Extended work-energy theorem: Spring

- The work-kinetic energy theorem can be extended to include potential energy. If **only gravitational force** does work

$$K_1 + U_{\text{grav},1} = K_2 + U_{\text{grav},2}$$

$$\tfrac{1}{2}mv_1^2 + mgy_1 = \tfrac{1}{2}mv_2^2 + mgy_2$$

$K + U$ does not change

- If we **only have elastic/spring force** and all work done by all rest forces are zero, then

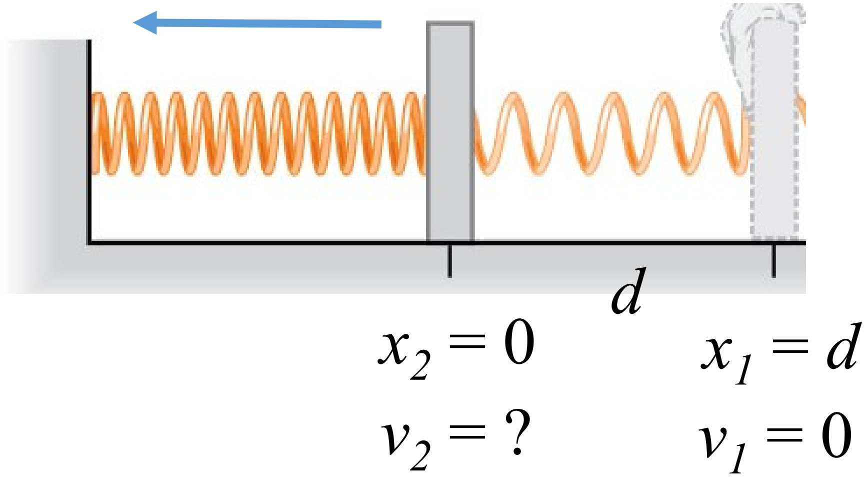
$$K_1 + U_{\text{el},1} = K_2 + U_{\text{el},2}$$

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$

$K + U$ does not change

Example: Review

A glider of mass **0.1 kg** is attached to the end of a horizontal air track by a spring with force constant **20.0 N/m**. The spring is pulled a distance ***d* = 0.106 m** out of equilibrium and released. Find the **maximum speed (in m/s)** achieved by the glider.



No friction: $K + U$ does not change

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2$$

$$\frac{1}{2}k(0.106)^2 + \frac{1}{2}m(0)^2 = \frac{1}{2}k0 + \frac{1}{2}mv_2^2$$

$v_2 = 1.5$ m/s

Example: Extended work-energy theorem

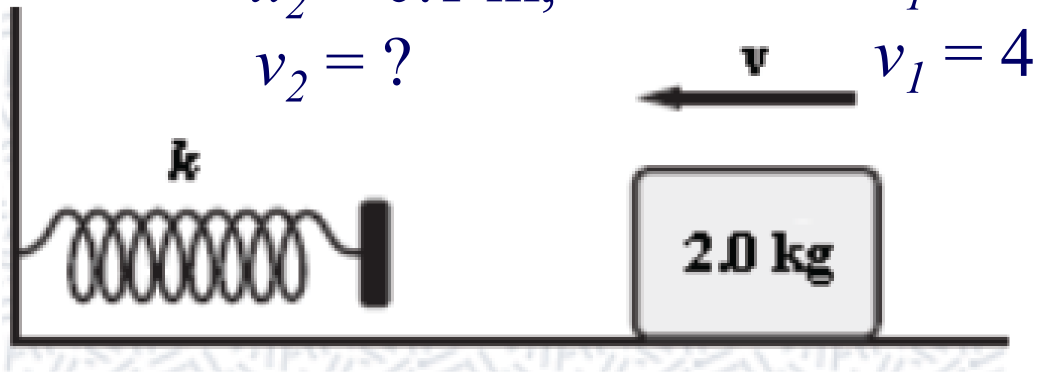
A **2 kg** block slides with no friction and with an initial speed of **4 m/s**. It hits a spring with spring constant ***k* =1400 N/m**. The block compresses the spring in a straight line for a distance **0.1 m**. What is the block's speed, in m/s, at that point?

No friction $K_1 + U_{el,1} = K_2 + U_{el,2}$

$$\tfrac{1}{2}mv_1^{\,2} + \tfrac{1}{2}kx_1^{\,2} = \tfrac{1}{2}mv_2^{\,2} + \tfrac{1}{2}kx_2^{\,2}$$

$x_2 = 0.1$ m,
$v_2 = ?$

$x_1 = 0$ m,
$v_1 = 4$ m/s



$$\frac{1}{2}(2\,kg)\left(4\frac{m}{s}\right)^2 + \frac{1}{2}\left(1400\frac{N}{m}\right)(0\,m)^2 = \frac{1}{2}(2\,kg)(v_2)^2 + \frac{1}{2}\left(1400\frac{N}{m}\right)(0.1\,m)^2$$

$v_2 = 3$ m/s

Extended work-energy theorem: elastic + grav

- The work-kinetic energy theorem can be extended to include **gravitational potential energy and elastic potential energy**:

