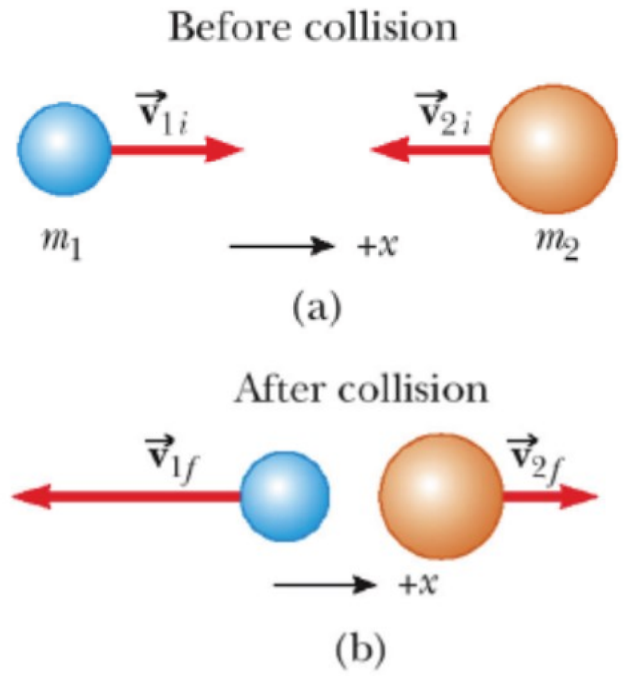
# Elastic Collisions




- Conservation of momentum and kinetic energy.

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \rightarrow m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

- A simpler equation can be use in place of kinetic energy.

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2) \qquad m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})$$

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

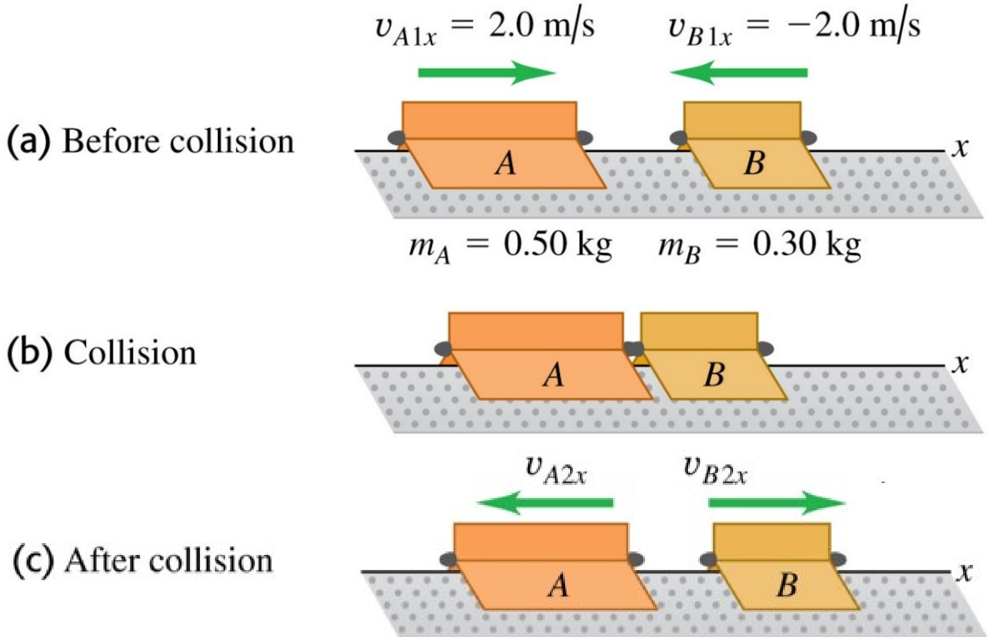
$$v_{1f} - v_{2f} = -(v_{1i} - v_{2i})$$

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

Momentum is vector: Be sure to have the **correct sign**.

# Example: colliding along a straight line

Two gliders with different masses move toward each other on a **frictionless** air track. Initial velocity of A is **+2.0 m/s** and initial velocity is **-2.0 m/s**. Suppose the collision is **elastic**. What is the final velocity of glider A and B?




For Elastic Collisions, we have:

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

known known

$$v_{1f} - v_{2f} = -(v_{1i} - v_{2i})$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

known known Masses are known

# Example: colliding along a straight line

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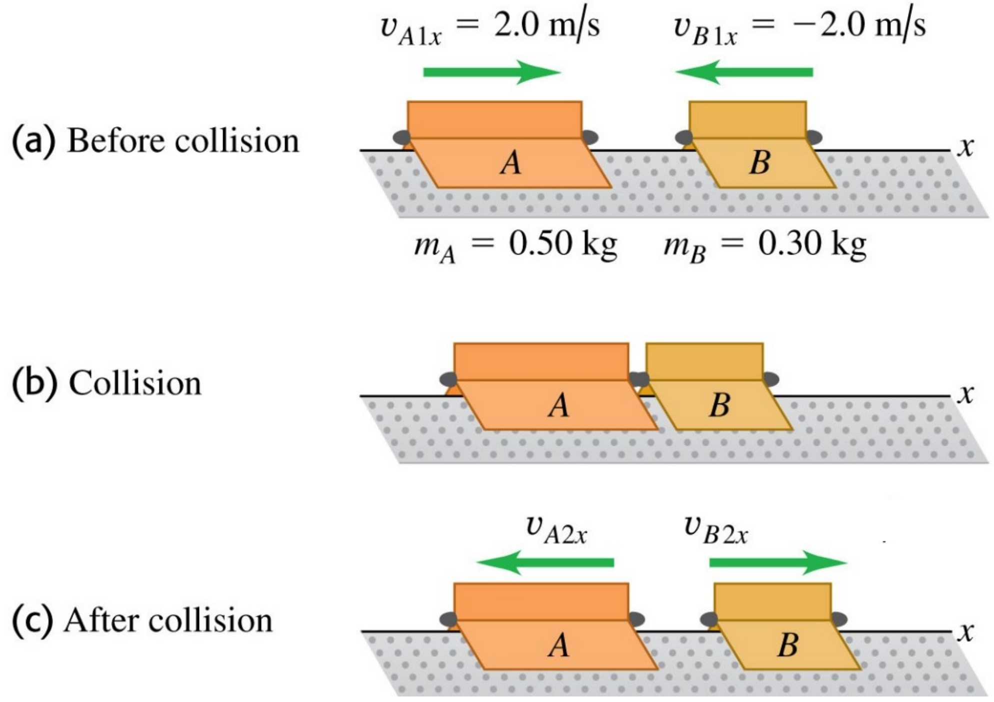



