Magnetism

[1] Pauli's paramagnetism (no ordering)
[2] Landau diamagnetism (no ordering)
[3] 1D Ising mode (no ordering with $T > 0$)

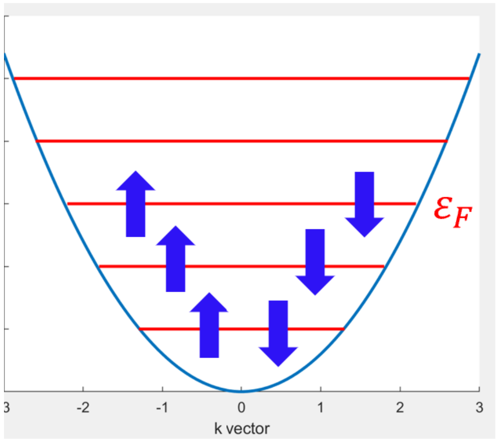
Magnetic Phase Transition:
[4] Mean Field Approximation.
[5] Landau theory

Pauli Paramagnetism: Discussion

- **Pauli's (quantum) paramagnetic: at low T with lowest order ($T = 0$).**

Fermi gas under magnetic field: metals, e.g., Al, Cu in magnetic field

Let's consider the simplest case: at $T = 0$.



Q1: For $H = 0$, how do the electrons occupy the energy level?

Q2: How does the plot look like under H field (up = +)?

Net magnetic moment!

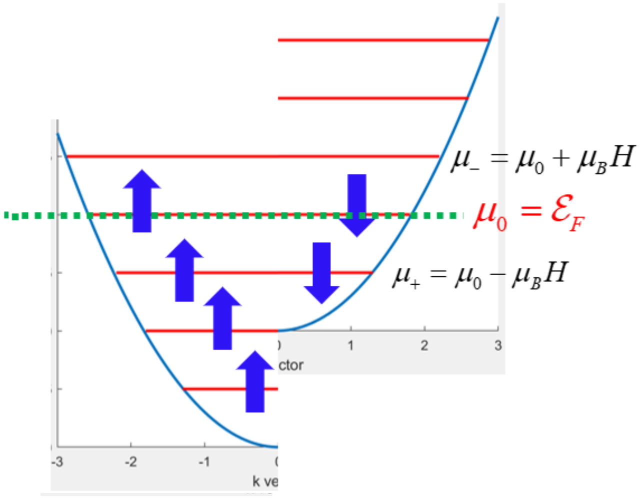
Q3: How can we find out the net moment size?

$$M = \mu_B (N_- - N_+) / V$$

Q4: Which variables determine the N_- and N_+ value?

μ_0 and $\mu_B H$ Fermi vectors: k_- and k_+

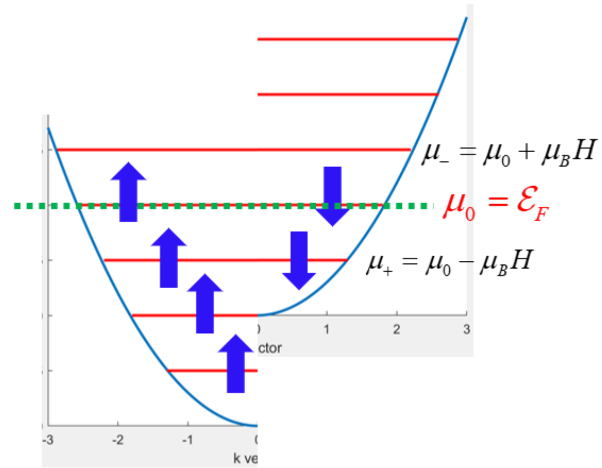
$$N_- = \frac{V}{h^3}\frac{4\pi}{3}(\hbar k_-)^3 \qquad N_+ = \frac{V}{h^3}\frac{4\pi}{3}(\hbar k_+)^3$$



Pauli Paramagnetism: Discussion

- **Pauli's (quantum) paramagnetic: at low T with lowest order ($T = 0$).**

$$M = \frac{\mu_B}{V}(N_- - N_+) \qquad N_- = \frac{V}{h^3}\frac{4\pi}{3}(\hbar k_-)^3 \qquad N_+ = \frac{V}{h^3}\frac{4\pi}{3}(\hbar k_+)^3$$



Q4: Which variables determine the k_- and k_+ value?