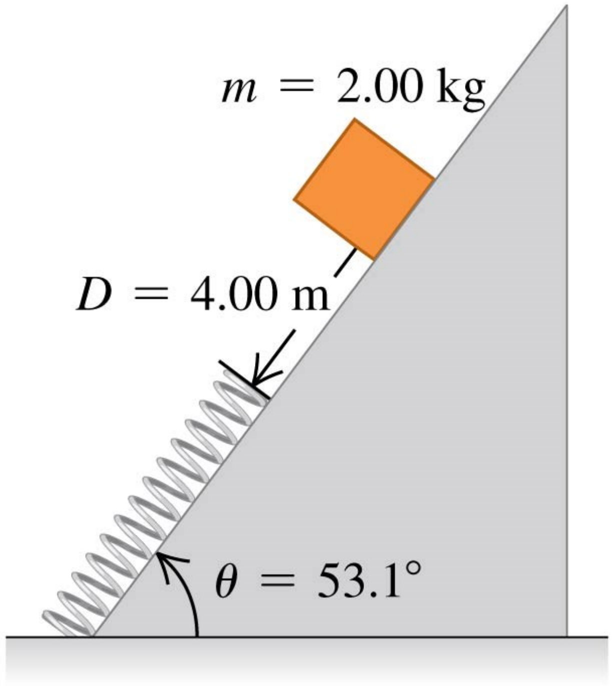
$$K_1 + U_{\text{grav},1} = K_2 + U_{\text{grav},2} \qquad K_1 + U_{\text{el},1} = K_2 + U_{\text{el},2}$$

$$K_1 + U_{grav,1} + U_{el,1} = K_2 + U_{grav,2} + U_{el,2}$$

$K + U$ does not change

$$\frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}kx_2^2$$

Example: A block is released from rest on a frictionless incline as shown. When the moving block is in contact with the spring and compressing it, what is happening to the gravitational potential energy U_{grav} and the elastic potential energy U_{el}?



(A). U_{grav} and U_{el} are both increasing.

(B). U_{grav} and U_{el} are both decreasing.

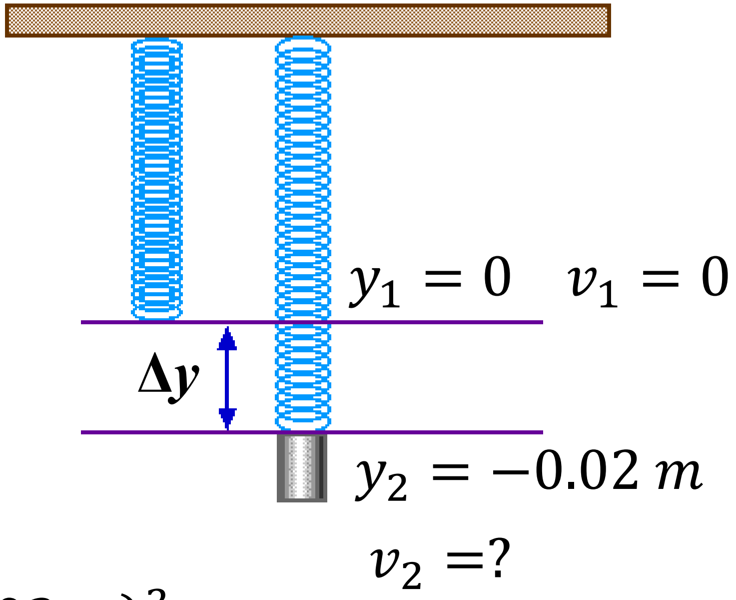
(C). U_{grav} is increasing; U_{el} is decreasing.

✓ (D). U_{grav} is decreasing; U_{el} is increasing.

(E). The answer depends on how the block's speed is changing.

Example: Potential energy in spring

An ideal spring of negligible mass is **12.00 cm** long when nothing is attached to it, and the spring constant is **2205 N/m**. You hang a **3.15 kg weight** on it and release the weight. What is the speed of the weight if the spring length is **14 cm**?



Δy = 14 cm - 12 cm = 2 cm = 0.02 m

$$\frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}kx_2^2$$

$$\frac{1}{2}m(0)^2 + mg(0) + \frac{1}{2}k(0\,m)^2$$

$$= \frac{1}{2}(3.15\,kg)(v_2)^2 + (3.15\,kg)(9.8)(-0.02\,m) + \frac{1}{2}\left(2205\frac{N}{m}\right)(0.02\,m)^2$$

$$v_2 = 0.33\,m/s$$

Extended work-energy theorem: grav + friction

- When we have other work (e.g., friction), the work-kinetic energy theorem can be extended to:

$$W_{tot} = W_{grav} + W_{other} = K_2 - K_1 = \Delta K$$

$$W_{\text{grav}} = U_{\text{grav},1} - U_{\text{grav},2}$$

$$W_{other} + U_{grav,1} - U_{grav,2} = K_2 - K_1$$

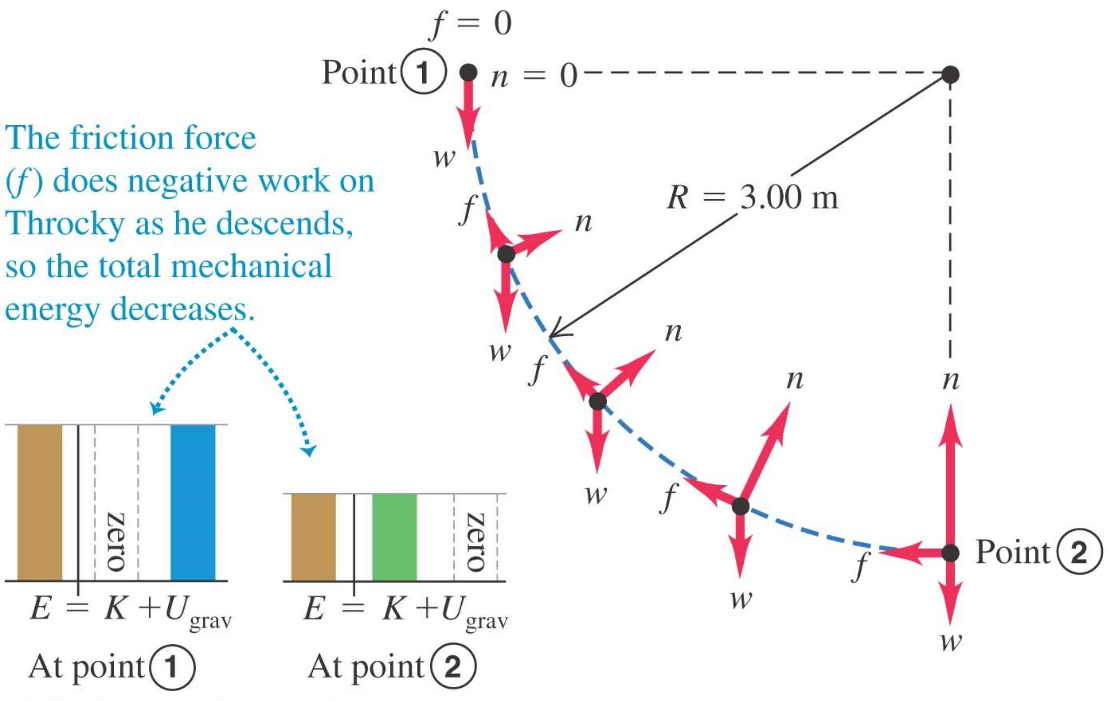
$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$$