$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$(2\text{ m/s}) + v_{Af} = v_{Bf} + (-\,2\text{ m/s}) \qquad (4\text{ m/s}) + v_{Af} = v_{Bf}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(0.5\text{ kg})(2\text{ m/s}) + (0.3\text{ kg})(-\,2\text{m /s}) = (0.5\text{ kg})v_{Af} + (0.3\text{ kg})v_{Bf}$$

$$0.4\text{ kg*m/s} = (0.5\text{ kg})v_{Af} + (0.3\text{ kg})v_{Bf}$$

$$0.4\text{ kg*m/s} = (0.5\text{ kg})v_{Af} + (0.3\text{ kg})[(4\text{ m/s}) + v_{Af}]$$

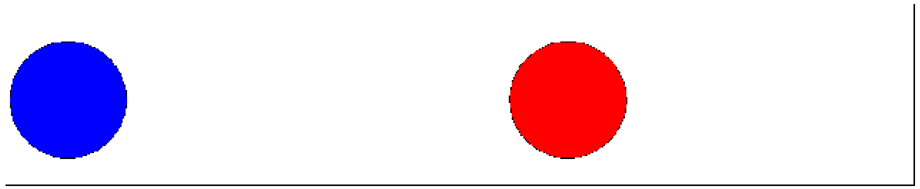
$$v_{Af} = -\,1.0\text{ m/s} \qquad v_{Bf} = +3.0\text{ m/s}$$

# Special case: One body initially at rest

$v_{1i} + v_{1f} = v_{2f} + \cancel{v_{2i}}$ $\quad$ $m_1 v_{1i} + \cancel{m_2 v_{2i}} = m_1 v_{1f} + m_2 v_{2f}$ $\quad$ Suppose $v_{2i} = 0$

$v_{1i} + v_{1f} = v_{2f}$ $\quad$ $m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$

If $m_1 = m_2$, $v_{1i} - v_{1f} = v_{2f}$, so $v_{1f} = 0$, and $v_{1i} = v_{2f}$

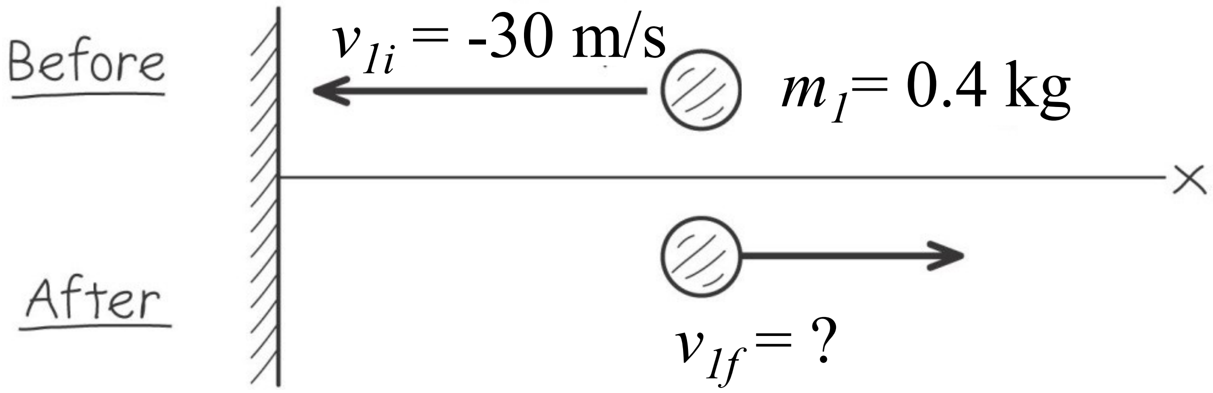


If $m_1 >> m_2$, $v_{2f} \sim 2v_{1i}$, $v_{1f} \sim v_{1i}$ $\quad$ Example: tennis ball colliding with a truck

If $m_1 << m_2$, $v_{2f} \sim 0$, $v_{1f} \sim - v_{1i}$ $\quad$ Example: A ball hit on a wall

# Example : elastic collision with one body at rest

You throw a ball with a mass of **0.4 kg** against a brick wall. It hits the wall moving horizontally to the left at **30 m/s** and rebounds horizontally to the right. Suppose that the collision is **elastic**. Find the final velocity of the ball.




$v_{1i}$ = -30 m/s $v_{1f}$ = ? m/s $m_1$ = 0.4 kg

$v_{2i}$ = 0 m/s $v_{2f}$ = 0 m/s $m_2$ = ?