Chemical potential: μ_- and μ_+

$$\frac{(\hbar k_-)^2}{2m_e} = \mu_{H=0} + \mu_B H = \mu_- \qquad \Rightarrow \hbar k_- = \left[2m_e\left(\mu_0 + \mu_B H\right)\right]^{1/2}$$

$$\frac{(\hbar k_+)^2}{2m_e} = \mu_{H=0} - \mu_B H = \mu_+ \qquad \Rightarrow \hbar k_- = \left[2m_e\left(\mu_0 - \mu_B H\right)\right]^{1/2}$$

$$M = \frac{\mu_B}{V}(N_- - N_+) = \frac{\mu_B}{V}\frac{V}{h^3}\frac{4\pi}{3}\left[(\hbar k_-)^3 - (\hbar k_+)^3\right] = \frac{\mu_B}{h^3}\frac{4\pi}{3}\left[(\hbar k_-)^3 - (\hbar k_+)^3\right]$$

$$= \frac{\mu_B}{h^3}\frac{4\pi}{3}(2m_e)^{3/2}\left[(\mu_0 + \mu_B H)^{3/2} - (\mu_0 - \mu_B H)^{3/2}\right]$$

For small H, $(\mu_0 \pm \mu_B H)^{3/2} \approx \mu_0^{3/2}\left(1 \pm \frac{3}{2}\frac{\mu_B H}{\mu_0}\right)$

$$\approx \frac{\mu_B}{h^3}\frac{4\pi}{3}(2m_e\mu_0)^{3/2}\left[3\frac{\mu_B H}{\mu_0}\right]$$

$$\mu_0 = \varepsilon_F = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

$$= \frac{3\mu_B^2 NH}{2\mu_0 V} = \frac{4\pi\mu_B^2 m_e}{h^2}\left(\frac{3n}{\pi}\right)^{1/3} H$$

M does not depend on T, since $T = 0$

Pauli Paramagnetism

- **Pauli's paramagnetic model for metals.** Spin-1/2 Fermions (electrons) ideal gas. Taking the field along the z-axis this reads, the single particle Hamiltonian is

$$\mathcal{H}_{\pm} = \mathcal{E}_i(\vec{k}) \pm \frac{g\mu_B}{\hbar} s_i^z H$$

$g = 2$ (gyromagnetic ratio)
μ_B the Bohr magneton

Q: Grand canonical partition function?

Grand canonical partition function **without field**

$$\mathcal{Q}_{-}(T,\mu) = \prod_{\vec{k}} \sum_{\{n_{\vec{k}}=0,1\}}^{-} \exp\left[-\beta(\mathcal{E}(\vec{k})-\mu)n_{\vec{k}}\right] = \prod_{\vec{k}} \sum_{\{n_{\vec{k}}=0,1\}}^{-} z^{n_{\vec{k}}} \exp\left(-\beta\varepsilon_{\vec{k}} n_{\vec{k}}\right) \qquad z = e^{\beta\mu}$$

Grand canonical partition function **with field**

$$\mathcal{Q}_{-}(T,\mu) = \prod_{\vec{k}} \sum_{\{n_{\vec{k}}=0,1\}}^{-} \exp\left[-\beta(\mathcal{H}_{-}-\mu)n_{\vec{k}}\right] \sum_{\{n_{\vec{k}}=0,1\}}^{-} \exp\left[-\beta(\mathcal{H}_{+}-\mu)n_{\vec{k}}\right]$$

$$= \prod_{\vec{k}} \left\{\sum_{n_{\vec{k}}} z^{n_{\vec{k}}} \exp\left[(-\beta\varepsilon_{\vec{k}} + \beta\mu_B H)n_{\vec{k}}\right]\right\}\left\{\sum_{n_{\vec{k}}} z^{n_{\vec{k}}} \exp\left[(-\beta\varepsilon_{\vec{k}} - \beta\mu_B H)n_{\vec{k}}\right]\right\}$$

$$= \prod_{\vec{k}} \prod_{\sigma=+,-} \sum_{n_{\vec{k}}} z_{\sigma}^{n_{\vec{k}}} \exp(-\beta\varepsilon_{\vec{k}} n_{\vec{k}}) \qquad z_{\pm} = e^{\beta\mu \pm \beta\mu_B H}$$

Pauli Paramagnetism

- **Pauli's paramagnetic model for metals.** Spin-1/2 Fermions (electrons) ideal gas. Taking the field along the z-axis this reads.