W_{other} can be work done by friction or some other forces.

$$\frac{1}{2}mv_1^2 + mgy_1 + W_{other} = \frac{1}{2}mv_2^2 + mgy_2$$

Example: Motion with friction

A person+skateboard (total mass **25.0 kg**) from rest down a curved, with friction. If we treat them as a particle, they moves through a quarter-circle with radius R = **3.00 m**. **Work** was done on them by the **friction** force is **285 J**. What is the **speed** at bottom?



Friction $K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$

$K_1 = 0$ $U_{grav,2} = 0$

$U_{grav,1} = mgR = (25.0 \text{ kg})(9.8 \text{ m/s}^2)(3.00 \text{ m}) = 735 \text{ J}$

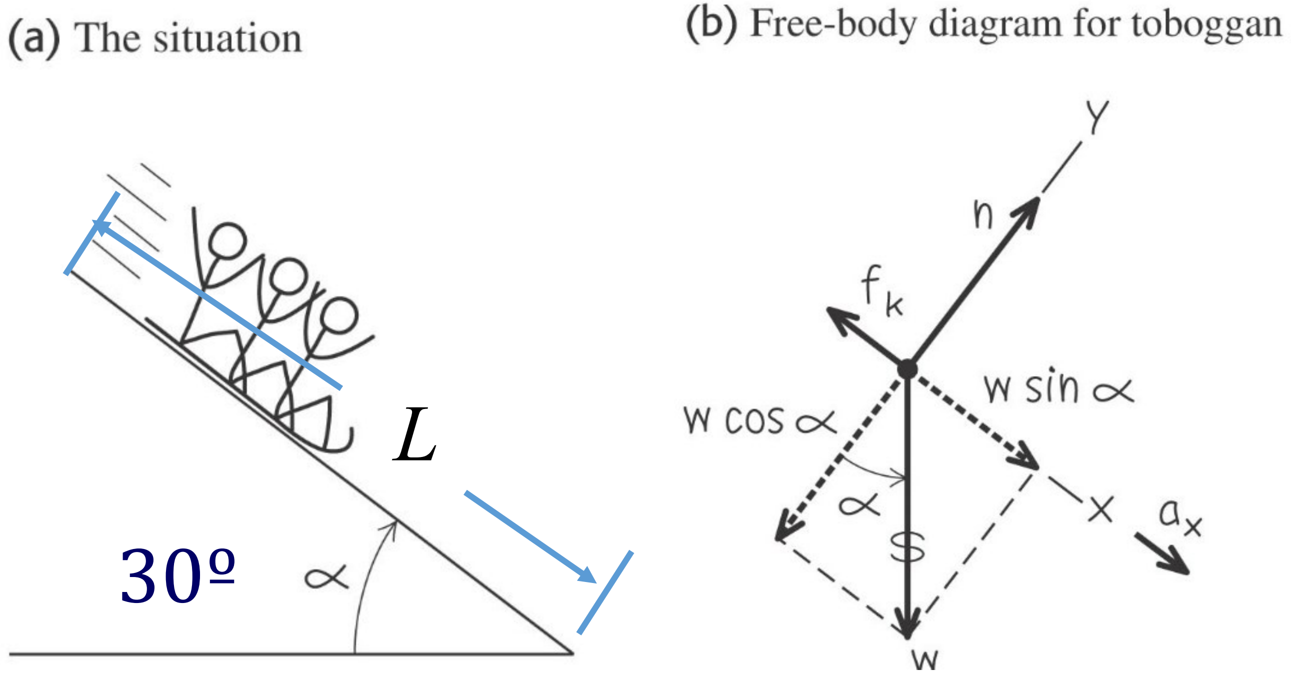
$W_f = W_{other} = -285 \text{ J}$

$K_2 = U_{grav,1} + W_{other} = 735 - 285 = 450 \text{ J}$

$$= \frac{1}{2} m v_2^2 = \frac{1}{2}(25.0 \text{ kg})(v_2)^2 = 450 \text{ J}$$

$v_2 = 6.00 \text{ m/s}$

Example: Consider the toboggan (**100 kg**) on a hill with **friction**. It slides down from rest at top. The length of the ramp is **20 m**, and the angle of the slope is **30º**. **Friction** force performs a work **5000 J**. Find out the **speed** at the bottom of the hill. $g = 10\ \text{m/s}^2$.