If $m_1 << m_2$, $v_{2f} \sim 0$, $v_{1f} \sim - v_{1i}$

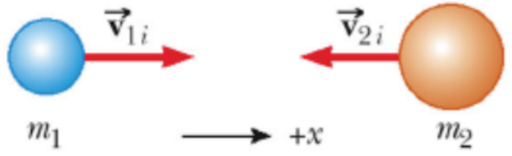
$$v_{1f} = +30 \text{ m/s}$$

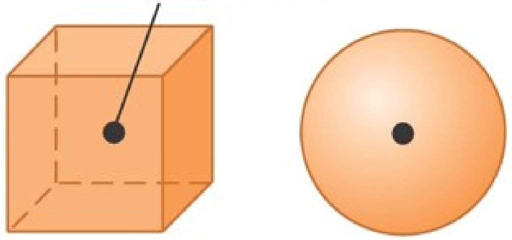
$$p_{1i} = mv_{1i} = (0.4 \text{ kg})(-30 \text{ m/s}) = -12 \text{ kg·m/s}$$

$$p_{1f} = mv_{1f} = (0.4 \text{ kg})(+30 \text{ m/s}) = +12 \text{ kg·m/s}$$

# Center of mass

- Another important concept is the **center of mass**.
- Suppose we have several particles with masses $m_1$, $m_2$, and so on.
- We define the center of mass of the system as the point at the position given by:




Position vectors of individual particles

Position vector of **center of mass** of a system of particles

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$$

Masses of individual particles

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

# Conservation of Momentum

- The center of mass of the cube in Figure at the right moves as though all the **mass were concentrated there**.

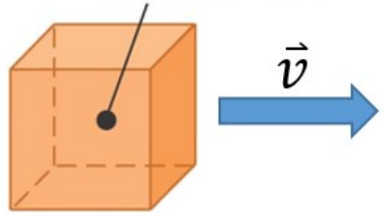
$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$v_{cm-x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x}}{m_1 + m_2 + m_3 + \cdots}$$

$$v_{cm-y} = \frac{m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y}}{m_1 + m_2 + m_3 + \cdots}$$

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$




The **total momentum** equals to the **total mass** times the **velocity of the center of mass**.

Total mass of a system of particles

Momenta of individual particles

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P}$$

Velocity of center of mass

**Total momentum of system**

No external force: $\sum \vec{F} = 0$

$\vec{v}_{cm}$ does not change, so $\vec{P}$ is conserved.

That is the total momentum is conserved.

# Example:

Find the center of mass for the four balls shown in the figure.

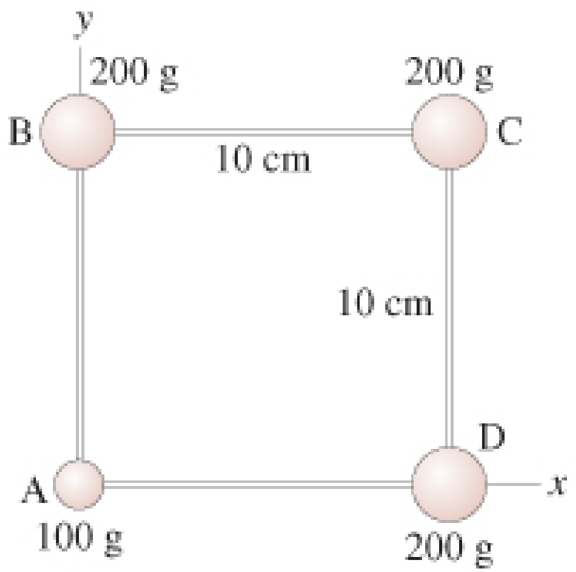
(A). $x_{cm} = 5$ cm, $y_{cm} = 5$ cm,

(B), $x_{cm} = 5.7$ cm, $y_{cm} = 5.7$ cm

(C), $x_{cm} = 5.7$ cm, $y_{cm} = 0$ cm

(D), $x_{cm} = 0$ cm, $y_{cm} = 5.7$ cm

(E), $x_{cm} = 0$ cm, $y_{cm} = 0$ cm




Ball A: (0 m, 0 m), 0.1 kg

Ball B: (0 m, 0.1 m), 0.2 kg

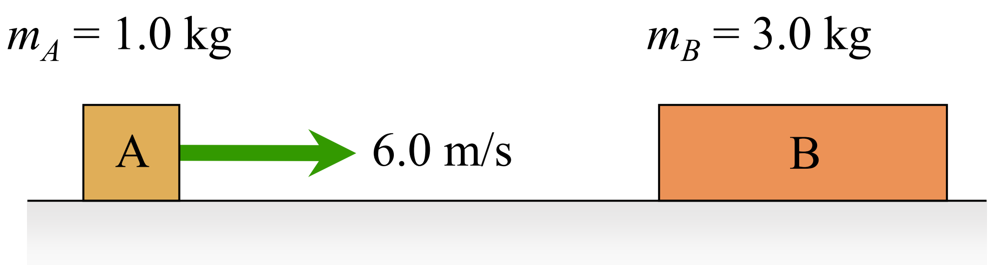
Ball C: (0.1 m, 0.1 m), 0.2 kg

Ball D: (0.1 m, 0.0 m), 0.2 kg

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

**Example:** Block *A* has mass 1.0 kg. Block *B* has mass 3.0 kg. Block *A* is initially moving to the right at 6.0 m/s, while block *B* is initially at rest. The surface they move on is level and frictionless. What is the velocity of the center of mass of the two blocks *after* the blocks collide?




