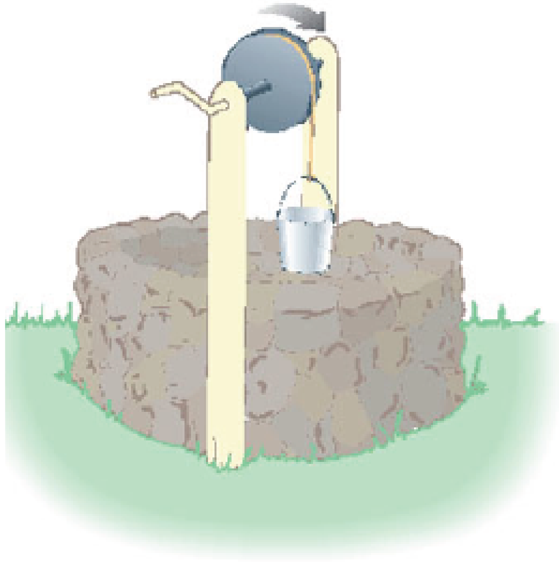
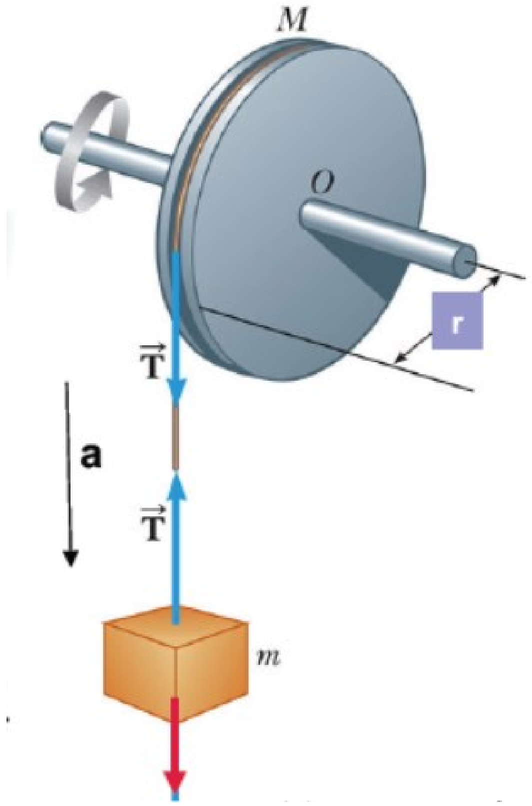
Example: Newton's second law for rotation

- A solid, frictionless cylindrical reel of mass $M = 2.5$ kg and radius $R = 0.2$ m is used to draw water from a well. A bucket of mass $m = 1.2$ kg is attached to a cord that is wrapped around the cylinder.

(a) Find the tension T in the cord and acceleration a of the bucket.

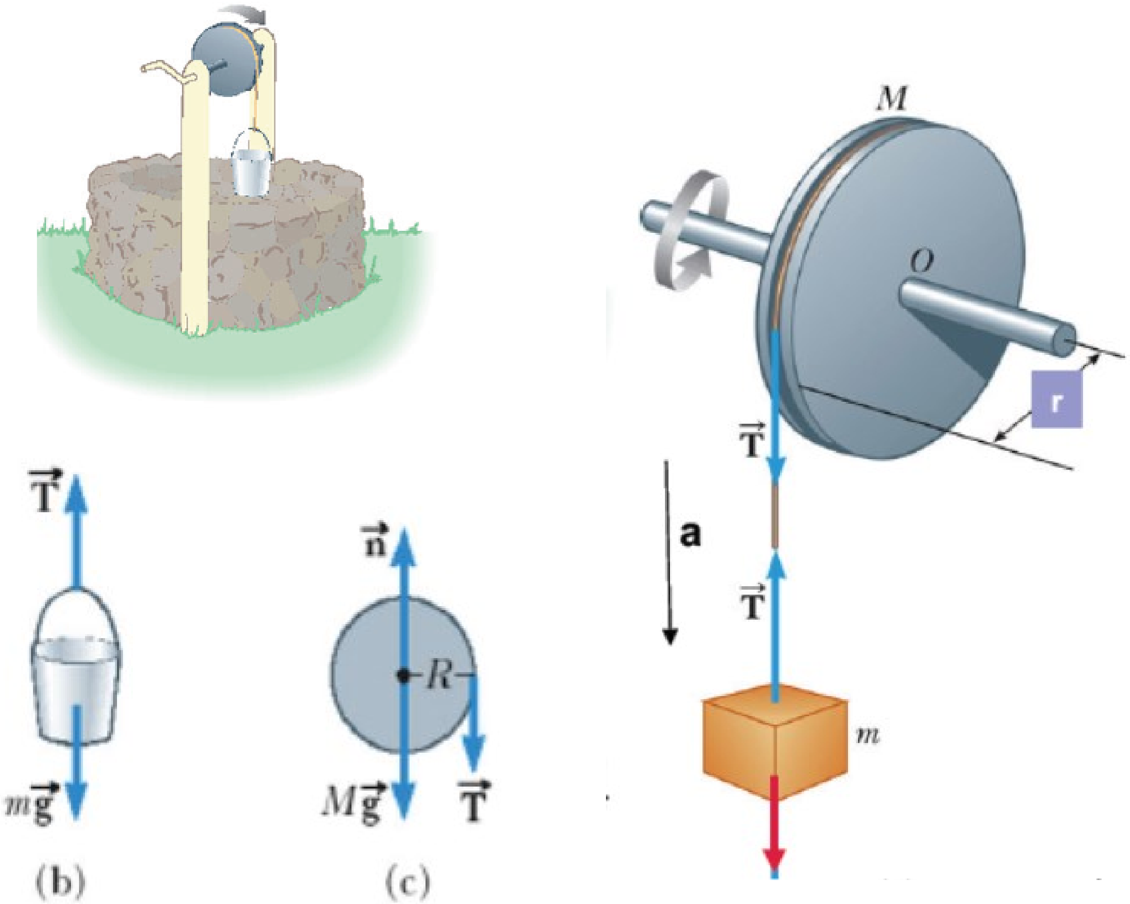
(b) If the bucket starts from rest at the top of the well and falls for 3.0 s before hitting the water, how far does it fall?