Friction $K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$

$K_1 = 0$ J $K_2 = 0.5mv^2$

$W_{other} = -5000$ J $U_2 = 0$ J

$U_1 = mg(y) = mg(20 \cdot sin30º) = 10^4$

$K_2 = 5000\ \text{J} = 0.5mv^2$

$v = 10$ m/s

Mechanical energy conservation for all forces

- **The total mechanical energy (potential + kinetic)**:

$$E = K + U = K + U_{grav} + U_{el}$$

- If other forces do work, $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$

If W_{other} is positive, $E = K + U$, increases

If W_{other} is negative, $E = K + U$, decreases

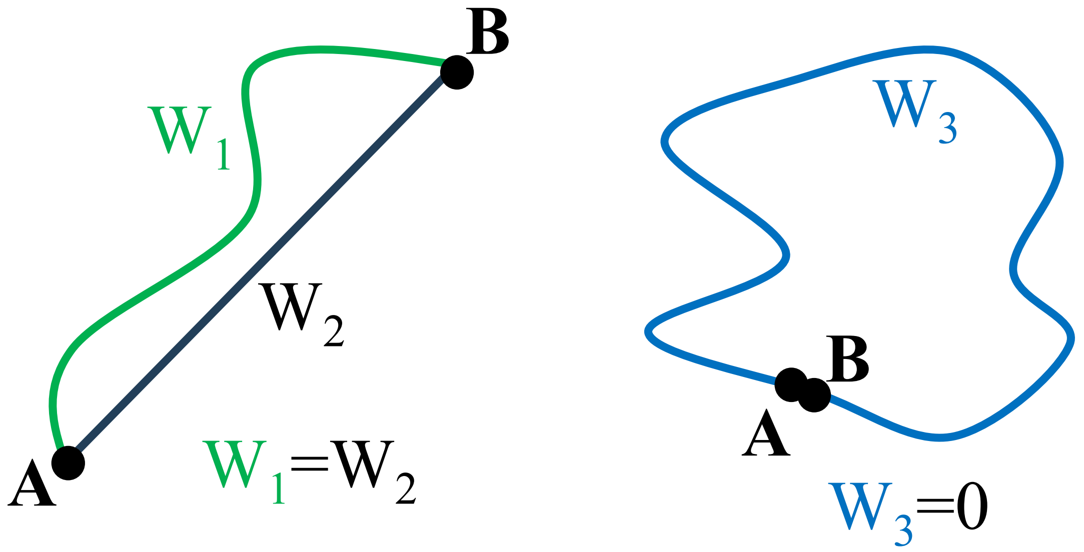
If W_{other} is zero, $E = K + U$, constant

"Other" forces are different from the gravitational and spring forces.

Conservative forces

- **Conservative forces**: Work and energy associated with the forces could be recovered. Total mechanical energy is conserved. Example: *gravity*, *spring force*, *EM force*.
- **Any conservative force can have a potential energy function associated with it. Example: elastic potential, gravitational potential, EM potential.**

- **Conservative force**: the work it does on an object moving between two points is **independent of the path** the objects take between the points. Like *mgy*
- The work depends only upon the **initial** and **final** positions of the object



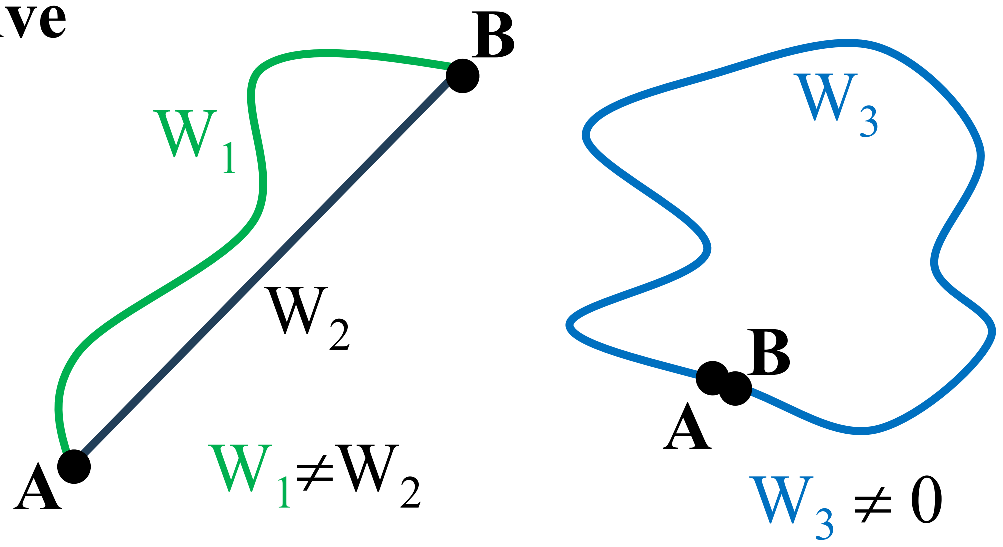
Nonconservative forces

- **Nonconservative forces**: The forces are general **dissipative** and total mechanical energy is NOT conserved. Example: *kinetic friction*, *air drag force*, *tension*…

  "Dissipative" means the energy may be converted to heat or sound.

  For a nonconservative force, potential energy can NOT be defined

- A force is **nonconservative** if the **work** it does on an object **depends on the path** taken by the object between its final and starting points.