(A). 6.00 m/s, to the right

(B), 3.00 m/s, to the right

(C), 1.50 m/s, to the right ✓

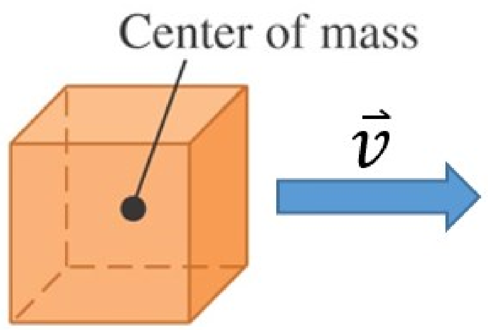
(D), zero

$$v_{\mathrm{cm}-x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$v_{cm-x} = \frac{(1.00 \times 6.00) + (3.00 \times 0)}{1.00 + 3.00} = 1.5\ m/s$$

# Center of mass: Newton's 2nd law

- The center of mass in Figure at the right moves as though all the mass were concentrated there.




Total mass of a system of particles

Momenta of individual particles

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P}$$

Velocity of center of mass

**Total momentum of system**

With external force:

$$M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt}$$

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots$$

$$\sum\vec{F} = \sum\vec{F}_{ext} + \sum\vec{F}_{int} = M\vec{a}_{cm}$$

# Summary of types of collisions

- **Elastic collision**: both momentum and kinetic energy are conserved

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

- **Completely inelastic collision**: objects stick together after collision

  Final velocities are the same;

  $$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

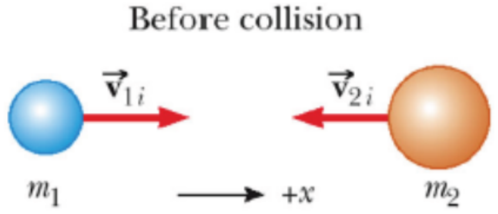
  Momentum is conserved; kinetic energy is not.

- **More general case**: inelastic collision.

  Momentum is conserved; kinetic energy is not.

  $$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

# Summary of problem-solving strategy




- **Set up a coordinate** axis: define the positive direction.
- **Plot a diagram**: label all known quantities (velocities and masses…) and unknown quantities.
- **Conservation of momentum**: before/after the collision.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

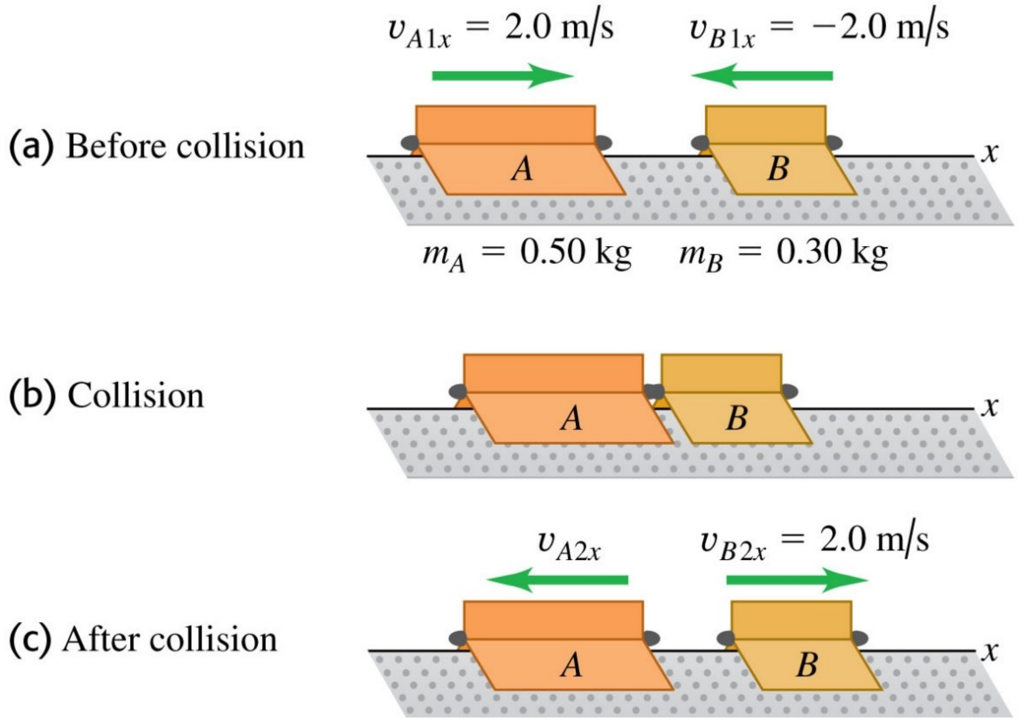
- **Confirm the type of collision**: elastic, completely inelastic or general inelastic?

  Conservation of kinetic energy? $v_{1i} + v_{1f} = v_{2f} + v_{2i}$

- **Analyze both *x* and *y*** directions if it is a 2D problem.
- Solve the problem.

# Example: colliding along a straight line

Two gliders with different masses move toward each other on a **frictionless** air track. After they collide, the final velocity of B is **2.0 m/s**. What is the final velocity of glider A?




**Step 1:** Coordinate: horizontal, right is positive $x$.

**Step 2:** Draw diagram.

**Step 3:** Conservation of momentum.

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

**Step 4:** Type of collision? Not sure.

Can NOT use conservation of kinetic energy.