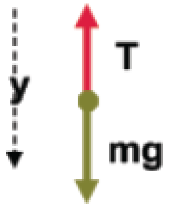
Example: Newton's second law for rotation

- Draw free body diagrams of each object
- Only the cylinder is rotating, so apply $\sum \tau = Tr = I\alpha_z$
- The bucket is falling, but not rotating, so apply $\sum F = mg - T = ma$
- Remember that $a = \alpha r$ and solve the resulting equations.



(A) Find the tension T and acceleration a

For mass m:

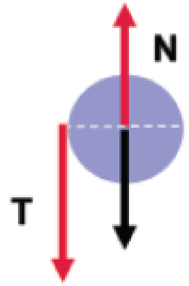


$$\sum F_y = ma = mg - T$$

$$\boxed{T = m(g-a)}$$

Unknowns: T, a

FBD for disk, with axis at "o":



$$\sum \tau_0 = +Tr = I\alpha \qquad I = \frac{1}{2}Mr^2$$

$$\boxed{\alpha = \frac{Tr}{I} = \frac{m(g-a)r}{\frac{1}{2}Mr^2}}$$

Unknowns: a, α

So far: 2 Equations, 3 unknowns →Need a constraint: $\boxed{a = +\alpha r}$ **Cord does not slip**

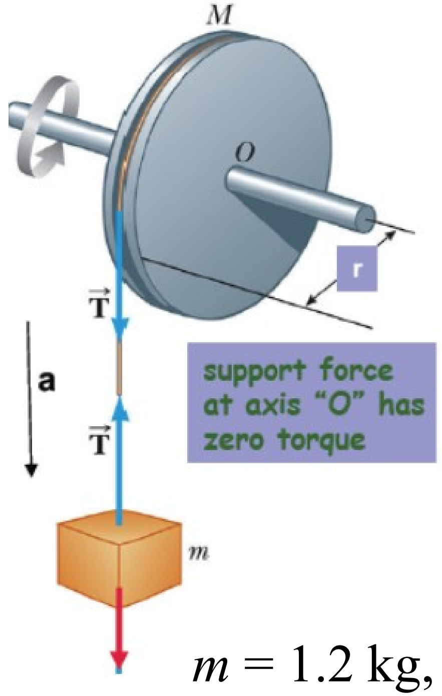
Substitute and solve:

$$\alpha = \frac{2mgr}{Mr^2} - \frac{2m\alpha r^2}{Mr^2} \qquad \alpha\left(1 + 2\frac{m}{M}\right) = \frac{2mg}{Mr} \qquad \boxed{\alpha = \frac{mg}{r(m+M/2)} \ (= 24\,\text{rad/s}^2)}$$

$$\alpha = \frac{mg}{r(m+M/2)} \ (= 24\,\text{rad/s}^2) \qquad \boxed{a = +\alpha r}$$

$$a = \frac{mg}{(m+M/2)} \ (= 4.8\,\text{m/s}^2)$$

$$\boxed{T = m(g-a) = 1.2(9.8-4.8) = 6\text{N}}$$



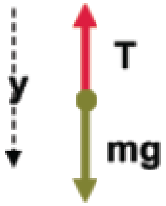
$m = 1.2$ kg,

$M = 2.5$ kg,

$r = 0.2$ m

(B) Bucket falls for 3 s (from rest), how far does it fall?

For mass m:

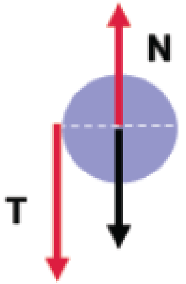


$$\sum F_y = ma = mg - T$$

$$\boxed{T = m(g-a)}$$

Unknowns: T, a

FBD for disk, with axis at "o":



