Quiz

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t}$$

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Comparison between rotational and linear equations

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

Example:

A disc is slowing down **to stop**. The **slowing down** process takes **10 s**, and the disc makes **100 full revolutions**. Find the magnitude of the **initial angular velocity** of the disc. Suppose the slowing down has a constant angular acceleration.

A. ✓ $\omega_0 = 40\pi$ rad/s,

B. $\omega_0 = 20\pi$ rad/s,

C. $\omega_0 = 10\pi$ rad/s,

D. $\omega_0 = 100\pi$ rad/s,

E. Not enough info to calculate.

$\alpha_z =$ constant
$\omega_z = \omega_{0z} + \alpha_z t$
$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2$
$\omega_z^{\ 2} = \omega_{0z}^{\ 2} + 2\alpha_z(\theta - \theta_0)$
$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

100 revolutions = $200\ \pi$

$\theta - \theta_0 = 200\ \pi$

$\omega_z = 0$ rad/s $t = 10$ s

$\omega_0 = ?$

$200\ \pi = 0.5\ (0 + \omega_0)(10\ \text{s})$

$\omega_0 = 40\pi$ rad/s

Relating linear and angular kinematics

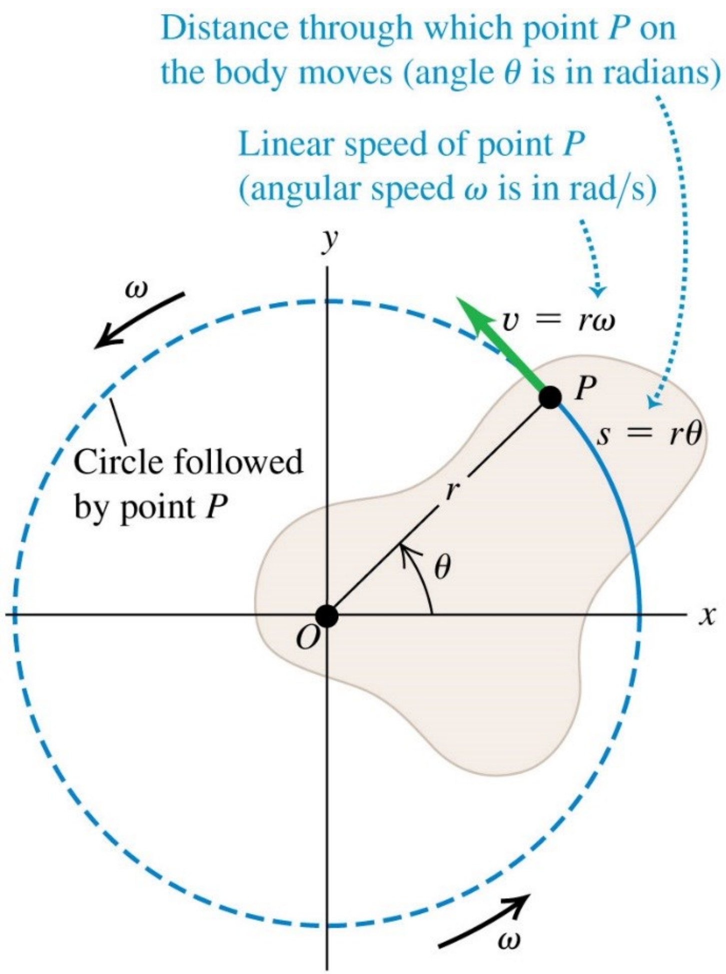
- The **linear velocity** is always tangent to the circular path called the **tangential velocity**.
- The tangential velocity vs angular velocity

$$v = \frac{\Delta s}{\Delta t} \qquad \Delta s = r\Delta\theta \qquad v = \frac{\Delta\theta}{\Delta t} r = \omega r \qquad \boxed{\omega = \frac{v}{r}}$$

- The **tangential acceleration** vs angular acceleration

$$a_{tan} = \frac{\Delta v}{\Delta t} = \frac{\Delta\omega}{\Delta t} r = \alpha r \qquad \boxed{a_{tan} = \alpha r}$$

Any other acceleration?



Relating linear and angular kinematics

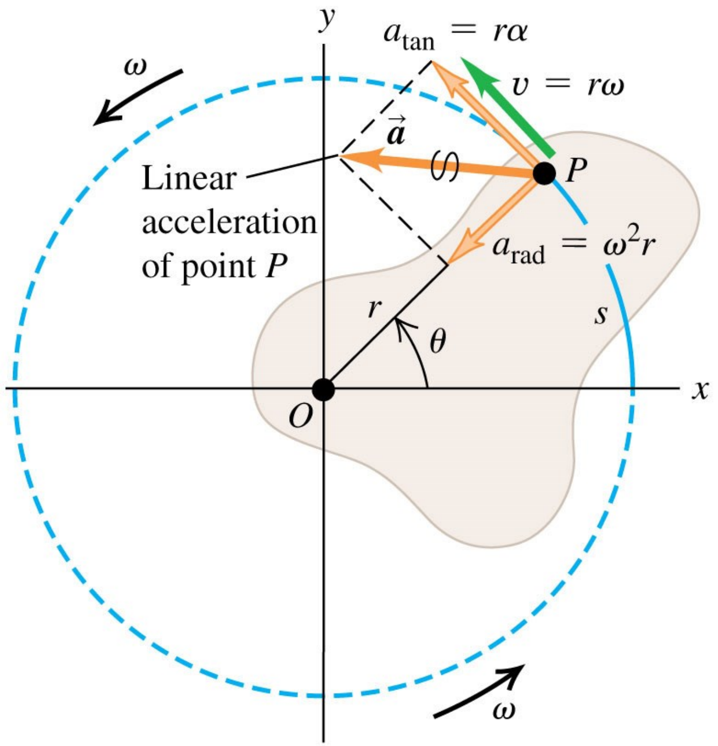
- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration.
  -- its centripetal (radial) acceleration is