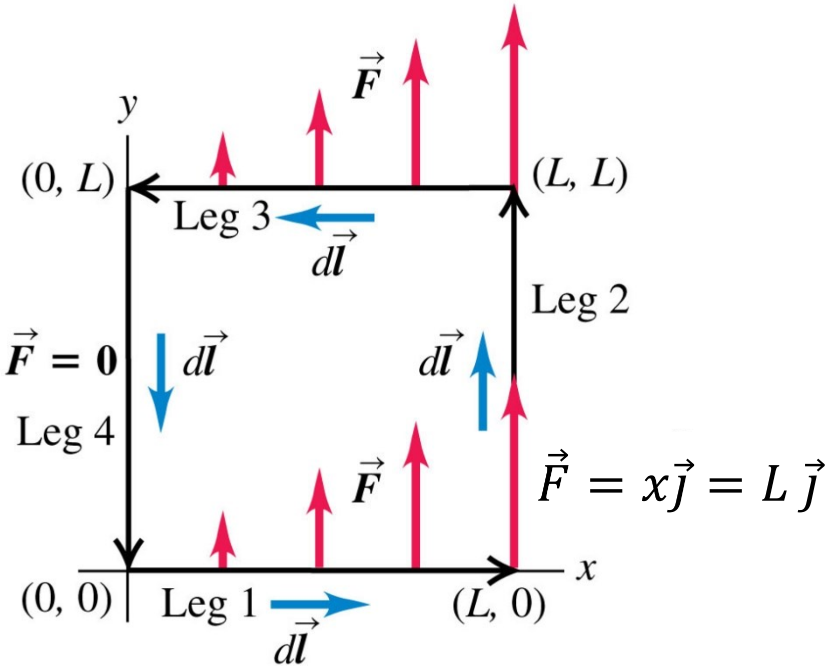
Work done by a Nonconservative force

$$W_{nc} = \sum \vec{F} \cdot \vec{d}$$

$$= \int_{P_1}^{P_2} \vec{F} \cdot d\,\vec{l}$$

Example: Conservative or nonconservative force?

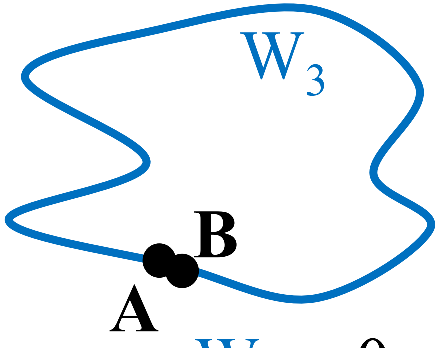
In a region of space the force on an electron is $\vec{F} = x\vec{j}$. The electron moves around a square loop in the *xy*-plane. Calculate the work done on the electron by the force $\vec{F}$ during a counterclockwise trip around the square. Is this force conservative or nonconservative?



Q: Conservative or nonconservative, criterion?

Work done by a nonconservative force depends on the path.

We choose a “loop” path and check if the work done is zero.



$$W_{nc} = \sum \vec{F} \cdot \vec{d} = \int_A^B \vec{F} \cdot d\,\vec{l}$$

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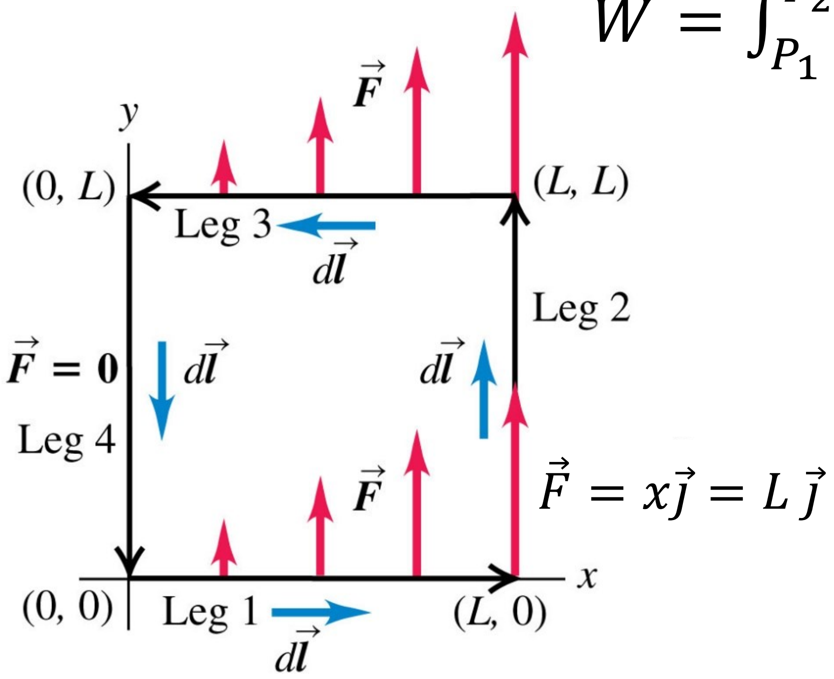


$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\,\vec{l},$$

Leg 1, 3 and 4: $\vec{F} \cdot d\vec{l} = 0$, so $W_1 = W_3 = W_4 = 0$

Leg 2: $W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\,\vec{l} = \int_{y=0}^{y=L} L dy = L^2$

$$W = W_1 + W_2 + W_3 + W_4 = L^2$$

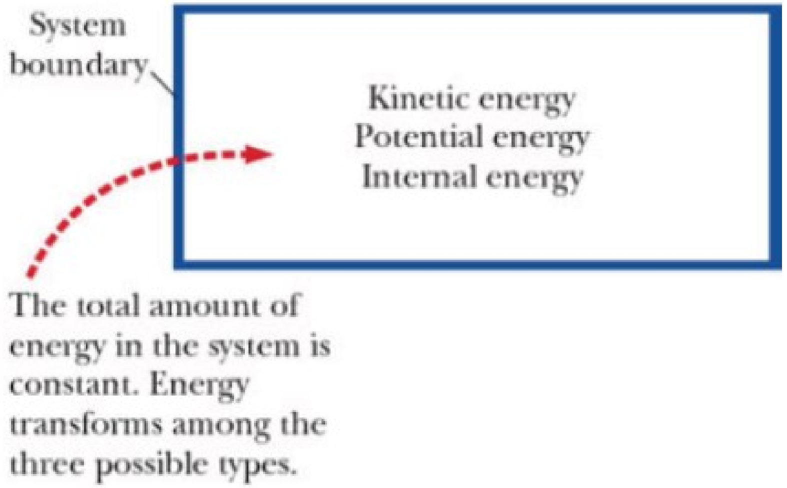
Starting point and ending point are same