$$\sum \tau_0 = +Tr = I\alpha \qquad I = \frac{1}{2}Mr^2$$

$$\boxed{\alpha = \frac{Tr}{I} = \frac{m(g-a)r}{\frac{1}{2}Mr^2}}$$

Unknowns: a, α

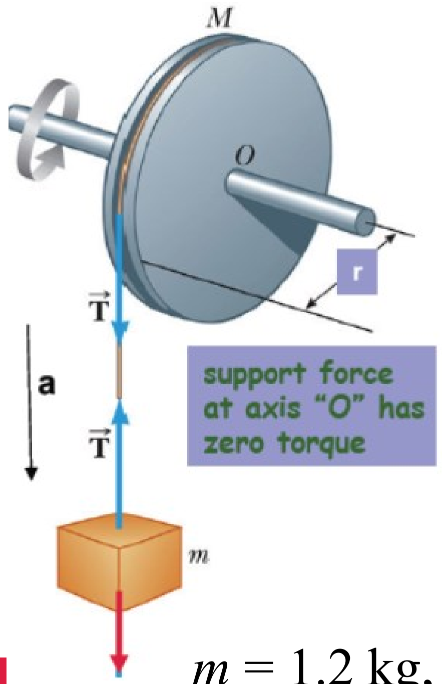
$$\boxed{a = +\alpha r}$$ **Cord does not slip**

$$\boxed{\alpha = \frac{mg}{r(m+M/2)} \quad (= 24\,\text{rad/s}^2)} \qquad a = \frac{mg}{(m+M/2)} \quad (= 4.8\,\text{m/s}^2)$$

$$\boxed{T = m(g-a) = 1.2(9.8 - 4.8) = 6\text{N}}$$

a_x = constant
$v_x = v_{0x} + a_x t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$

$$\boxed{x_f - x_f = v_i t + \frac{1}{2}at^2 = 0 + \frac{1}{2}\times 4.8\times 3^2 = 21.6\text{m}}$$