$$v = \omega r \qquad a_{rad} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \qquad \boxed{a_{rad} = r\omega^2}$$

- Total acceleration: $\vec{a} = \vec{a}_{tan} + \vec{a}_{rad}$ $\qquad a_{tan} = \alpha r$

$$|\vec{a}| = \sqrt{a_{tan}^2 + a_{rad}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

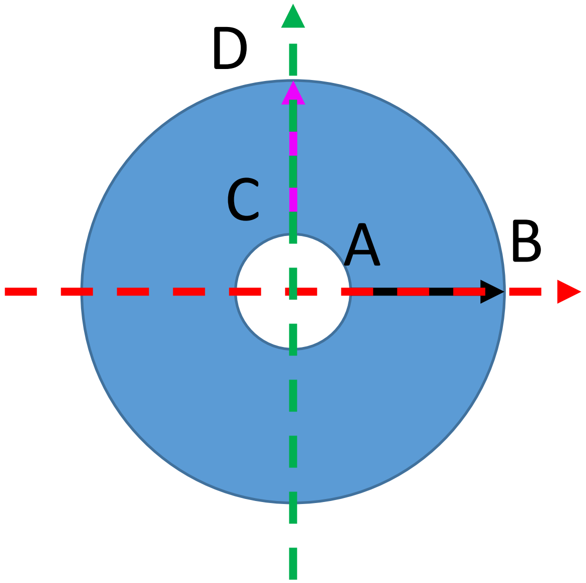
Radial and tangential acceleration components:
- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).



Relating linear and angular kinematics

- **Every point** on the rotating object has the **same angular motion** (angular displacement, angular velocity, angular acceleration)

- Every point on the rotating object does **NOT** have the same linear motion.

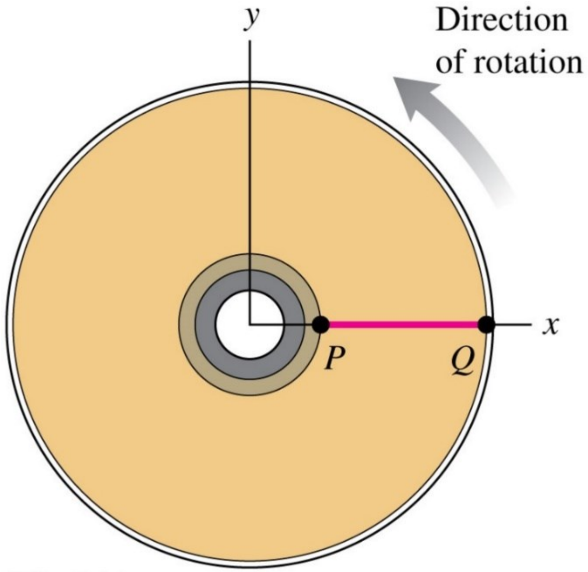


Displacement: $s_t = \theta r$ Velocity: $v_t = \omega r$

Acceleration: $a_t = \alpha r$ $a_{rad} = \frac{v^2}{r} = r\omega^2$

Example:

A DVD is rotating with an ever-increasing speed. How do the centripetal acceleration a_{rad} and tangential acceleration $a_{\tan}$ compare at points P and Q?



(A). P and Q have the same a_{rad} and $a_{\tan}$.

(B). Q has a greater a_{rad} and a greater $a_{\tan}$ than P. ✓

(C). Q has a smaller a_{rad} and a greater $a_{\tan}$ than P.

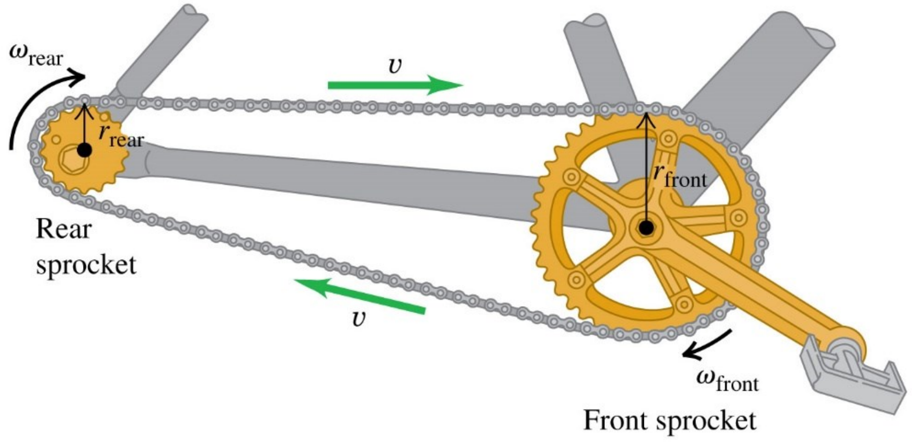
(D). P and Q have the same a_{rad}, but Q has a greater $a_{\tan}$ than P.

$$a_t = r\alpha$$

$$a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

Example:

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



(A). a faster linear speed and a faster angular speed.