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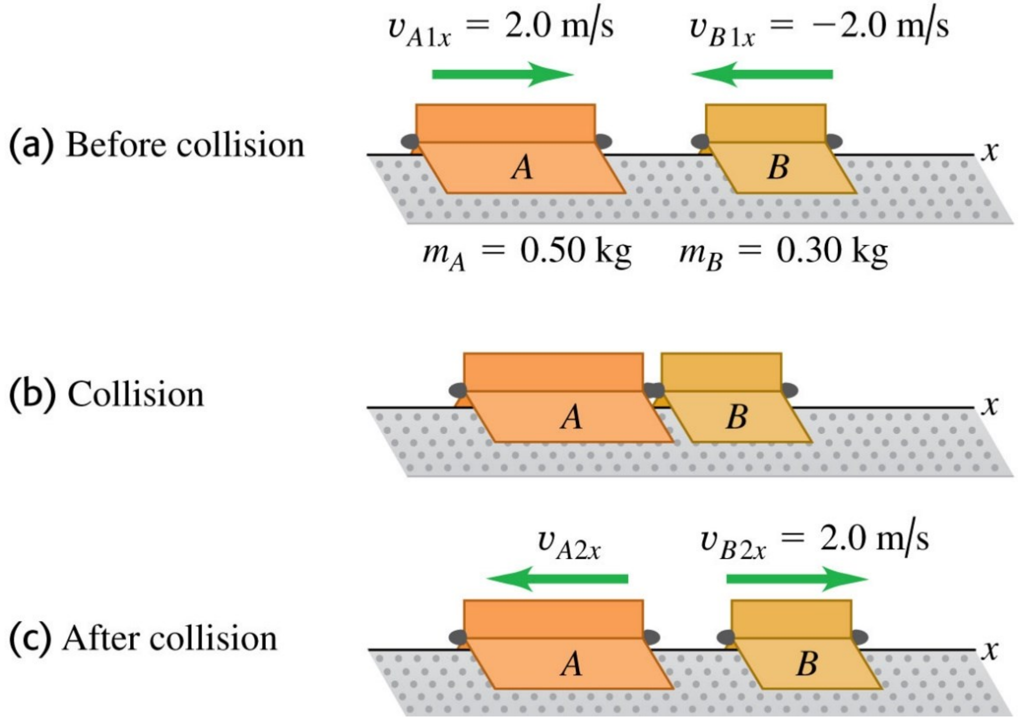



**Step 3:** Conservation of momentum.

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$(0.50\,\mathrm{kg})(2.0\,\mathrm{m/s}) + (0.30\,\mathrm{kg})(-2.0\,\mathrm{m/s}) = (0.50\,\mathrm{kg})v_{A2x} + (0.30\,\mathrm{kg})v_{B2x}$$

$$0.50 v_{A2x} + 0.30 v_{B2x} = 0.40\,\mathrm{m/s}$$

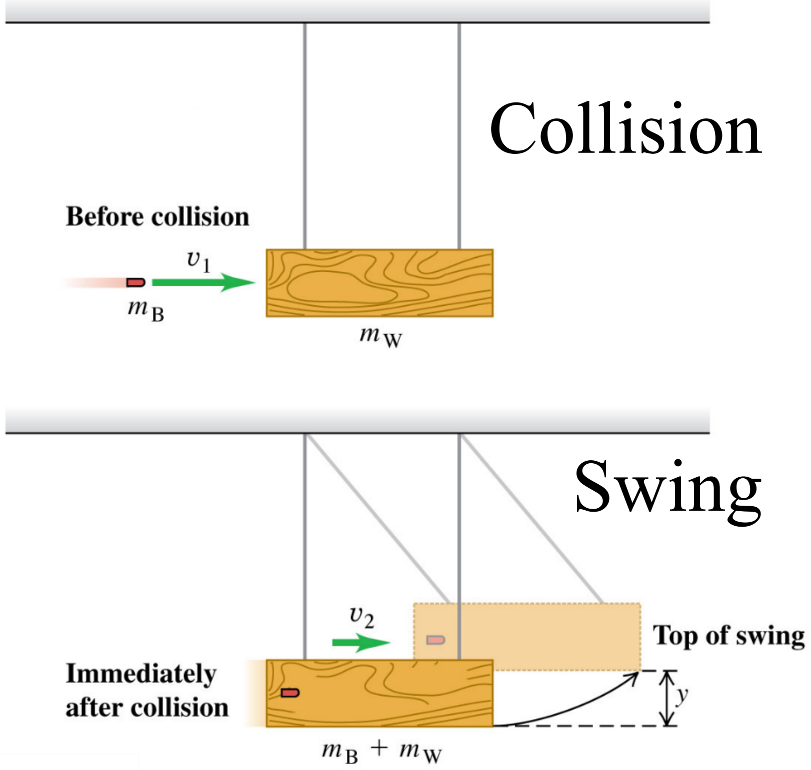
$v_{B2x} = 2.0$ m/s $\quad v_{A2x} = -0.4$ m/s $\quad$ Inelastic or elastic?

$K_i = 0.5(0.5\text{ kg})(2\text{ m/s})^2 + 0.5(0.3\text{ kg})\,(-2\text{ m/s})^2 = 1.6\text{ J}$

$K_f = 0.5(0.5\text{ kg})(-0.4\text{ m/s})^2 + 0.5(0.3\text{ kg})\,(2\text{ m/s})^2 = 0.64\text{ J}$

# Example: a ballistic pendulum

A bullet of mass $\boldsymbol{m_B}$ makes a completely inelastic collision with a block of wood of mass $\boldsymbol{m_W}$, which is suspended like a pendulum. After the impact, the block swings up to a maximum height $\boldsymbol{h}$. In terms of $\boldsymbol{h}$, $\boldsymbol{m_B}$, and $\boldsymbol{m_W}$, what is the initial speed $\boldsymbol{v_1}$ of the bullet?