$m = 1.2$ kg,

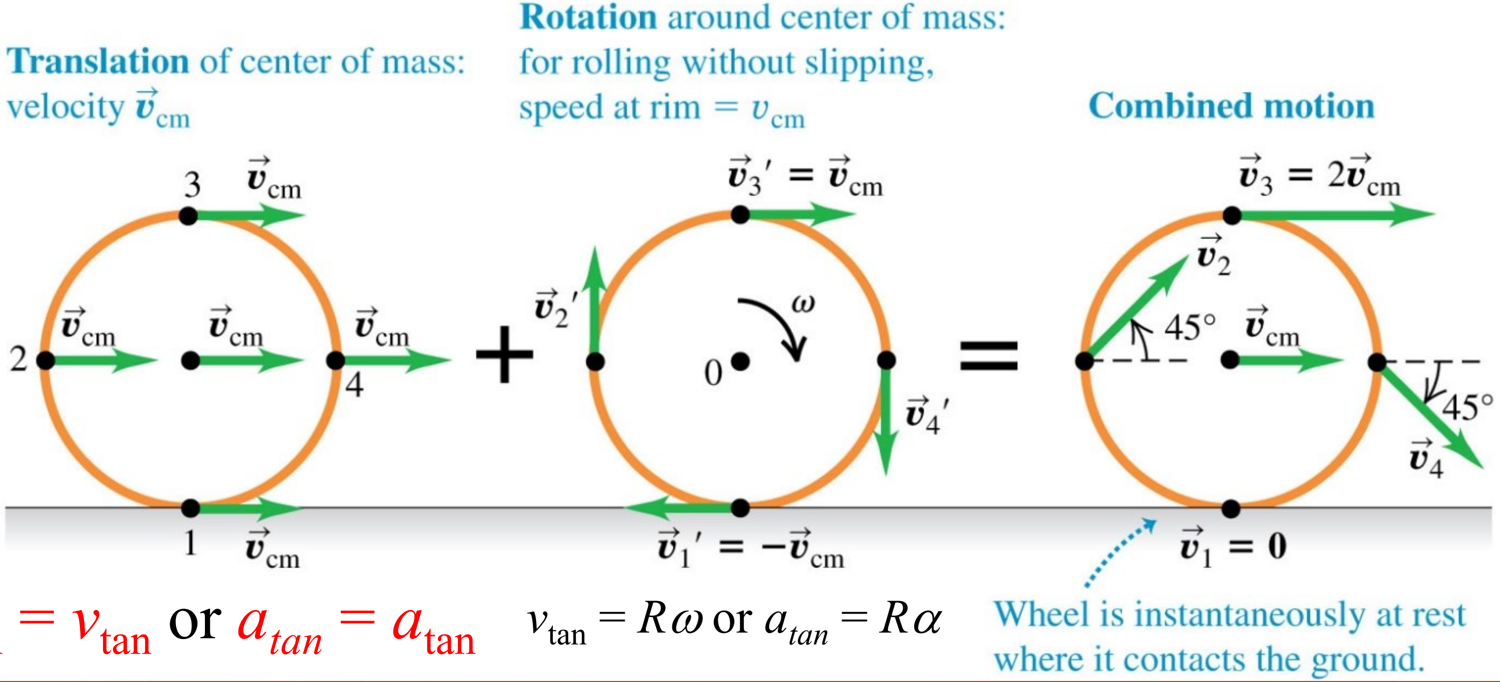
$M = 2.5$ kg,

$r = 0.2$ m

Disk's rotational inertia slows linear accelerations

Rolling without slipping

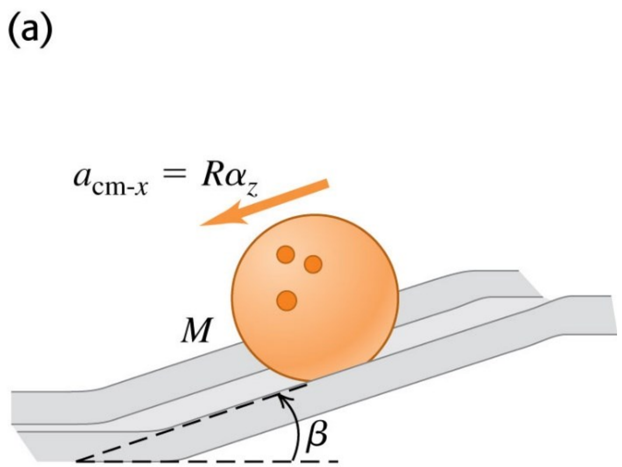
- The condition for rolling without slipping is $v_{cm} = R\omega$ or $a_{cm} = R\alpha$
- We need to consider the combined motion of points on a rolling wheel.
- Which force makes the object rolling? Static friction! Does zero work!

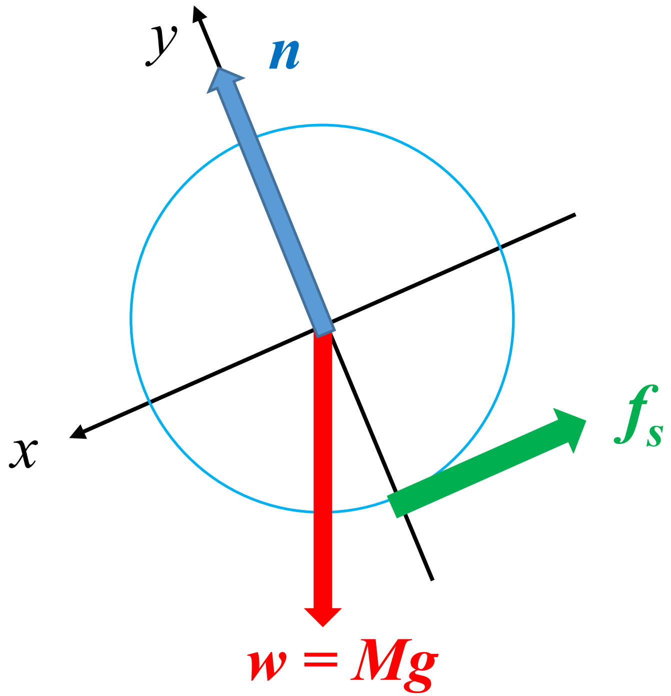


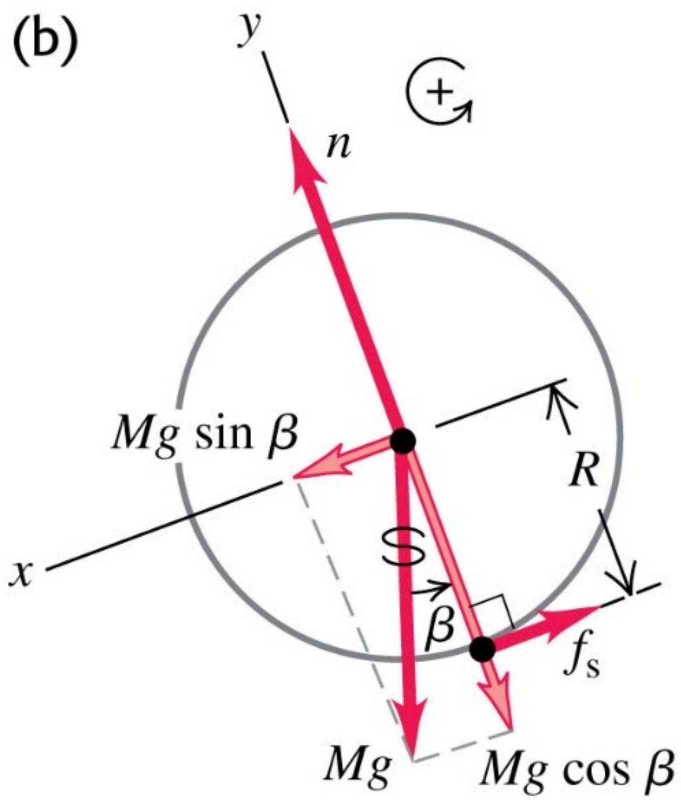
$v_{cm} = v_{tan}$ or $a_{tan} = a_{tan}$ $\quad v_{tan} = R\omega$ or $a_{tan} = R\alpha$

Example: Rolling without slipping

A bowling ball rolls without slipping down a ramp, which is inclined at an angle β to the horizontal. What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.