(B). the same linear speed and a faster angular speed.

(C). a slower linear speed and the same angular speed.

✓(D). the same linear speed and a slower angular speed.

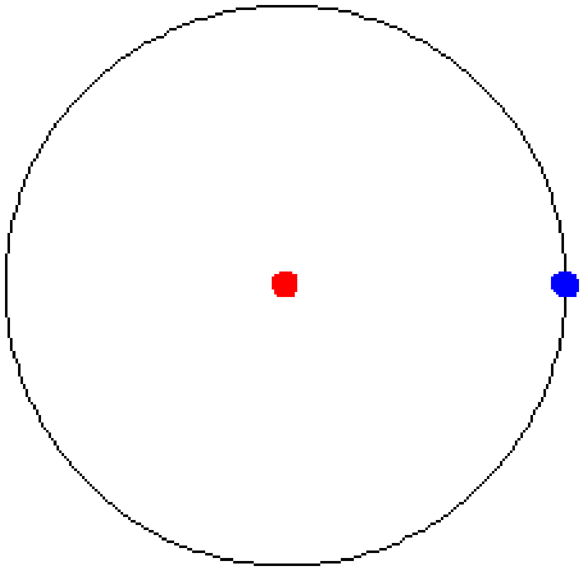
(E). none of the above

$$\omega = \frac{v}{r} \quad or \quad v = r\omega$$

Rotational kinetic energy

- The linear velocity (of one particle) is always tangent to the circular path: tangential velocity.
- The magnitude is defined by the **tangential velocity**

$$\omega = \frac{v}{r} \quad \text{or} \quad v = \omega r$$



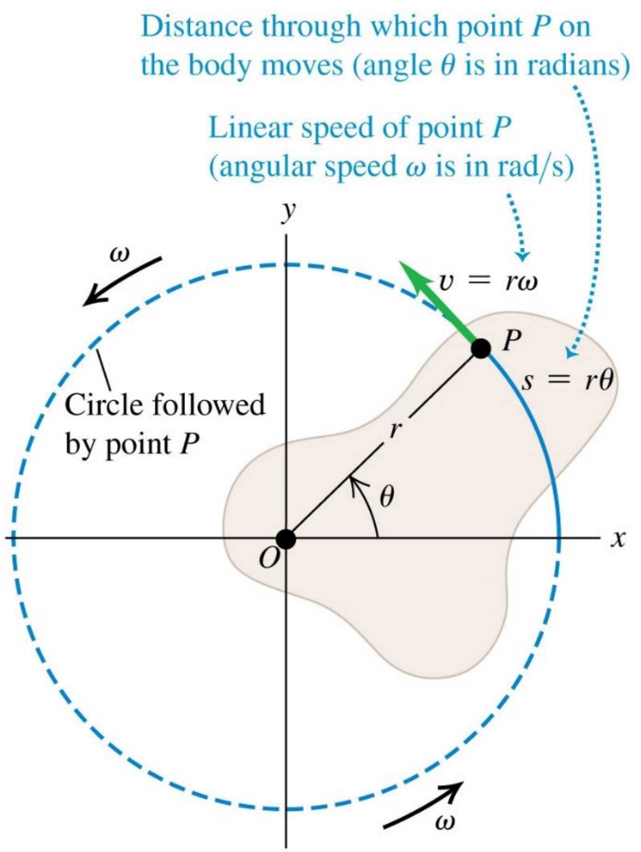
- Kinetic energy (of one particle): $K = \frac{1}{2}mv^2$

Rotational kinetic energy

- The linear velocity (of one particle) is always tangent to the circular path: tangential velocity.
- The magnitude is defined by the tangential velocity

$$\omega = \frac{v}{r} \quad \text{or} \quad v = \omega r$$

- Kinetic energy (of one particle): $K = \frac{1}{2}mv^2$
- An "**continuous**" object rotating about an axis with an angular speed ω, has **rotational kinetic energy**.
- Each particle has a kinetic energy of $K_i = \frac{1}{2}m_i v_i^2$
- Since $v = r\omega$, for each particle: $K_i = \frac{1}{2}m_i r_i^2 \omega_i^2$