**Step 1:** Coordinate? Collision: horizontal, right is positive *x*.

Swing: vertical, up is positive *y*.

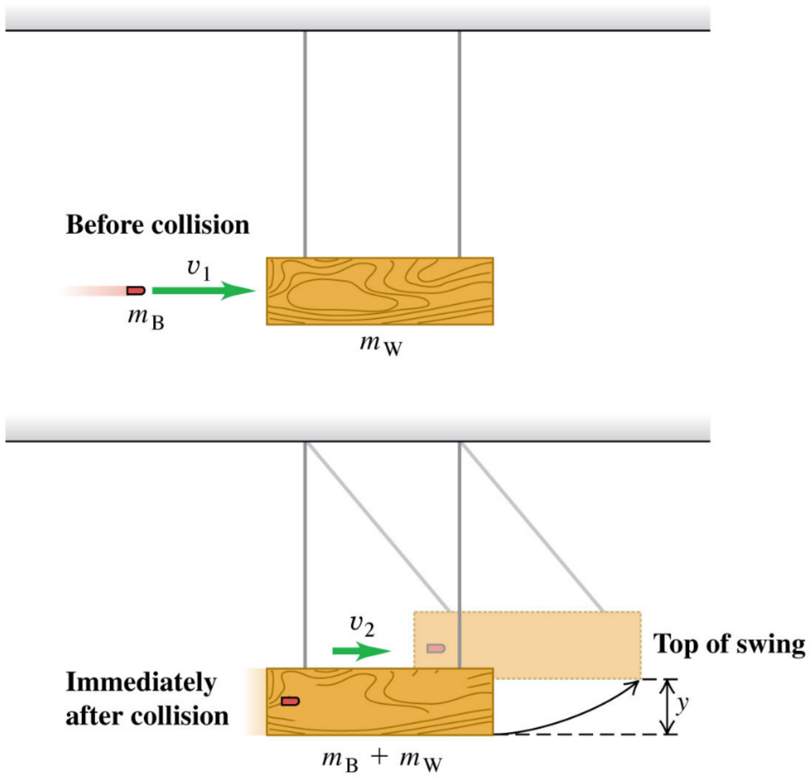
**Step 2:** Draw diagram.

**Step 3 and 4:** For collision, conservation of momentum, inelastic.

$$m_B v_1 = (m_B + m_W) v_2 \qquad v_1 = \frac{m_B + m_W}{m_B} v_2$$

# Example: a ballistic pendulum

A bullet of mass $m_B$ makes a completely inelastic collision with a block of wood of mass $m_W$, which is suspended like a pendulum. After the impact, the block swings up to a maximum height $h$. In terms of $h$, $m_B$, and $m_W$, what is the initial speed $v_1$ of the bullet?




**Step 3 and 4:** Conservation of momentum, inelastic.

$$m_B v_1 = (m_B + m_W) v_2 \qquad v_1 = \frac{m_B + m_W}{m_B} v_2$$

**Step 5:** Swing process: conservation of total mechanical energy.

$$\frac{1}{2}(m_B + m_W) v_2^2 = (m_B + m_W) g h \qquad v_2 = \sqrt{2gh}$$

$$v_1 = \frac{m_B + m_W}{m_B} \sqrt{2gh}$$

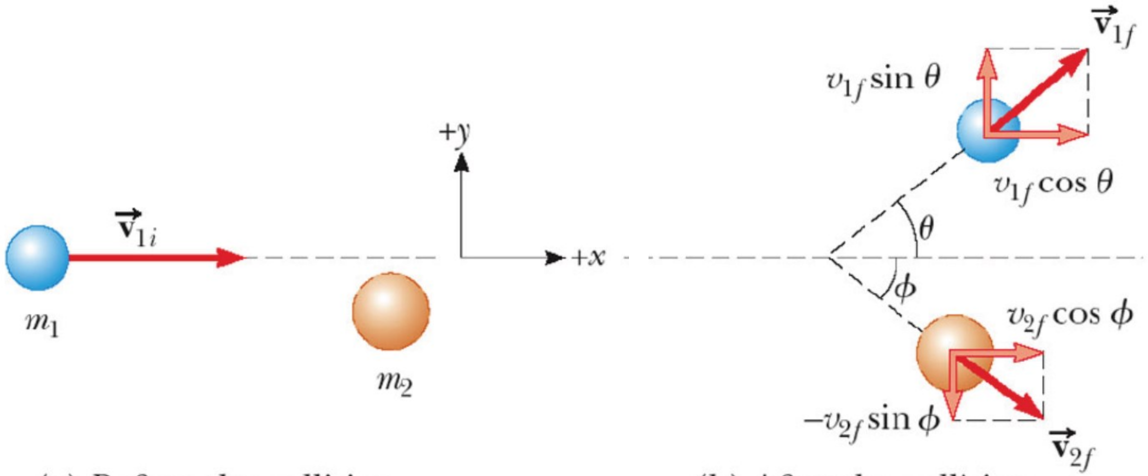
# Two-dimensional collision

- More general: two objects collide in two-dimensional.