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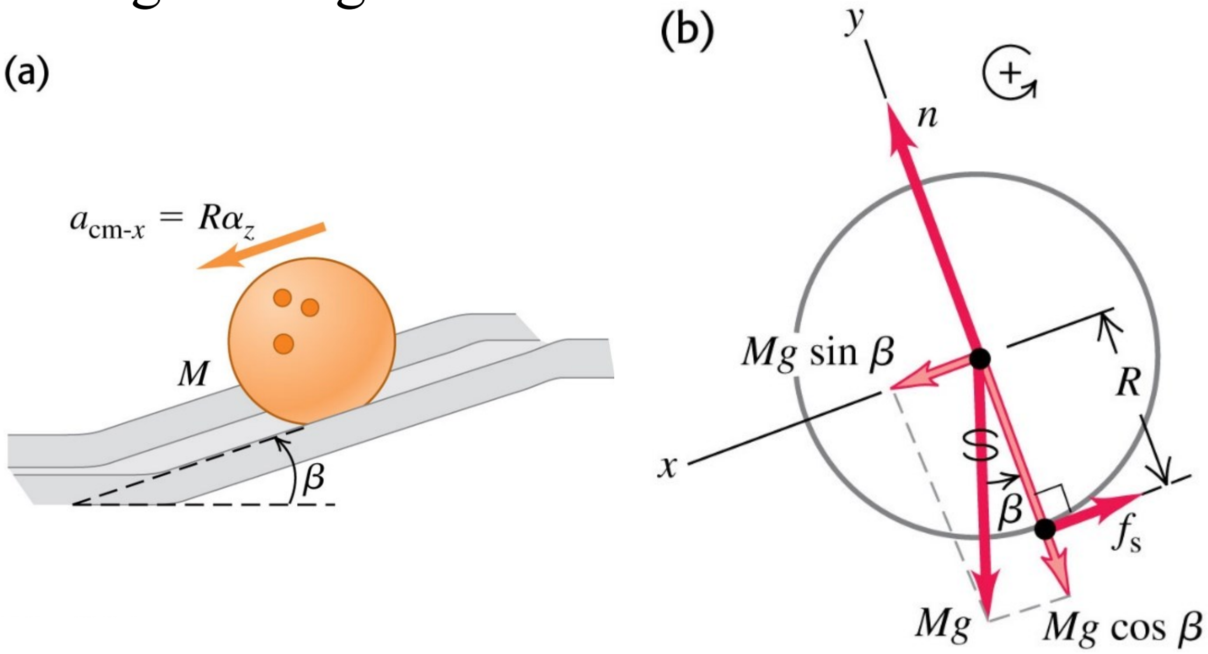


$$\sum \tau_z = fR = I_{cm}\alpha_z = \left(\frac{2}{5} MR^2\right)\alpha_z$$

$$I_{cm} = \frac{2}{5} MR^2$$

$$fR = \frac{2}{5} MRa_{cm-x}$$

$$a_{cm} = R\alpha$$

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{cm-x}$$

$$a_{cm-x} = \frac{5}{7} g\sin\beta$$

$$f = \frac{2}{7} Mg\sin\beta$$

Summary: Rolling without sliding

$$\sum \tau = I\alpha \qquad \sum F = ma$$

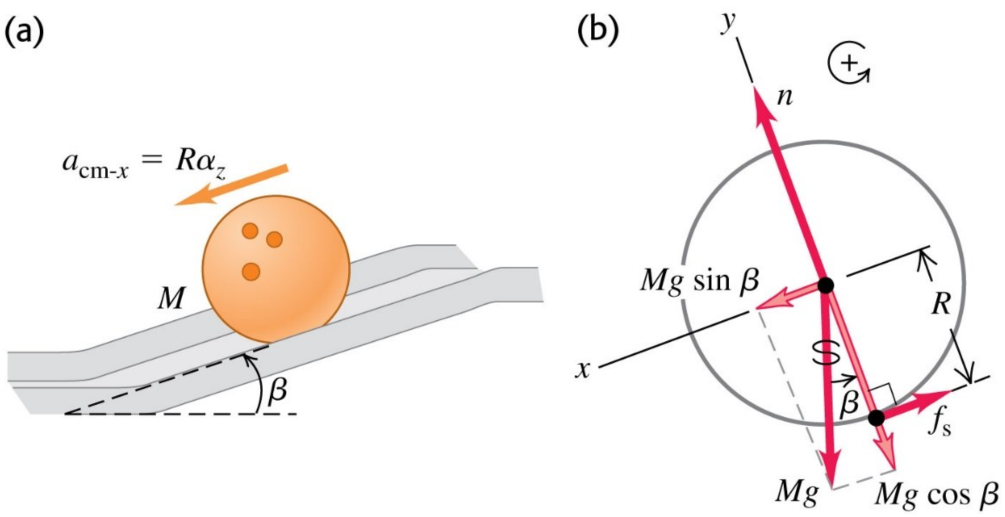
- The condition for rolling without sliding is