Rotational kinetic energy

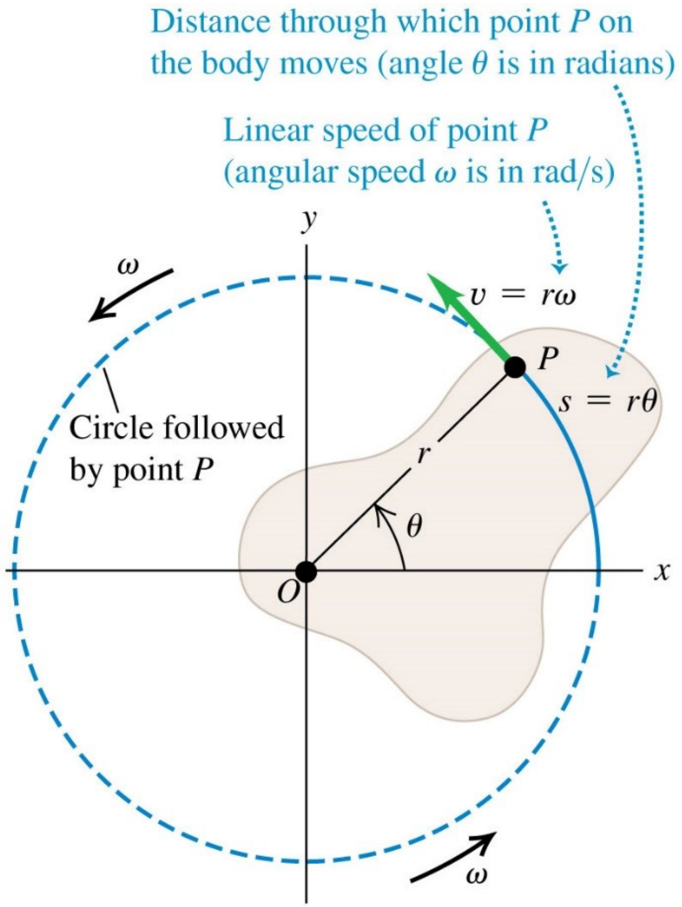
- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega_i^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

- We can rewrite this equation:

$$K_R = \frac{1}{2} \left(\sum_i m_i r_i^2\right) \omega^2 = \frac{1}{2} I \omega^2$$

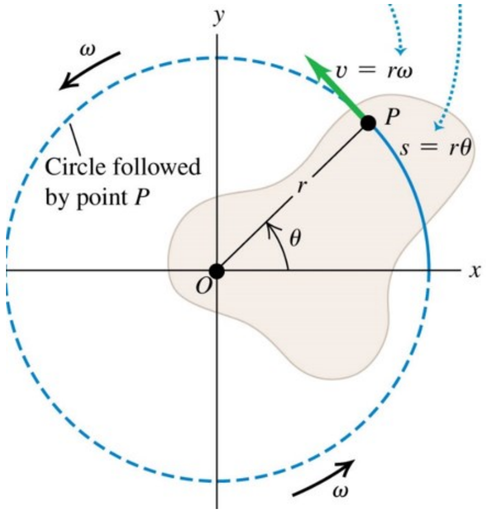
- Compare it with $K = \frac{1}{2} m v^2$.
- ***I*** is called the **moment of inertia**. $\boldsymbol{I = \sum_i m_i r_i^2}$



Rotational kinetic energy

- There is an analogy between the kinetic energies associated with linear motion and the kinetic energy associated with rotational motion.

$$K = \frac{1}{2}mv^2 \qquad \text{vs} \qquad K_R = \frac{1}{2}I\omega^2$$



- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object.

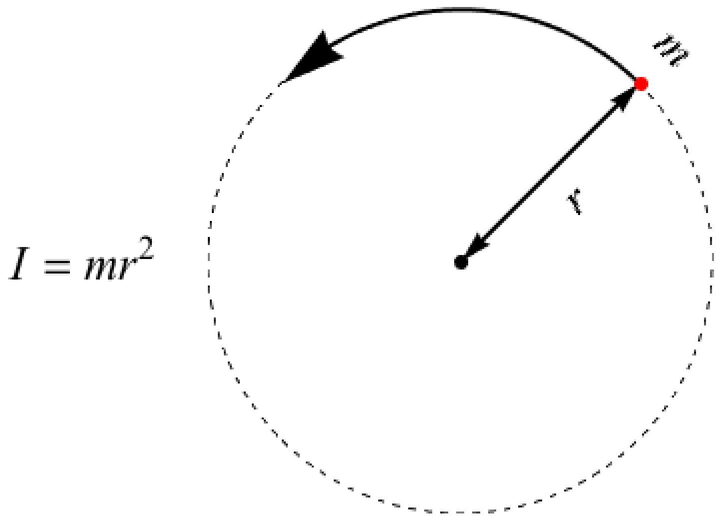
- Units of rotational kinetic energy are Joules (J).