Conservation of momentum: $m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$

Component form: $m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

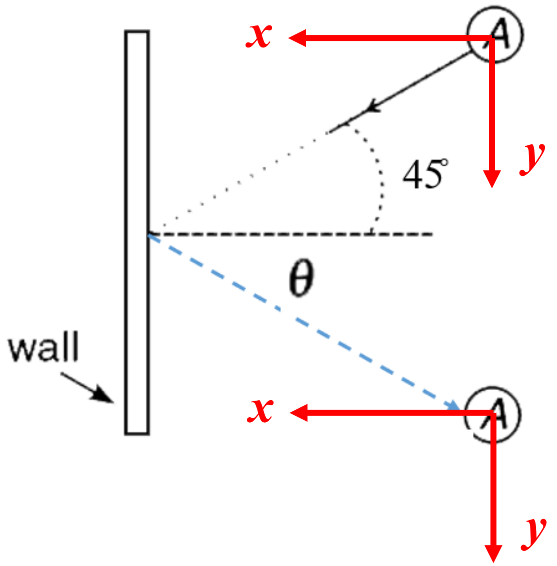


(a) Before the collision (b) After the collision

$v_{1i} + ~~v_{1f} = v_{2f}~~ + v_{2i}$

# Example : elastic collision with one body at rest

You throw a ball with a mass of **0.4 kg** against a brick wall. It hits the wall moving to the left at **30 m/s** (45 degrees below horizontal) and rebounds to the down right. Suppose that the collision is **elastic**. Find the angle ***θ*** and the impulse on the wall.




**Step 1 and 2:** Setup coordinate and draw diagram

**Step 3:** Conservation of momentum?

Along *y*, no external force, true. $p_{fy} = mv_{fy} = p_{iy}$

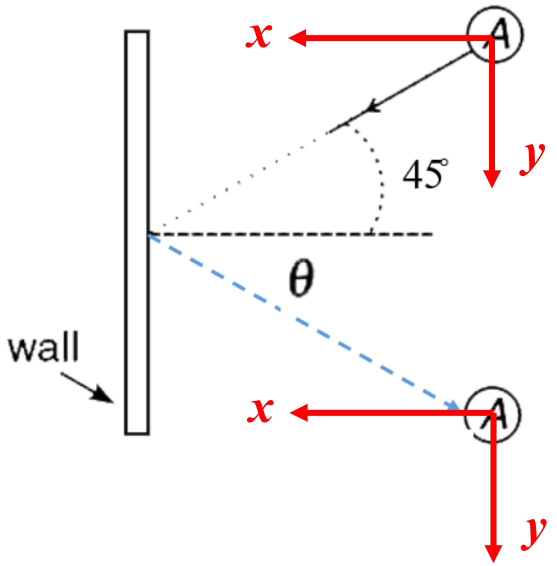
Along *x*, external force, not sure.

**Step 4:** Type of collision?

Elastic: conservation of kinetic energy $KE_f = KE_i$

# Example : elastic collision with one body at rest

You throw a ball with a mass of **0.4 kg** against a brick wall. It hits the wall moving to the left at **30 m/s** (45 degrees below horizontal) and rebounds to the down right. Suppose that the collision is **elastic**. Find the angle $\boldsymbol{\theta}$ and the impulse on the wall.




$$p_{ix} = mv_{ix} = (0.4\ \text{kg})(30\ \text{m/s})(cos45°) = +8.484\ \text{kg·m/s}$$

$$p_{iy} = mv_{iy} = (0.4\ \text{kg})(30\ \text{m/s})(sin45°) = +8.484\ \text{kg·m/s}$$

$$KE_i = 0.5\times(0.4\ \text{kg})\times(30\ \text{m/s})^2 = 180\ \text{J}$$

$$p_{fy} = mv_{fy} = p_{iy} = +8.484\ \text{kg·m/s}$$

$$v_{fy} = (+8.484\ \text{kg·m/s})/(0.4\ \text{kg}) = +21.21\ \text{m/s}$$

$$KE_f = KE_i = 180\ \text{J} = 0.5\times(0.4\ \text{kg})\times(+21.21\ \text{m/s})^2 + 0.5\times(0.4\ \text{kg})\times(v_{fx})^2$$