nonzero work (depends on *L*), nonconservative force

Conservation of energy in general

- For any system: Total energy = Kinetic + Potential + Internal

$$E = K + U_p + U_i$$



$$\Delta E = E_2 - E_1 = (K_2 - K_1) + (U_{p2} - U_{p1}) + (U_{i2} - U_{i1})$$

$$\Delta E = \Delta K + \Delta U_p + \Delta U_i$$

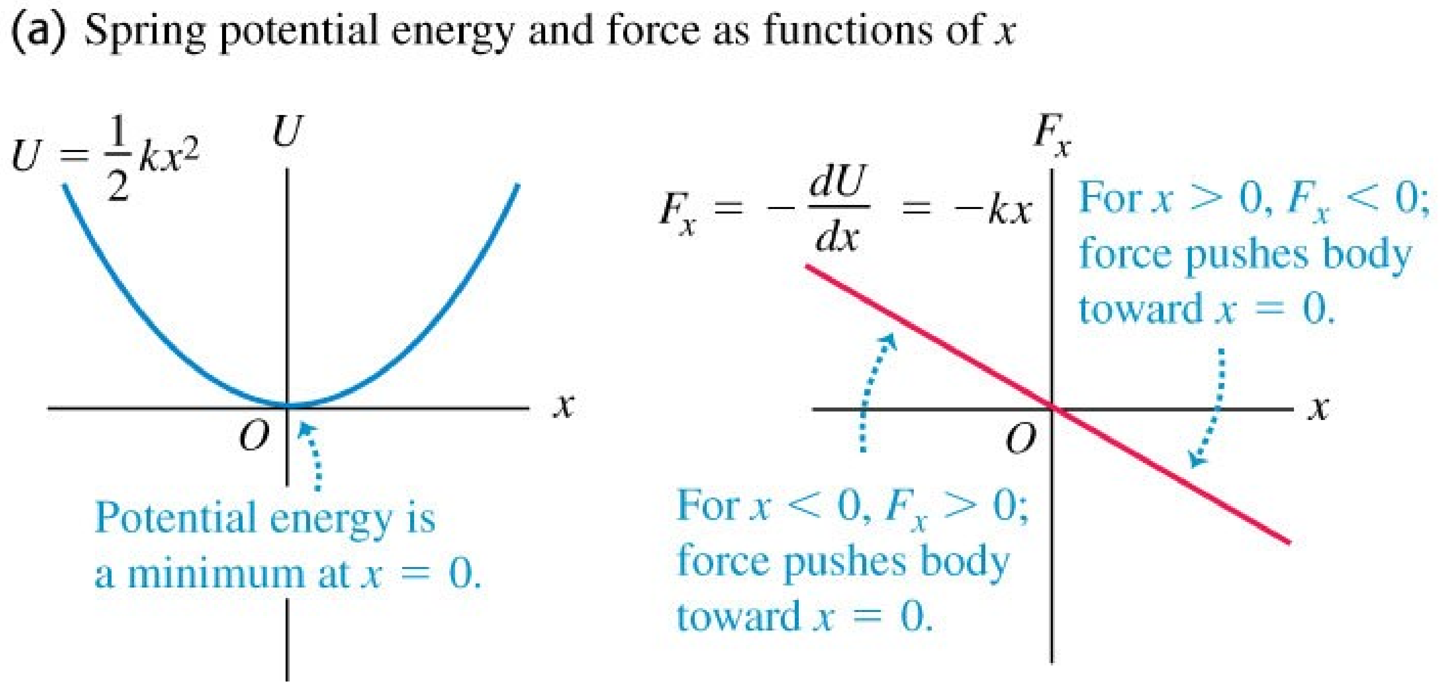
- Law of conservation of energy $\Delta K + \Delta U + \Delta U_i = 0$

- **Energy is never created or destroyed. It only changes form.**

Force and potential energy in one dimension

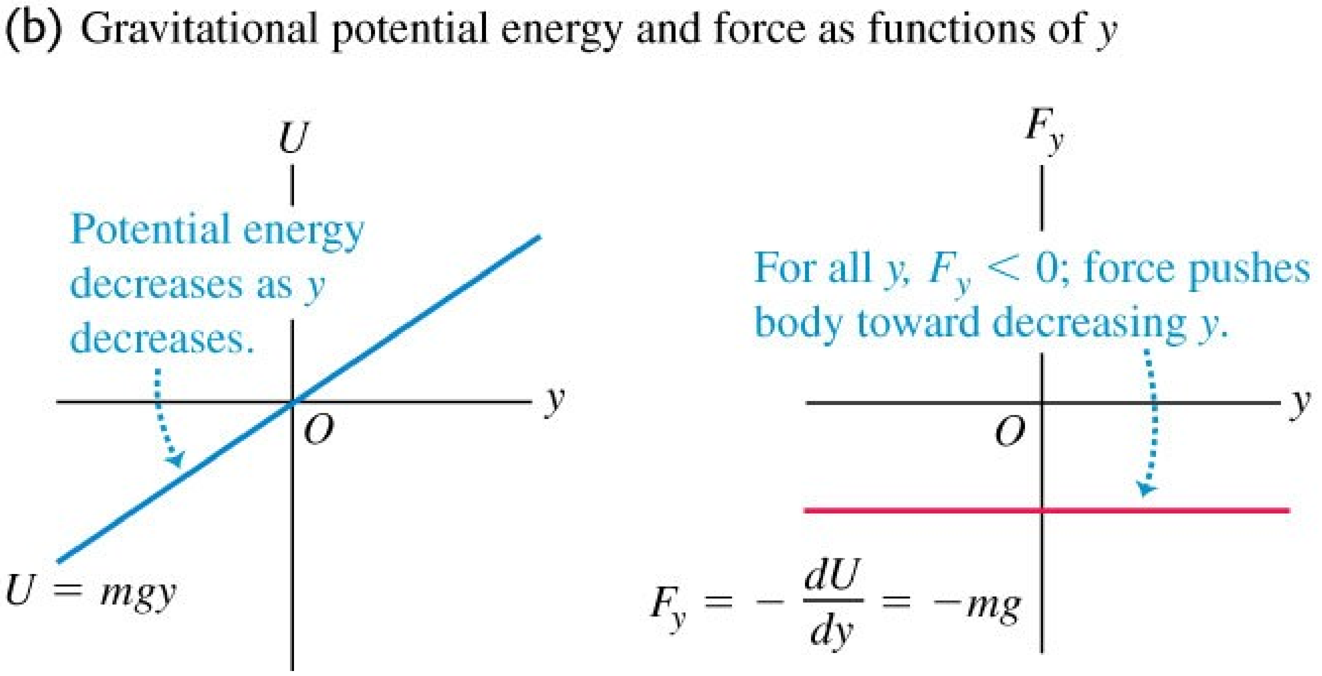
- In one dimension, $dW = F_x(x) \cdot dx = -dU(x)$ So, $\boldsymbol{F_x(x) = -dU(x)/dx}$