Grand canonical partition function and grand potential function **without field**

$$\mathcal{Q}_-(T,\mu) = \prod_{\vec{k}} \sum_{\{n_{\vec{k}}=0,1\}}^{-} z^{n_{\vec{k}}} \exp\left(-\beta\varepsilon_{\vec{k}} n_{\vec{k}}\right) \qquad \mathcal{G} = -k_B T \ln \mathcal{Q}_- = -\frac{k_B T V}{\lambda^3} f_{5/2}^-(z)$$

Grand canonical partition function and grand potential function **with field**

$$\mathcal{Q}_-(T,\mu) = \prod_{\vec{k}} \prod_{\sigma=+,-} \sum_{n_{\vec{k}}} z_\sigma^{n_{\vec{k}}} \exp(-\beta\varepsilon_{\vec{k}} n_{\vec{k}}) \qquad \mathcal{G} = -\frac{k_B T V}{\lambda^3}\left[f_{5/2}^-(z_+) + f_{5/2}^-(z_-)\right]$$

$$z_\pm = e^{\beta\mu \pm \beta\mu_B H}$$

We determine the magnetization

$$M = -\frac{1}{V}\frac{\partial \mathcal{G}}{\partial H} = \frac{k_B T}{\lambda^3}\sum_\sigma \frac{\partial}{\partial H} f_{5/2}^-(z_\sigma) = \frac{k_B T}{\lambda^3}\sum_\sigma \frac{\partial z_\sigma}{\partial H}\frac{\partial}{\partial z} f_{5/2}^-(z_\sigma) = \frac{\mu_B}{\lambda^3}\sum_\sigma \sigma z_\sigma \frac{\partial}{\partial z} f_{5/2}^-(z_\sigma)$$

$$= \frac{\mu_B}{\lambda^3}\sum_\sigma \sigma f_{3/2}^-(z_\sigma) = \frac{\mu_B}{\lambda^3}\left\{f_{3/2}^-(z_+) - f_{3/2}^-(z_-)\right\}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} f_m^\pm(z) = \frac{1}{z} f_{m-1}^\pm(z)$$

Pauli Paramagnetism

- **Pauli's (quantum) paramagnetic model for metals.** Spin-1/2 Fermions (electrons). Field along the z-axis this reads, the grand potential function

$$\mathcal{Q}_{-}(T,\mu)=\prod_{\vec{k}}\prod_{\sigma=+,-}\sum_{n_{\vec{k}}} z_{\sigma}^{n_{\vec{k}}}\exp(-\beta\varepsilon_{\vec{k}}n_{\vec{k}}) \qquad \mathcal{G}=-\frac{k_BTV}{\lambda^3}\left[f_{5/2}^{-}(z_+)+f_{5/2}^{-}(z_-)\right] \qquad z_{\pm}=e^{\beta\mu\pm\beta\mu_B H}$$

The magnetization $M=-\frac{1}{V}\frac{\partial\mathcal{G}}{\partial H}=\frac{\mu_B}{\lambda^3}\left\{f_{3/2}^{-}(z_+)-f_{3/2}^{-}(z_-)\right\}=\mu_B(n_+-n_-)$

Q: How do we understand the M?

$$n=\frac{N}{V}=\frac{2}{\lambda^3}f_{3/2}^{-}(z)$$

The density of Fermions is given by?

$$n=\frac{N}{V}=\frac{1}{\lambda^3}\left\{f_{3/2}^{-}(z_+)+f_{3/2}^{-}(z_-)\right\}=n_++n_-=\frac{2}{\lambda^3}\cdot f_{3/2}^{-}(z)\Big|_{H=0}$$

The spin susceptibility for zero magnetic field, given by

$$\chi=\frac{\partial M}{\partial H}\bigg|_{H=0}=\frac{\mu_B^2}{\lambda^3k_BT}2z\frac{\partial f_{3/2}^{-}(z)}{\partial z}\bigg|_{H=0}=\frac{2\mu_B^2}{\lambda^3k_BT}f_{1/2}^{-}(z)$$

Pauli Paramagnetism

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

- **Pauli's (quantum) paramagnetic: at low *T* and low *H*.**

$$M = \frac{\mu_B}{\lambda^3}\left\{ f_{3/2}^{-}\left(z_+\right) - f_{3/2}^{-}\left(z_-\right)\right\} \qquad \chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} = \frac{2\mu_B^2}{\lambda^3 k_B T} f_{1/2}^{-}(z) \qquad z_\pm = e^{\beta\mu \pm \beta\mu_B H}$$