$$v_{\text{cm}} = R\omega \quad \text{or} \quad a_{\text{cm}} = R\alpha$$

- Static friction does NO work!
- Conservation of mechanical energy

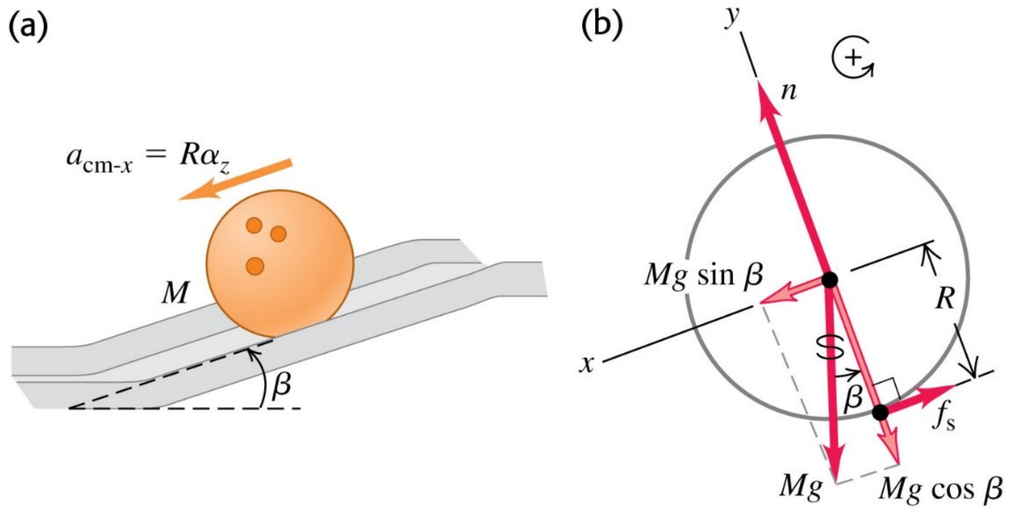
$$K_1 + U_1 = K_2 + U_2 \qquad U = Mgy_{cm}$$

- The kinetic energy of a rotating and translating rigid body is $K = 1/2\, Mv_{\text{cm}}^2 + 1/2\, I_{\text{cm}}\omega^2$.



Example: Rolling without slipping

A bowling ball rolls without slipping down a ramp, which is inclined at an angle β to the horizontal. It rolls down a distance L meter along the ramp. What is the linear speed of the center of the bowling ball?



Conservation of energy?

$$\Delta U = Mg \times L \times sin\beta = \Delta K$$

$$\Delta K = 1/2\ Mv_{cm}^2 + 1/2\ I_{cm}\omega^2.$$

$$v_{cm} = R\omega \qquad I_{cm} = \frac{2}{5}MR^2$$

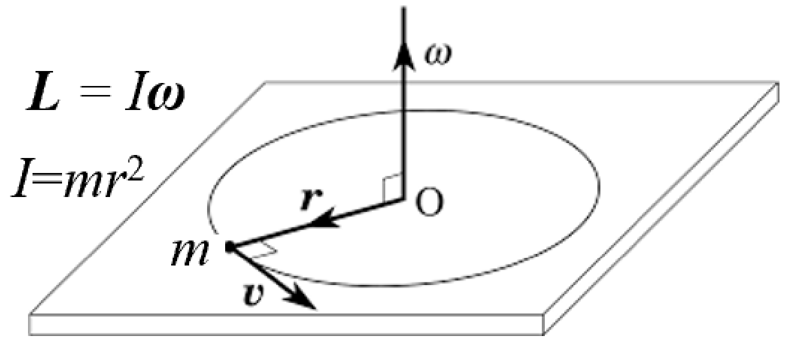
$$MgLsin\beta = 1/2\ Mv_{cm}^2 + 1/5\ (MR^2)(v_{cm}/R)^2.$$

Friction does zero work. $v_{cm} = \sqrt{\frac{10}{7} gLsin\beta}$

Angular Momentum

- Compare linear motion with rotational motion:
  - ➢ ***F*** becomes $\boldsymbol{\tau}$
  - ➢ *m* becomes *I*
  - ➢ ***a*** becomes $\boldsymbol{\alpha}$
  - ➢ ***v*** becomes $\boldsymbol{\omega}$
  - ➢ ***x*** becomes $\boldsymbol{\theta}$

Anything else?



- Linear momentum defined as $\boldsymbol{p} = m\boldsymbol{v}$
- What if mass of center of object is not moving, but it is rotating?
- Angular momentum $\boldsymbol{L} = I\boldsymbol{\omega}$

- ✓ ***L*** has the same direction as ω
- ✓ ***L*** is positive when object rotates in CCW
- ✓ ***L*** is negative when object rotates in CW.
- ✓ Multiply a factor of "*I*".
- ✓ Units: kg m^2/s

Angular Momentum of a particle

- A second definition of angular momentum of a particle

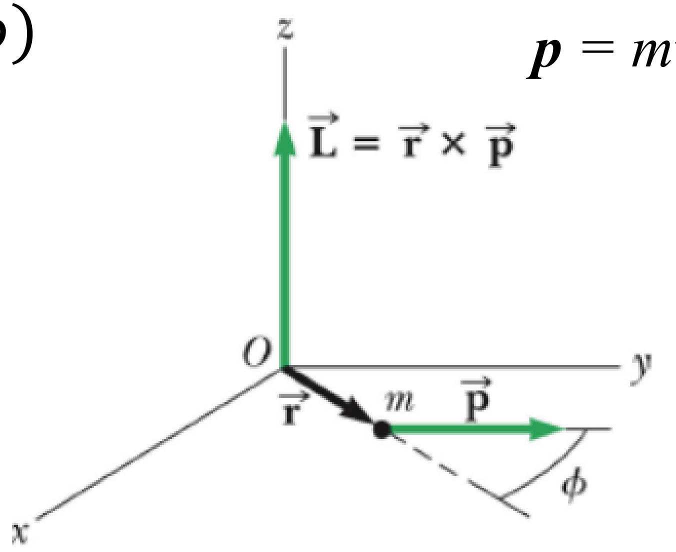
$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}} = \boldsymbol{m}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) \qquad L = r(mvsin\phi) = r(psin\phi)$$