A conservative force can be obtained from its potential energy function



Force and potential energy in one dimension

- In one dimension, $F_x(x) = -dU(x)/dx$

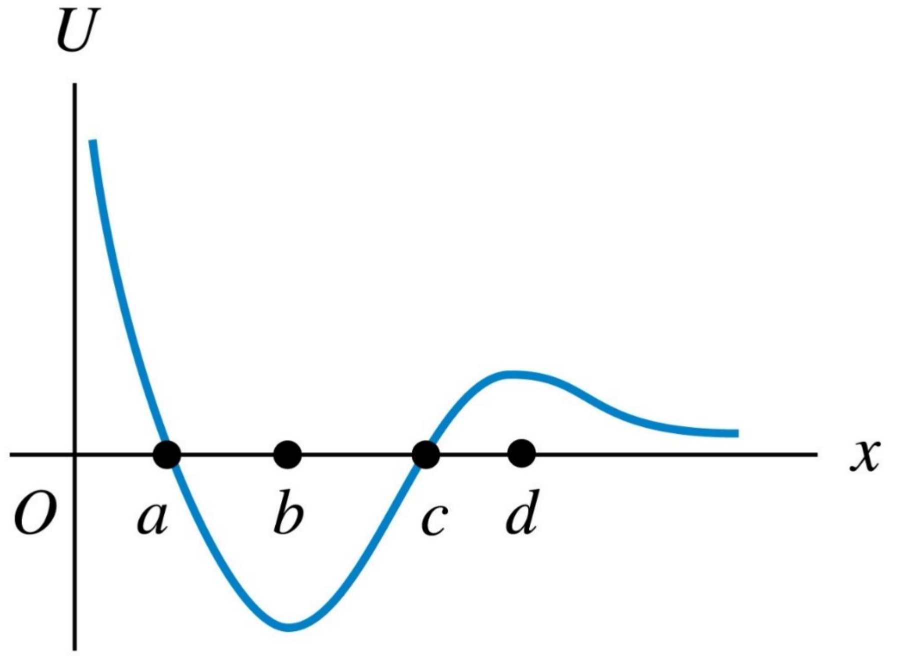


- In two dimension, the components of a conservative force can be obtained from its potential energy function using

$$F_x = -\partial U/dx \quad \textbf{and} \quad F_y = -\partial U/dy$$

Example: The graph shows the potential energy U for a particle that moves along the x-axis. At which of the labeled x-coordinates is there *zero* force on the particle?

$$F_x(x) = -dU(x)/dx$$



A. at $x = a$ and $x = c$

B. at $x = b$ only

C. at $x = d$ only

D. at $x = b$ and d ✓

E. misleading question—there is a force at all values of x

Example: Force and potential energy

A puck with coordinates x and y slides on a level, frictionless air-hockey table. It is acted on by a conservative force described by the potential-energy function

$$U(x, y) = \frac{1}{2}k(x^2 + y^2)$$

Find a vector expression for the force acting on the puck, and find an expression for the magnitude of the force.

$$\boldsymbol{F_x = -\partial U/dx} \quad \textbf{and} \quad \boldsymbol{F_y = -\partial U/dy}$$

The x- and y-components of $\vec{F}$ are

$$F_x = -\partial U/dx = -kx \qquad \text{and} \qquad F_y = -\partial U/dy = -ky$$

The vector expression for the force is

$$\vec{F} = (-kx)\,\hat{\imath} + (-ky)\,\hat{\jmath} = -\,k(x\hat{\imath} + y\hat{\jmath})$$

The magnitude of the force is

$$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

Problem solving

- Define the system to see if it includes non-conservative forces (especially friction, drag force …)
- Without non-conservative forces $\frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}kx_f^2 = \frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}kx_i^2$
- With non-conservative force $W_{nc} = (K_f+U_f) - (K_i+U_i)$

$$-fd + \sum W_{other} = \left(\frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}kx_f^2\right) - (\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}kx_i^2)$$