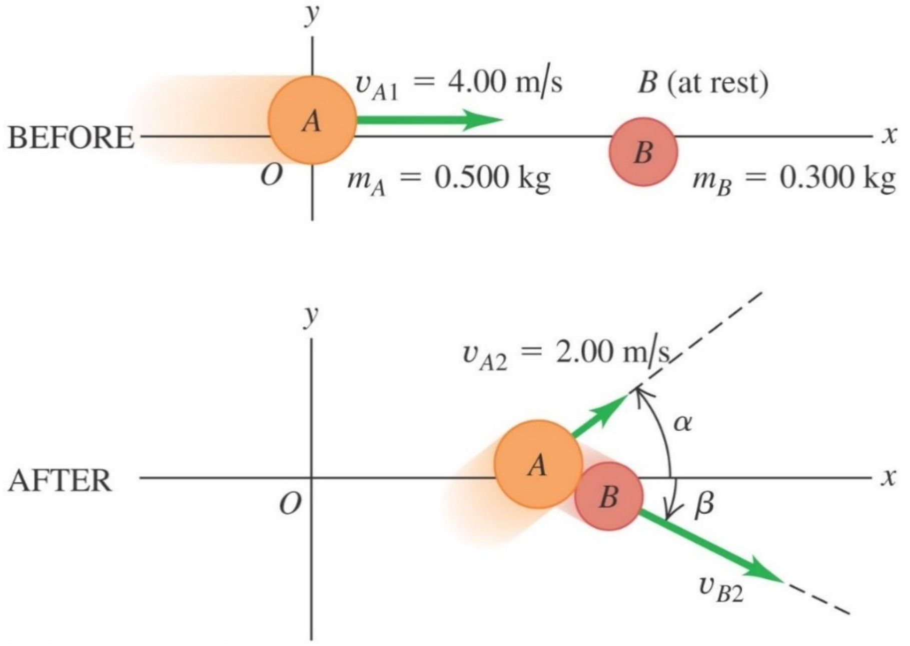
$$v_{fx} = -21.21\ \text{m/s}$$

$$\boldsymbol{\theta} = 45°$$

$$J_x = p_{fx} - p_{ix} = (-8.484) – (+8.484) = 16.968\ \text{kg·m/s}$$

# Example: two-dimensional elastic collision

The following figure shows an **elastic** collision of two pucks (masses $\boldsymbol{m_A}$ **= 0.5 kg** and $\boldsymbol{m_B}$ **= 0.3 kg**) on a **frictionless** air-hockey table. Puck A has an initial velocity of **4.0 m/s** in the positive *x*-direction and a final velocity of **2.0 m/s** in the unknown direction $\boldsymbol{\alpha}$. Puck B is initially **at rest**. Find the final speed $v_{B2}$ of puck B and the angles $\alpha$ and $\beta$




**Step 1 and 2:** Setup coordinate and draw diagram

**Step 3:** Conservation of momentum?

No external force, true for both *x* and *y*.

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$0 = m_A v_{A2y} + m_B v_{B2y}$$

**Step 4:** Type of collision?

Elastic: conservation of total kinetic energy

$$\frac{1}{2} m_A {v_{A1}}^2 = \frac{1}{2} m_A {v_{A2}}^2 + \frac{1}{2} m_B {v_{B2}}^2$$

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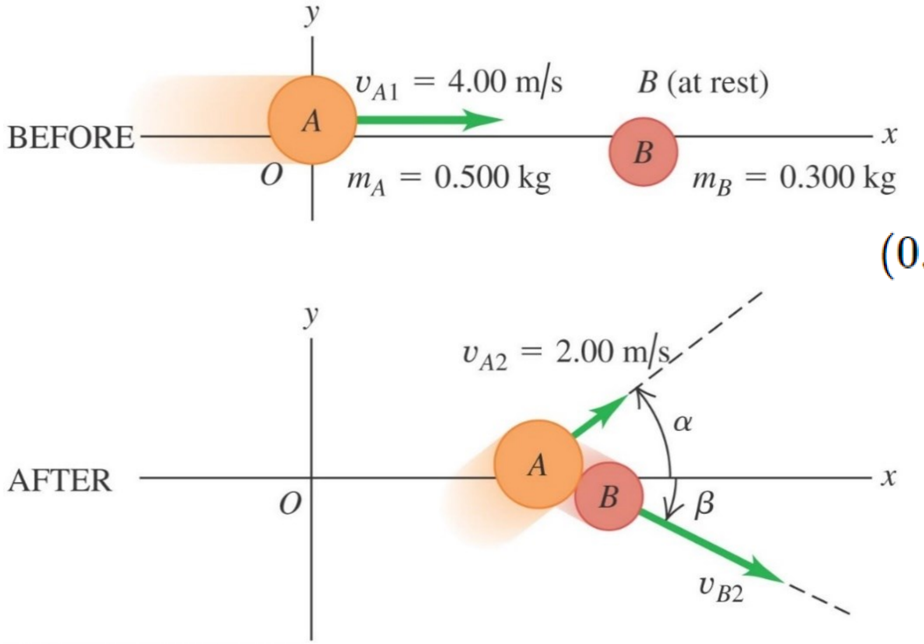


$$\frac{1}{2}m_A v_{A1}{}^2 = \frac{1}{2}m_A v_{A2}{}^2 + \frac{1}{2}m_B v_{B2}{}^2$$

$$v_{B2}{}^2 = \frac{m_A v_{A1}{}^2 - m_A v_{A2}{}^2}{m_B}$$

$$v_{B2}{}^2 = \frac{(0.500\,\text{kg})(4.0\,\text{m/s})^2 - (0.500\,\text{kg})(2.00\,\text{m/s})^2}{0.300\,\text{kg}} \qquad v_{B2} = 4.47\,\text{m/s}$$

Conservation of the $x$- and $y$-components of total momentum gives

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$(0.500\,\text{kg})(4.00\,\text{m/s}) = (0.500\,\text{kg})(2.00\text{m/s})(\cos\alpha) + (0.300\,\text{kg})(4.47\,\text{m/s})(\cos\beta)$$

Get, $\cos\alpha = 2 - 1.341\cos\beta$ (1)

$$0 = m_A v_{A2y} + m_B v_{B2y}$$

$$0 = (0.500\,\text{kg})(2.00\,\text{m/s})(\sin\alpha) - (0.300\,\text{kg})(4.47\,\text{m/s})(\sin\beta)$$

So, $\sin\alpha = 1.341\sin\beta$ (2)

$\sin^2\alpha + \cos^2\alpha = 1$ So, $(1.341\sin\beta)^2 + (2 - 1.341\cos\beta)^2 = 1$

$(1.341)^2(\sin^2\beta + \cos^2\beta) + 4 - 4\times1.341\cos\beta = 1$ so, $\cos\beta = 0.8945$

Because (1), $\cos\alpha = 2 - 1.341\cos\beta = 0.8$ $\alpha = 36.9^\circ$ $\beta = 26.6^\circ$