Moment of Inertia of Point Mass

- For a single particle, the definition of moment of inertia is

$$I = mr^2$$

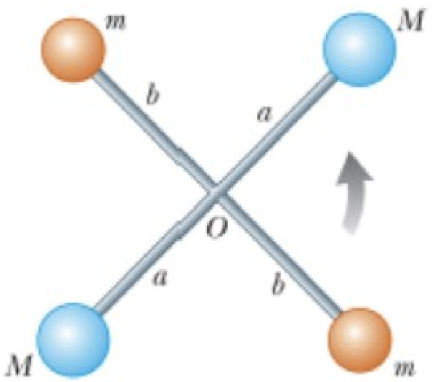
m is the mass of the single particle
r is the rotational radius



- SI unites of moment of inertia are $kg.m^2$

- Moment of inertia and mass of an object are different quantities.
- Moment of inertia depends on both the quantity of matter and its distribution (through the r^2 term)

Moment of Inertia of Point Mass



- For a composite particle, the definition of moment of inertia is

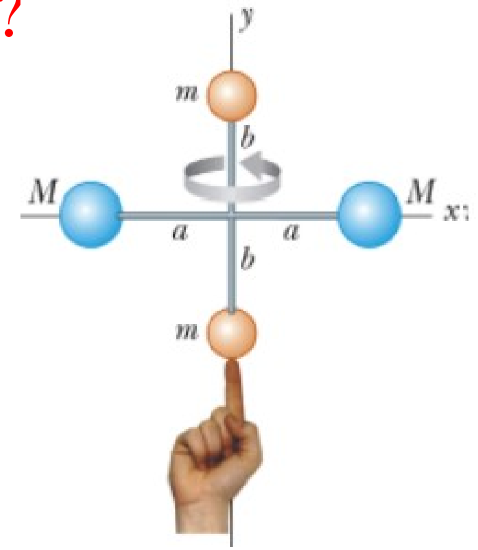
$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + \ldots$$

m_i is the mass of the *i*th single particle

r_i is the rotational radius of *i*th particle.

- The moment of inertia *I* about an *O* axis perpendicular to the page?

$$I = \sum_i m_i r_i^2 = mb^2 + Ma^2 + mb^2 + Ma^2 = 2mb^2 + 2Ma^2$$

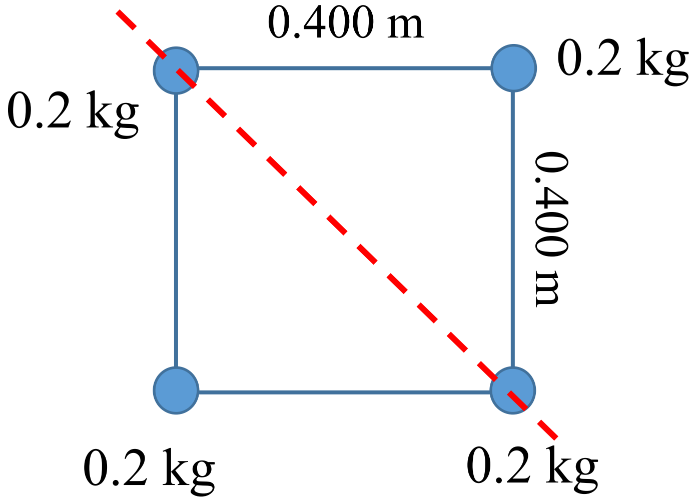


- The moment of inertia *I* about the *y* axis is?

$$I = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$$

Example:

Four small spheres, each of which you can regard as a point of mass 0.2 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods. Find the moment of inertia of the system about an axis that passes through the centers of the upper left and lower right spheres and through point *O*.



A. $I = 0.032$ kg.m^2 ✓

B. $I = 0.016$ kg.m^2

C. $I = 0.064$ kg.m^2

D. $I = 0.320$ kg.m^2

E. $I = 0.160$ kg.m^2

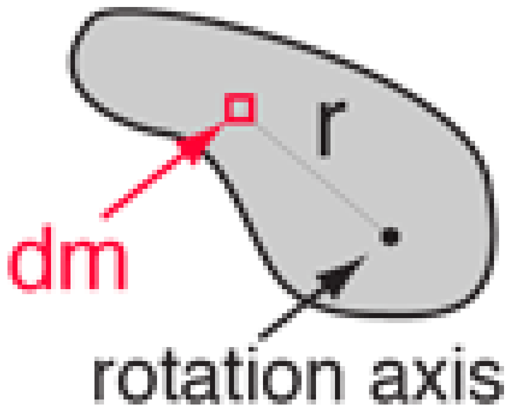
$$r = 0.2828 \text{ m}$$

$$I = \sum m_i r_i^2 = 2(0.200 \text{ kg})(0.2828 \text{ m})^2$$

$$I = 0.0320 \text{ kg} \cdot \text{m}^2$$

Moment of Inertia of Extended objects

- Divided the extended objects into many small volume elements, each of mass Δm_i.
- We can rewrite the expression for I in terms of Δm



$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

- With the small volume segment assumption, $dm = \rho dV$

$$I = \int \rho r^2 dV$$

- If is ρ constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known.