Expanding the results for small H, gives

$$f_{3/2}^{-}\left(z e^{\pm\beta\mu_B H}\right) \approx f_{3/2}^{-}\left[z\left(1 \pm \beta\mu_B H\right)\right] \approx f_{3/2}^{-}(z) \pm z\cdot\beta\mu_B H \frac{\partial}{\partial z} f_{3/2}^{-}(z)$$

$$M = \frac{\mu_B}{\lambda^3}\left(2\beta\mu_B H\right)\cdot f_{1/2}^{-}(z) = \frac{2\mu_B^2}{k_B T}\frac{1}{\lambda^3}\cdot H\cdot f_{1/2}^{-}(z) = \chi H$$

with z given by $n = \frac{2}{\lambda^3}\cdot f_{3/2}^{-}(z)$

Low temperature, use Sommerfeld expansion, and use the lowest order (?)

$$\lim_{z\to\infty} f_m^{-}(z) = \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{360}\frac{m(m-1)(m-2)(m-3)}{(\ln z)^4} + \cdots\right]$$

$(1/2)! = \sqrt{\pi}/2$,
$(3/2)! = 3\sqrt{\pi}/4$.

$$\frac{n\lambda^3}{2} = f_{3/2}^{-}(z) = \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2}\left[1 + \frac{\pi^2}{6}\frac{3}{2}\frac{1}{2}(\ln z)^{-2} + \cdots\right] \Rightarrow \ln z = \left(\frac{3n\sqrt{\pi}}{8}\lambda^3\right)^{2/3}$$

$$\lim_{z\to\infty} f_{1/2}^{-}(z) \approx \frac{2}{\sqrt{\pi}}(\ln z)^{1/2} = \lambda\left(\frac{3n}{\pi}\right)^{1/3} \qquad M = \frac{4\pi\mu_B^2 m_e}{h^2}\left(\frac{3n}{\pi}\right)^{1/3} H \qquad \chi = \frac{4\pi\mu_B^2 m_e}{h^2}\left(\frac{3n}{\pi}\right)^{1/3}$$

Does the M or χ depend on temperature? Why? Lowest order: $T = 0$

Pauli Paramagnetism

- **Pauli's (quantum) paramagnetic: high T limit?**

For high temperatures, the following result remains valid, since the expansion was performed under the assumption of a small H.

$$f_{3/2}^{-}\left(ze^{\pm\beta\mu_B H}\right) \approx f_{3/2}^{-}\left[z\left(1\pm\beta\mu_B H\right)\right] \approx f_{3/2}^{-}(z) \pm z\cdot\beta\mu_B H\,\frac{\partial}{\partial z} f_{3/2}^{-}(z)$$

$$M = \frac{\mu_B}{\lambda^3}\left\{f_{3/2}^{-}\left(z_{+}\right) - f_{3/2}^{-}\left(z_{-}\right)\right\} = \frac{\mu_B}{\lambda^3}\left(2\beta\mu_B H\right)\cdot f_{1/2}^{-}(z) = \frac{2\mu_B^2}{k_B T}\frac{1}{\lambda^3}\cdot H\cdot f_{1/2}^{-}(z) = \chi H$$

But we need to use the high temperature expansion for f_m function.

$$f_m^{-}(z) = z - \frac{z^2}{2^m} + \frac{z^3}{3^m} - \frac{z^4}{4^m} + \cdots \qquad \text{with } z \text{ given by } n = \frac{2}{\lambda^3}\cdot f_{3/2}^{-}(z)$$

Lowest order: $z = \frac{n\lambda^3}{2}$ $\qquad M = \frac{2\mu_B^2}{k_B T}\frac{1}{\lambda^3}\cdot H\cdot\frac{n\lambda^3}{2} = \frac{n\mu_B^2}{k_B T}\cdot H \qquad \chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} = \frac{n\mu_B^2}{k_B T}$

The magnetic susceptibility $\chi \propto 1/T$ is the well-known Curie law.

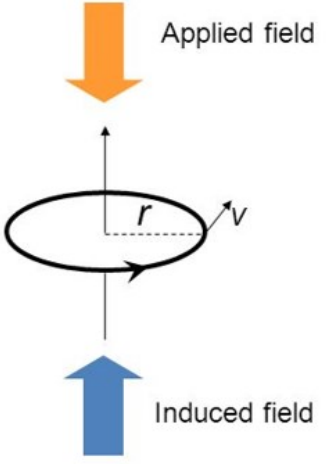
Landau Diamagnetism

- **Landau diamagnetism (quantum):**
- ✓ We have discussed the paramagnetic properties which come from the spin degree of freedom of electrons. (nuclear spins are 10^{-3} weaker than the electron spin.)
- ✓ Electrons have another degree of freedom, i. e., the orbital degree of freedom. The electrons, free or bound, move in quantized orbits in the magnetic field.
- ✓ This will induce diamagnetism, that is the magnetic moment is aligned antiparallelly to an external magnetic field. (Magnetic susceptibility is positive for paramagnetic and negative for diamagnetic.)
- ✓ In a physical substance, these two effects compete. Here, we will consider the pure diamagnetism and ignore the spin degree of freedom.