$$vsin\phi = v_{tan} = r\omega$$

$$L = r(mr\omega) = mr^2\omega = I\omega$$

$\boldsymbol{L} = I\boldsymbol{\omega}$

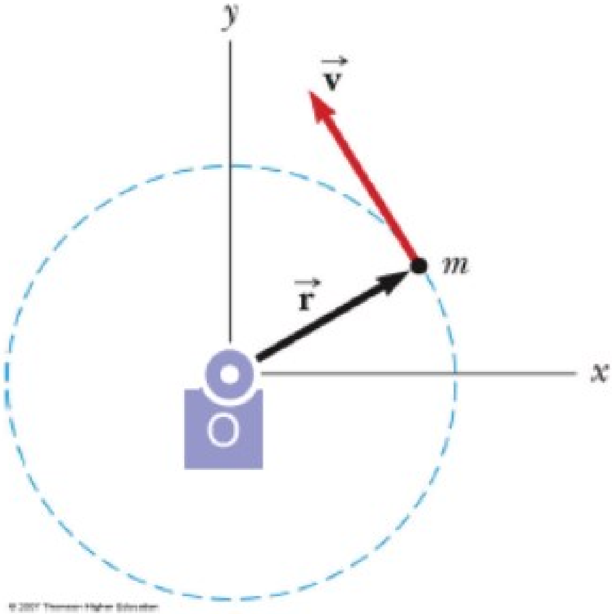
$\boldsymbol{p} = m\boldsymbol{v}$



- ***r*** is the particle's instantaneous position vector
- ***p*** is its instantaneous linear momentum
- ***r*** and ***p*** tail to tail form a plane, ***L*** is the perpendicular to this plane
- Only tangential momentum component contribute

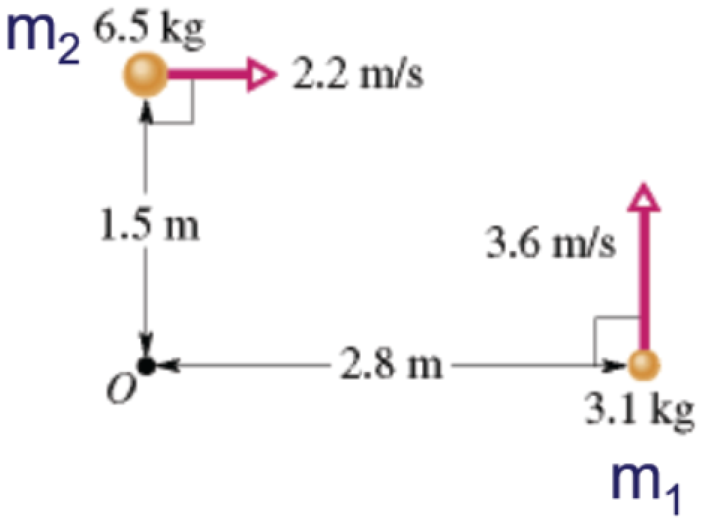
Examples: Angular Momentum

L for a single particle doing circular motion



$L = rpsin\phi = mvrsin(90°) = mvr$

L for two particles



$$\vec{L}_{net} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$L_{net} = r_1 m v_1 \sin\theta_1 - r_2 m v_2 \sin\theta_2$$

$$= r_1 m v_1 - r_2 m v_2$$

$$= 2.8 \times 3.1 \times 3.6 - 1.5 \times 6.5 \times 2.2$$

$$= 31.25 - 21.45 = 9.8\, kgm^2 / s$$

Angular Momentum and Torque

- Net torque acting on an object is equal to the time rate of change of the object's angular momentum

$$\sum \tau = I\alpha = I\frac{\Delta\omega}{\Delta t} = I\left(\frac{\omega - \omega_0}{\Delta t}\right) = \frac{I\omega - I\omega_0}{\Delta t} = \frac{\Delta L}{\Delta t}$$

$$\sum \tau = \frac{change\ in\ the\ angular\ momentum}{time\ interval}$$

- Torque cause the angular momentum to change. Net torque acting on a body is the time rate of change of its angular momentum

$$\vec{F}_{net} = \sum \vec{F} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \vec{\tau}_{net} = \sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum

- Isolated system: net external torque is zero.

$$\vec{\tau}_{net} = \sum \vec{\tau} = \frac{d\vec{L}_{tot}}{dt} = 0 \qquad \vec{L}_{tot} = constant \ or \ \vec{L}_i = \vec{L}_f$$

- ***L*** is conserved separately for *x, y, z* direction

- For an isolated system consisting of particles,

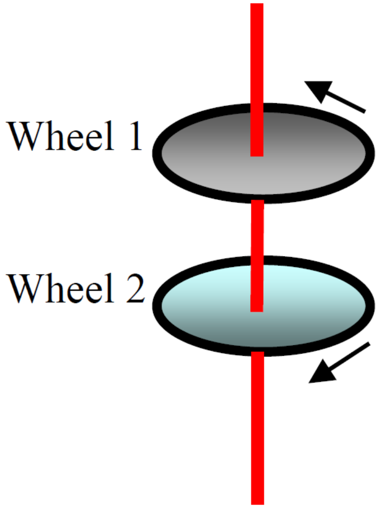
$$\vec{L}_{tot} = \sum \vec{L}_n = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots = constant$$

Example:

One wheel of rotational inertia $\boldsymbol{I_1}$ **= 2 kgm²** is rotating freely at **20 rad/s** in **counterclockwise** direction on a shaft whose rotational inertia is negligible. A second wheel of rotational inertia $\boldsymbol{I_2}$ **= 5 kgm²**, rotating freely at **15 rad/s** in the **opposite** direction, is suddenly coupled along the same shaft to the first wheel. Afterwards, the coupled wheel system rotates together. What is the angle velocity?