Example: Uniform Rigid Rod

- The shaded area has a mass d$m = \rho\, dx$, $\rho = M/L$.

$$dm = \rho dx = (M/L)dx$$

- The moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{M}{L}\left(\frac{1}{3}x^3 \Big|_{-\frac{L}{2}}^{+\frac{L}{2}}\right) = \frac{M}{3L}\left((+\frac{L}{2})^3 - (-\frac{L}{2})^3\right)$$

$$= \frac{M}{3L} 2(\frac{L^3}{8}) = \frac{1}{12} M L^2$$

$$I = \frac{1}{12} M L^2$$

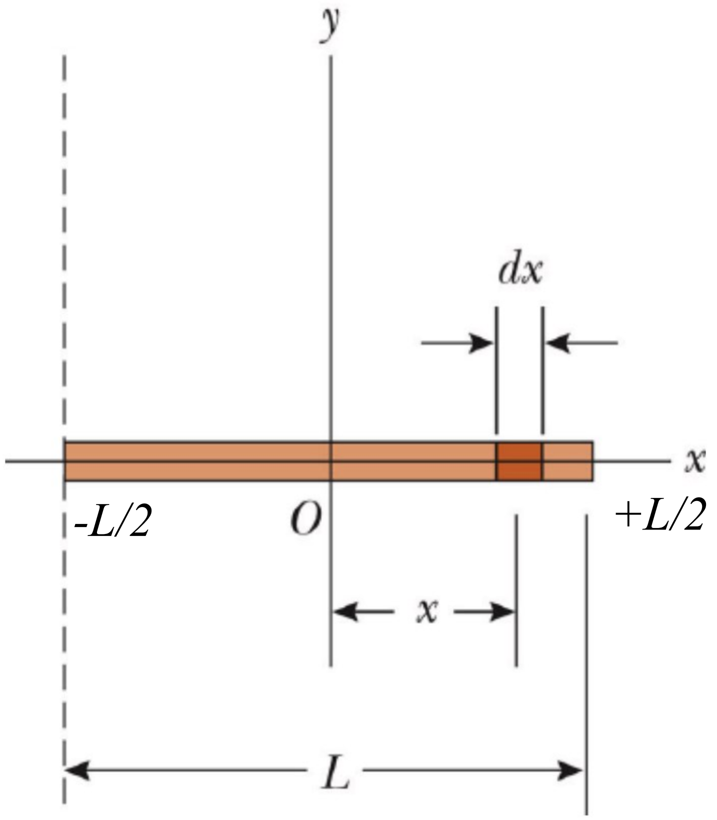


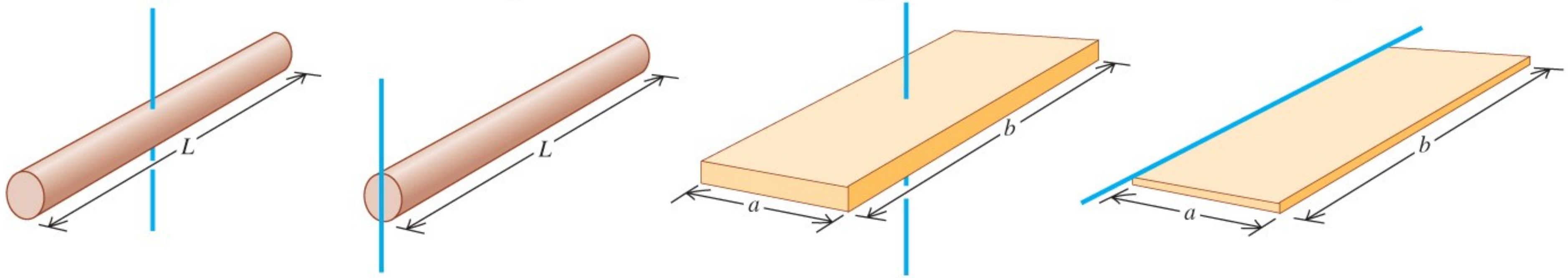
TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center: $I = \frac{1}{12}ML^2$

(b) Slender rod, axis through one end: $I = \frac{1}{3}ML^2$

(c) Rectangular plate, axis through center: $I = \frac{1}{12}M(a^2 + b^2)$

(d) Thin rectangular plate, axis along edge: $I = \frac{1}{3}Ma^2$



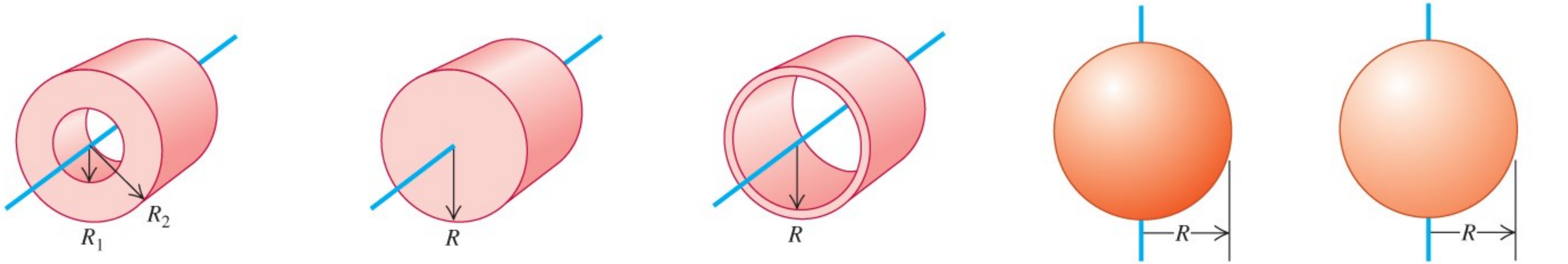
(e) Hollow cylinder: $I = \frac{1}{2}M(R_1^2 + R_2^2)$