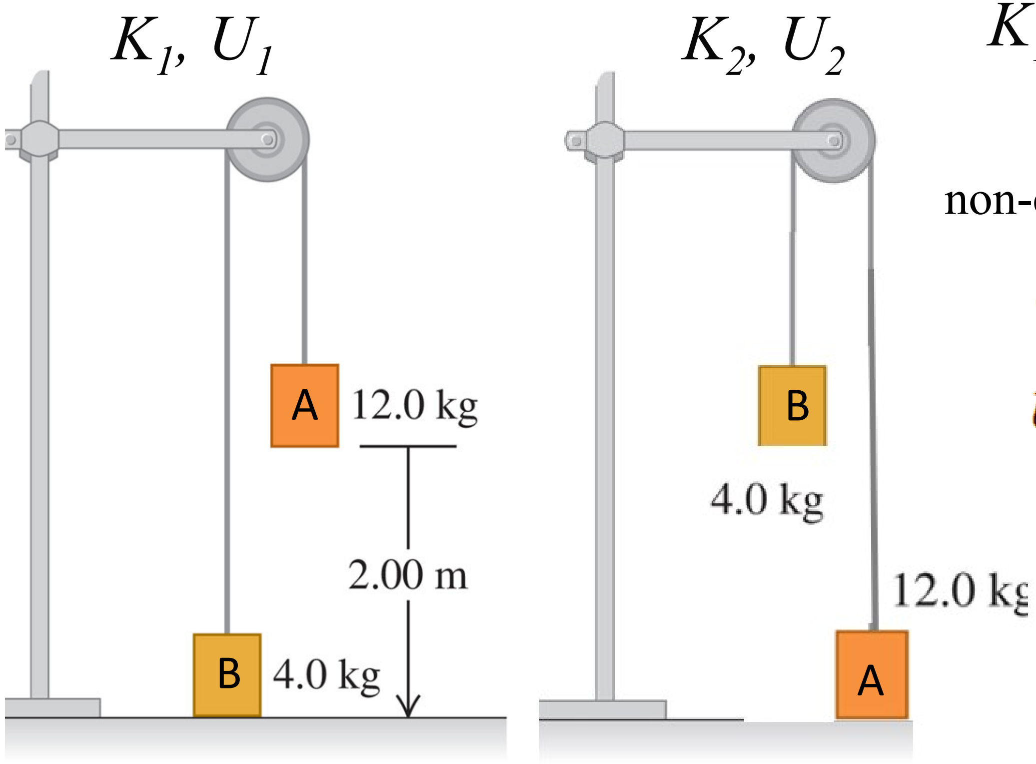
- Select the location of zero potential energy: Do not change this location while solving the problem
- Identify **two points** the object of interest moves between:

One point should be where information is given

The other point should be where you want to find out something.

Example:

A system of two paint buckets connected by a lightweight rope is released from rest with the 12.0 kg bucket h = 2.0 m above the floor. Use the principle of conservation of energy to find the speed with which this bucket strikes the floor. Ignore friction and the mass of the pulley.



Reference level

$K_1 = K_{A1} + K_{B1},\ U_1 = U_{A1} + U_{B1}$ $\quad$ $K_2 = K_{A2} + K_{B2},\ U_2 = U_{A2} + U_{B2}$

Without non-conservative forces $\cancel{K_1} + U_{\text{grav},1} = K_2 + U_{\text{grav},2}$ gives $U_1 = K_2 + U_2$.

$U_1 = m_A g y_{A,1} = (12.0\text{ kg})(9.80\text{ m/s}^2)(2.00\text{ m}) = 235.2\text{ J}.$ $\quad$ $U_{B1} = 0$

$U_2 = m_B g y_{B,2} = (4.0\text{ kg})(9.80\text{ m/s}^2)(2.00\text{ m}) = 78.4\text{ J}.$ $\quad$ $U_{A2} = 0$

$$K_2 = \tfrac{1}{2}(m_A + m_B)v_2^2 = (8.00\text{ kg})v_2^2.$$

$235.2\text{ J} = (8.00\text{ kg})v_2^2 + 78.4\text{ J}.$ $\quad$ $v_2 = \sqrt{\dfrac{235.2\text{ J} - 78.4\text{ J}}{8.00\text{ kg}}} = 4.4\text{ m/s}$

A metal ball of mass 0.25 kg on a frictionless surface is attached to a spring with spring constant k = 2500 N/m. The ball is pushed against the spring until the spring is compressed 2.0 cm, and then released. Find the maximum speed of the ball in m/s.

$$U_{el} = \frac{1}{2}kx^2 = \frac{1}{2}(2500\,\mathrm{N/m})(0.02\,\mathrm{m})^2 = 0.5\,\mathrm{J}$$

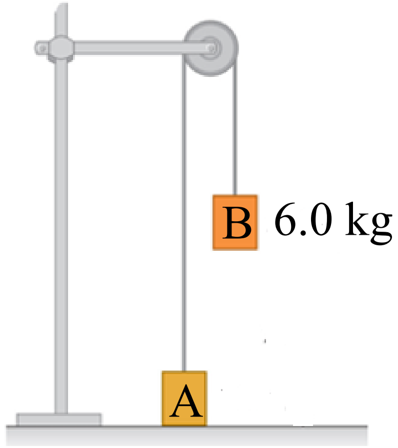
$$U_{el} = K = \frac{1}{2}mv^2 = \frac{1}{2}(0.25\,\mathrm{kg})(v)^2 = 0.5\,\mathrm{J} \qquad v = 2\,\mathrm{m/s}$$

2-kg apple is attached to the end of a spring. As a result, the length of the spring increases by 12 cm. Find the elastic potential energy, in J, stored in the spring.

$$w = F = kx \qquad 9.8 \times (2\,\mathrm{kg}) = k(0.12\,\mathrm{m}) \qquad k = 163.3$$

$$U_{el} = \frac{1}{2}kx^2 = \frac{1}{2}(163.3\,\mathrm{N/m})(0.12\,\mathrm{m})^2 = 1.18\,\mathrm{J}$$

Two unequal blocks of mass 2.0 kg and 6.0 kg are suspended from either end of a light rope over a frictionless pulley as it is shown in the figure. The blocks are released from rest. After the heavier block descends 0.2 m, what is the change in gravitational potential energy of the system?



2.0 kg

$$\Delta U_A = mgh = (2\text{ kg})(9.8)(+0.2) = +3.92\text{ J}$$

$$\Delta U_B = mgh = (6\text{ kg})(9.8)(-0.2) = -11.76\text{ J}$$

$$\Delta U = \Delta U_A + \Delta U_B = -7.84\, J$$

A 4.0 kg stone is dropped from a 50.0 m tall building. Neglecting the air resistance, what is the kinetic energy of the stone when it hits the ground?

$$\Delta U = mgh = (4.0\text{ kg})(9.8)(50.0\text{ m}) = 1960\text{ J} = K$$