Q1: What is the total initial L before coupling?

$$L = I_1\omega_1 - I_2\omega_2 = 2 \times 20 - 5 \times 15 = -35$$



Q2: What is the total final L *after* coupling?

No external force, L is conserved.

$$L = -35 = I\omega = (2+5)\omega$$

$\omega = 5$ rad/s, clockwise

Example:

A **4-g** object is dropped onto a record of rotational inertia ***I* = 200 gcm²** initially rotating freely at **78 revolutions per minute (rpm)**. The object adheres to the surface of the record at distance **5 cm** from its center. The angular velocity of the record is

Q1: What is the total initial *L* before coupling?

78 rpm = 8.17 rad/s $\quad \omega_1 = 8.17 \text{ rad/s}$

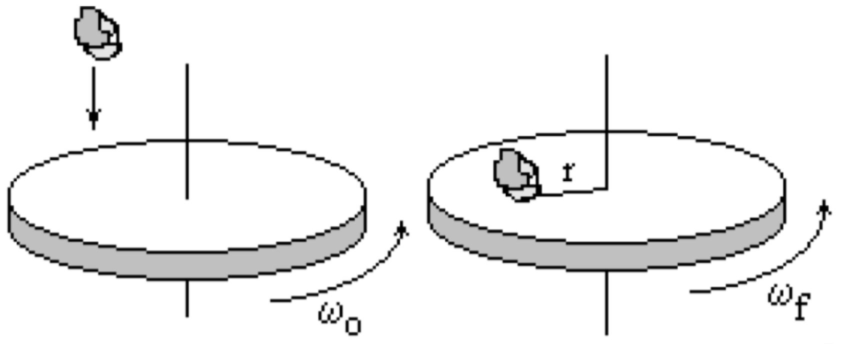
$$L = I_1\omega_1 = \left(2.0\times10^{-5}\text{kgm}^2\right)(8.17) = 16.34 \times 10^{-5}\text{ kgm}^2$$

Q2: Is the *L* conserved? Which quantity changes?

$$L = I_2\omega_2 = 16.34 \times 10^{-5}\text{ kgm}^2$$

$$I_2 = I_1 + mR^2 = 3 \times 10^{-5}\text{kgm}^2$$

$$\omega_2 = 5.45\text{ rad/s} = 52\text{ rpm}$$



Example:

A spinning skater pulls his arms in as he rotates on the ice. As he pulls his arms in, what happens to his angular momentum L and kinetic energy K?

(A) L and K both increase.

(B) L stays the same; K increases. ✓

(C) L increases; K stays the same.

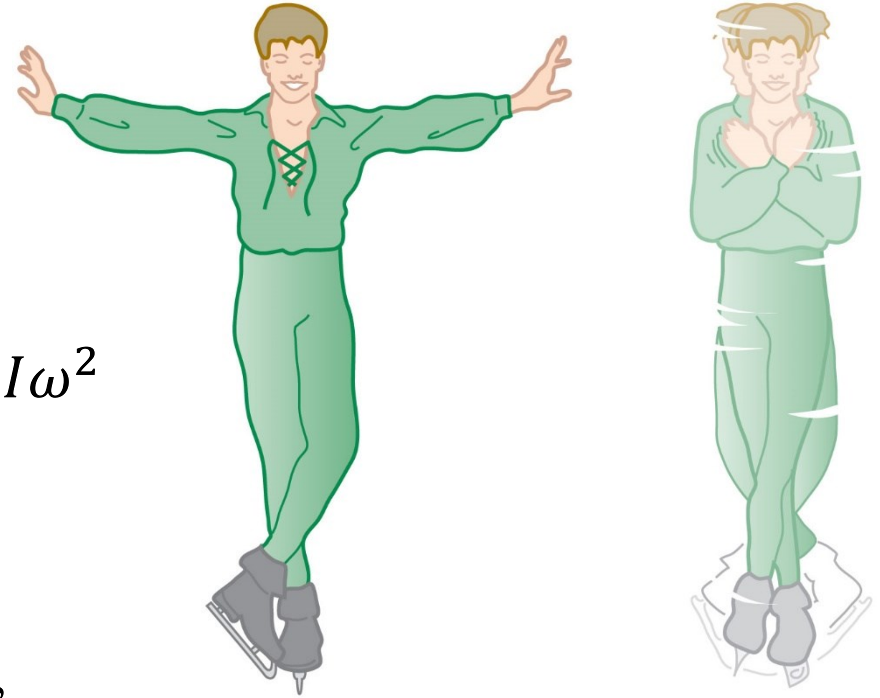
(D) L and K both stay the same.

$$I = mr^2$$

$$L = I\omega$$

$$K = (1/2)I\omega^2$$

$$K = L^2/2I$$



The extra kinetic energy comes from work done by the skater's own muscles while pulling arms inward