(f) Solid cylinder: $I = \frac{1}{2}MR^2$

(g) Thin-walled hollow cylinder: $I = MR^2$

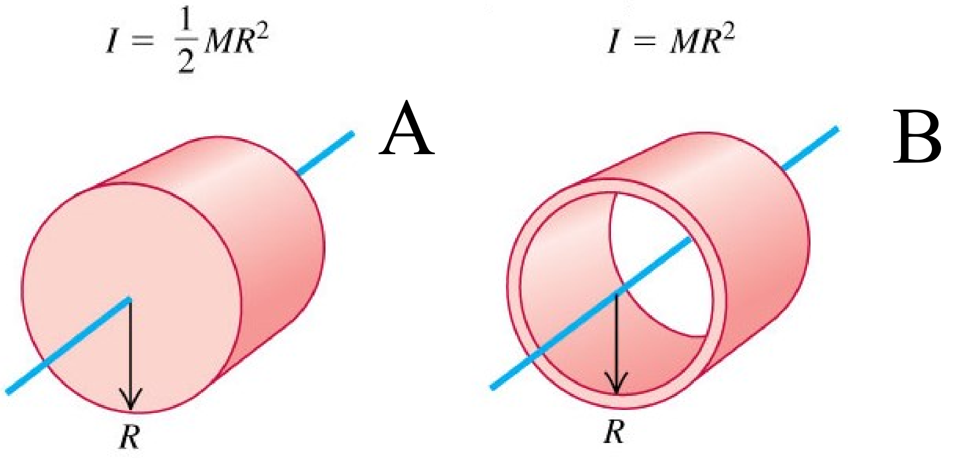
(h) Solid sphere: $I = \frac{2}{5}MR^2$

(i) Thin-walled hollow sphere: $I = \frac{2}{3}MR^2$



Example:

The two objects shown here all have the same mass M = 10 kg and radius R= 1 m. Each object is rotating about its axis of symmetry (shown in blue) with an angular velocity ω = 2 rad/s. What are their rotational kinetic energy?



A. $K_A = 20$ J, $K_B = 20$ J

B. $K_A = 10$ J, $K_B = 10$ J

C. $K_A = 10$ J, $K_B = 20$ J ✓

D. $K_A = 20$ J, $K_B = 10$ J

$$K_R = \frac{1}{2}I\omega^2$$

$$I_A = \frac{1}{2}MR^2 = \frac{1}{2}(10kg)(1m)^2 = 5$$

$$K_A = \frac{1}{2}I\omega^2 = \frac{1}{2}(5)(2)^2 = 10$$

$$I_B = MR^2 = (10kg)(1m)^2 = 10$$

$$K_B = \frac{1}{2}I\omega^2 = \frac{1}{2}(10)(2)^2 = 20$$

Example:

You want to double the radius of a rotating solid cylinder while keeping its kinetic energy constant. (The mass does not change.) To do this, the final angular velocity of the cylinder must be

$$K_R = \frac{1}{2} I\omega^2$$

(A) 4 times its initial value.

(B) twice its initial value.

(C) the same as its initial value.

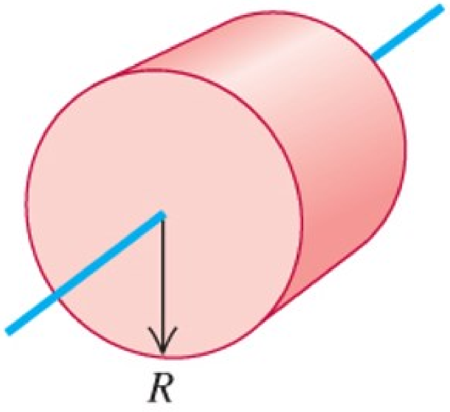
(D) 1/2 of its initial value. ✓

(E) 1/4 of its initial value.

$$I_f = \frac{1}{2} M(2R)^2 = 4I$$

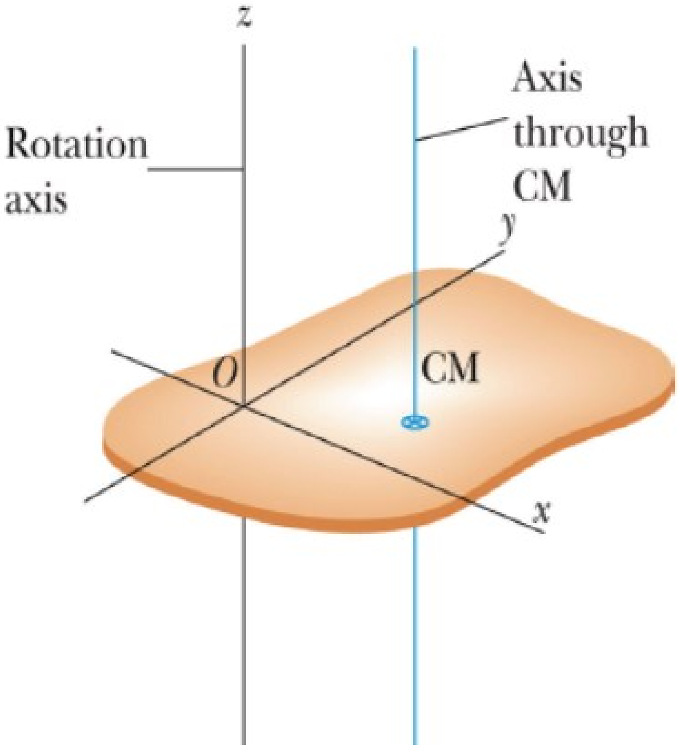
$$K_f = \frac{1}{2}(4I)\left(\frac{\omega}{2}\right)^2 = \frac{1}{2} I\omega^2$$

$I = \frac{1}{2}MR^2$



Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the **parallel-axis theorem** often simplifies calculations.



