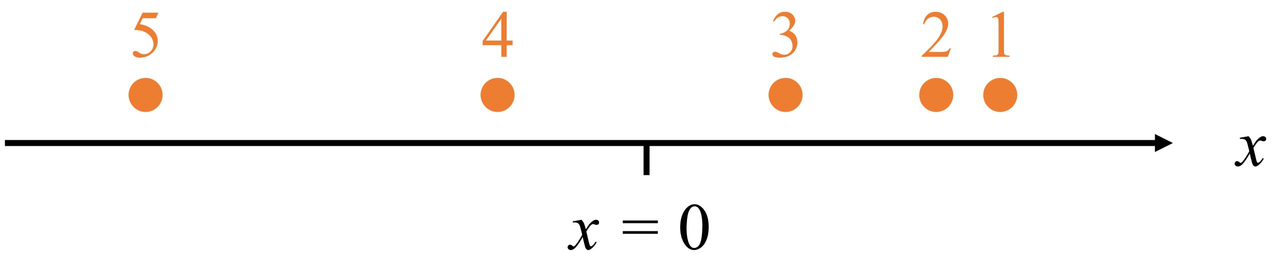


Q1: Which direction is this object moving to? Positive x or negative x? Negative.

Q2: What is the sign of the velocity? Positive or negative? Negative.

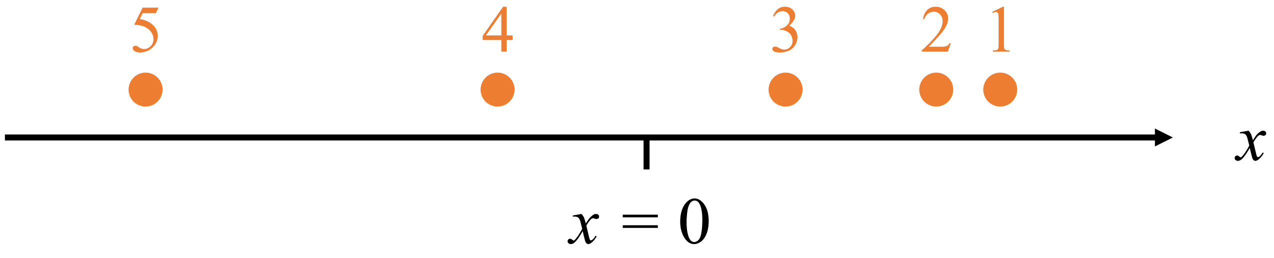
Q3: Doest the velocity change sign at any point?

Sign from 1 to 3? Sign from 3 to five?

No, always negative.

Example:

This is a motion diagram of an object moving along the x-direction with constant acceleration. The dots 1, 2, 3, … show the position of the object at equal time intervals Δt.



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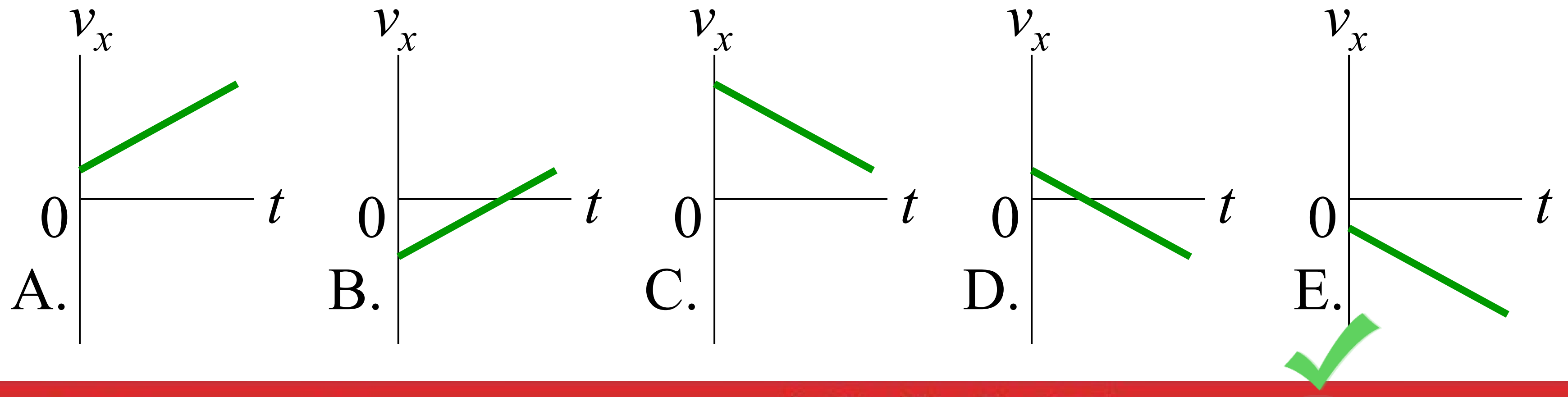

Kinematic equations

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Position $\xrightarrow{\frac{d}{dt}}$ Velocity $\xrightarrow{\frac{d}{dt}}$ Acceleration

- **Acceleration is uniform**: This lecture we need to *quantitatively* understand the relationship between acceleration, velocity and displacement.

$$v = v_0 + at$$

$$\Delta x = \bar{v}t = \frac{1}{2}(v_0 + v)t$$

$$\Delta x = v_0 t + \tfrac{1}{2} a t^2$$

$$v^2 = {v_0}^2 + 2a\Delta x$$

Derivation of the Equation (1) $v = v_0 + at$

- Given initial conditions: $a(t) = \text{constant} = a$, $v(t = 0) = v_0$, $x(t = 0) = x_0$
- Start with definition of average acceleration:

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - t_0} = \frac{v - v_0}{t - 0} = \frac{v - v_0}{t} = a$$

- We immediately get the first equation $v = v_0 + at$

- Shows velocity as a function of acceleration and time.
- Use this equation when you do not know and **are not asked** to find the **displacement**.