- The theorem states $I = I_{CM} + MD^2$

  I is about any axis parallel to the axis through the center of mass of the object

  I_{CM} is about the axis through the center of mass

  D is the distance from the center of mass axis to the arbitrary axis.

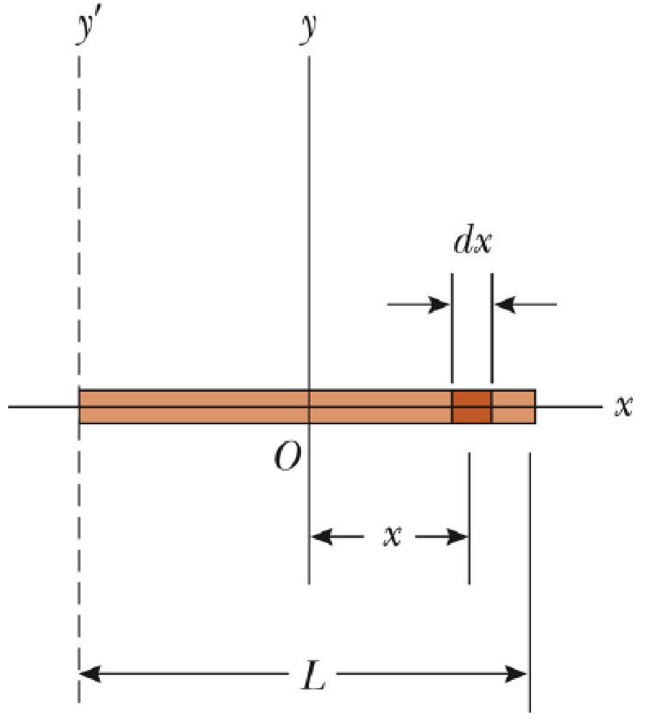
Example: Parallel-axis theorem for Rigid Rod

- The moment of inertia about *y* axis is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{1}{12} ML^2$$

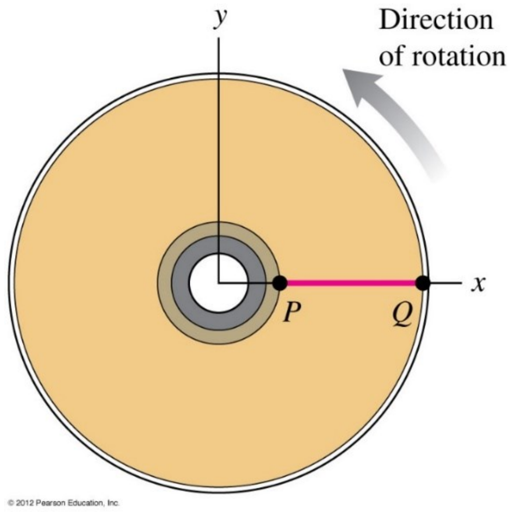
- Using the parallel-axis theorem, the moment of inertia about *y'* is

$$I_{y'} = I_{CM} + MD^2 \ = \frac{1}{12} ML^2 + M(\frac{L}{2})^2 = \frac{1}{3} ML^2$$



Example

Constant angular acceleration π rad/s^2, point P is 0.5 m from the rotational axis and initial at rest. What is the linear speed of point P when it complete one revolution?



$\alpha = \pi \quad \omega_0 = 0 \quad \theta\text{-}\theta_0 = 2\pi \quad \omega_z?$

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$$

$$= 0 + 2\pi(2\pi)$$
$$= 4\pi^2$$

$$\omega_z = 2\pi$$

$$v = r\omega = (0.5)(2\pi) = 3.14 \text{ m/s}$$

$\alpha_z =$ constant
$\omega_z = \omega_{0z} + \alpha_z t$
$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2$
$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

Example

A disk with radius R = 1.0 m. is spinning about its center. Initially the disc has an angular velocity of 100 rev/min, and is slowing down uniformly at a rate of 2.0 rad/s^2. By the time it stops spinning, the total number of revolutions the disk will make is?

$\alpha_z =$ constant
$\omega_z = \omega_{0z} + \alpha_z t$
$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2$
$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

θ-θ_0 = ?

$$\omega_0 = 100\frac{rev}{min} = \frac{100(2\pi)}{60\ s} = 10.47\ \text{rad/s}$$

α = -2.0 rad/s^2 $\quad$ $\omega_z = 0$

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad 0 = 10.47^2 - 2(2)\Delta\theta$$

$\Delta\theta = 27.4$ rad $\quad$ $\Delta\theta = 4.36$ rev