Derivation of the Equation (2)

$$\Delta x = v_{avg} t = \frac{1}{2}(v_0 + v)t$$

- Given initial conditions: $a(t) = \text{constant} = a$, $v(t = 0) = v_0$, $x(t = 0) = x_0$
- Start with definition of average velocity:

$$v_{avg} = \frac{x - x_0}{t} = \frac{\Delta x}{t} = \frac{1}{2}(v_0 + v)$$

- Since velocity changes at a constant rate, we have

$$\Delta x = v_{avg} t = \frac{1}{2}(v_0 + v)t$$

- Gives displacement as a function of velocity and time
- Use when you do not know and **are not asked** for the **acceleration**.

Derivation of the Equation (3)

$$\Delta x = x - x_0 = v_0 t + \frac{1}{2} a t^2$$

- Given initial conditions: $a(t) = \text{constant} = a$, $v(t=0) = v_0$, $x(t=0) = x_0$
- Start with two just-derived equations: $v = v_0 + at$ $\quad \Delta x = v_{avg} t = \frac{1}{2}(v_0 + v)t$
- We have $\Delta x = \frac{1}{2}(v_0 + v)t = \frac{1}{2}(v_0 + v_0 + at)t$ $\quad \Delta x = x - x_0 = v_0 t + \frac{1}{2} a t^2$
- Gives displacement as a function of all three quantities: time, initial velocity and acceleration.
- Use when you do not know and **are not asked** for the **final velocity**.

Derivation of Equation (4) $v^2 = v_0^{\ 2} + 2a\Delta x = v_0^{\ 2} + 2a(x - x_0)$

- Given initial conditions: $a(t) = \text{constant} = a,\ v(t = 0) = v_0,\ x(t = 0) = x_0$
- Rearrange the definition of average acceleration:

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t} = a$$ to find the time: $$t = \frac{v - v_0}{a}$$

- Use it to eliminate t in the second equation: $$\Delta x = \frac{1}{2}(v_0 + v)t = \frac{1}{2a}(v + v_0)(v - v_0) = \frac{v^2 - v_0^{\ 2}}{2a}$$

rearrange to get $$v^2 = v_0^{\ 2} + 2a\Delta x = v_0^{\ 2} + 2a(x - x_0)$$

- Gives velocity as a function of acceleration and displacement.
- Use when you do not know and **are not asked** for the **time**.

Problem-Solving Hints

- Read the problem: find out what quantities are **given**, and which are **unknown**.
- Draw a diagram
  - ✓ Choose a coordinate system, label **initial** and **final** points.
  - ✓ Indicate a **positive direction** for velocities and accelerations.
- Label all quantities, be sure all the units are consistent
  - ✓ Convert if necessary

- **Choose the appropriate kinematic equation**
- **Solve for the unknows**
  - ✓ You may have to solve two equations for two unknows.
- Check your results.

Example:

An airplane moves with an initial velocity v_o, and it accelerates with a constant acceleration and reaches a lift-off speed of 30 m/s after a take-off run of 300 m in 20 s. What is the **initial velocity v_o**?

Q1: What quantities are given, and which are unknown?

Given quantities

t = 20 s v = 30 m/s Δx = 300 m

Unknown quantities

v_0 = ? m/s a = ? m/s^2

Q2: Draw a diagram?

Q3: Choose the appropriate kinematic equation?

$$\Delta x = \bar{v}t = \frac{1}{2}(v_0 + v)t$$

300 m = 0.5×(v_0+ 30 m/s)×(20 s)

Q4: Solve unknown quantity?

v_0 = 0 m/s

$$v = v_0 + at$$

$$\Delta x = \bar{v}t = \frac{1}{2}(v_0 + v)t$$

$$\Delta x = v_0 t + \tfrac{1}{2}at^2$$

$$v^2 = v_0^2 + 2a\Delta x$$

Example:

An airplane moves with an initial velocity v_o, and it accelerates with a constant acceleration and reaches a lift-off speed of 30 m/s after a take-off run of 300 m in 20 s. What is the **constant acceleration**?

Q1: What quantities are given, and which are unknown?

Given quantities

t = 20 s v = 30 m/s Δx = 300 m

$\boldsymbol{v_0}$ **= 0 m/s**

Unknown quantities

v_0 = ? m/s a = ? m/s^2

Q2: Draw a diagram?

Q3: Choose the appropriate kinematic equation?

$$v = v_0 + at$$

30 m/s = 0 m/s + a×(20 s)

Q4: Solve unknown quantity?

1.5 m/s^2 = a

$$v = v_0 + at$$

$$\Delta x = \bar{v}t = \frac{1}{2}(v_0 + v)t$$

$$\Delta x = v_0 t + \tfrac{1}{2}at^2$$

$$v^2 = v_0{}^2 + 2a\Delta x$$