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}(\vec{r})\right)^2$$

$\vec{A}$ is the electromagnetic four-potential, e is the electron charge, magnetic field $\vec{B} = \nabla \times \vec{A}$

We consider a uniform external magnetic field B pointing along z axis, and choose the vector potential, $\vec{A} = (-By, 0, 0)$



diamagnetism

Landau Diamagnetism

- **Landau diamagnetism.** Then the Hamiltonian is

$$\vec{A} = (-By, 0, 0)$$

$$H = \frac{1}{2m}\left\{[-i\hbar\frac{\partial}{\partial x} - eBy]^2 + [-i\hbar\frac{\partial}{\partial y}]^2 + [-i\hbar\frac{\partial}{\partial z}]^2\right\}$$

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}(\vec{r})\right)^2$$

Solutions to the Schrődinger equation are simply plane waves in the x and z direction combined with $f(y)$,

$$\psi(\vec{r}) = e^{i(k_x x + k_z z)} f(y)$$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left(\hbar k_x - eBy\right)^2\right] f(y) = E' f(y)$$

Note that x is not in this Hamiltonian, so it is a cyclic coordinate, and p_x is conserved. The H commutes with p_x, so H and p_x have a common set of eigenstates, and the eigenstates of p_x are just the plane waves.

Define $\omega_c = eB/m$ and let $y_0 = \frac{\hbar k_x}{eB}$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_c^2\left(y - y_0\right)^2\right] f(y) = E' f(y)$$

This is simply the Schrődinger equation for a harmonic oscillator, jiggling with frequency ω_c and situated **at position y_0 which is a function of k_x.**

The eigen energy levels are also for harmonic oscillator.

$$E' = \left(j + \frac{1}{2}\right)\hbar\omega_c \qquad E = E' + \frac{\hbar^2 k_z^2}{2m}$$

Landau Diamagnetism

- **Flux quantum.** The discrete energy levels are known as *Landau levels*. They are highly degenerate.

✓ For a box L, that k_x is quantized in unit of $\Delta k_x = 2\pi/L$. This means that we can have a harmonic oscillator located every $\Delta y_0 = 2\pi\hbar/eBL$. $y_0 = \hbar k_x/eB$

✓ For a box L, the number of oscillators that we can fit into is $\frac{L}{\Delta y_0} = eBL^2/2\pi\hbar$.

✓ This is also the degeneracy of each level $g = \frac{eBL^2}{2\pi\hbar} = \frac{\Phi}{\Phi_0}$, where $\Phi = L^2 B$ is the magnetic flux through the system and $\Phi_0 = 2\pi\hbar/e$ is the flux quanta. (The result above does not include a factor of 2 for the spin degeneracy).

- **Grand potential.** Without p_z, i.e., 2D electron gas in magnetic field. Including the factor of 2 from electron spin and degeneracy g, we have

$$\mathcal{Q}_- = \prod_{\vec{k}} [1 + \exp(\beta\mu - \beta\mathcal{E}(\vec{k}))] \qquad \ln \mathcal{Q}_- = \sum_{\vec{k}} \ln[1 + \exp(\beta\mu - \beta\mathcal{E}(\vec{k}))]$$

$$\mathcal{G} = -k_B T 2g \sum_{j=0}^{\infty} \ln\left[1 + ze^{-\beta\varepsilon_j}\right], \text{ fugacity } z = e^{\beta\mu}, \text{ and } \varepsilon_j = \left(j + \frac{1}{2}\right)\hbar\omega_c$$

$\mu \sim \varepsilon_j$, i.e., $ze^{-\beta\varepsilon_j} \ll 1$, using $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, we get $\ln\left[1 + ze^{-\beta\varepsilon_j}\right] \approx ze^{-\beta\varepsilon_j}$

Landau Diamagnetism

$$\ln\left[1+ze^{-\beta\varepsilon_j}\right]\approx ze^{-\beta\varepsilon_j},\quad \varepsilon_j=\left(j+\frac{1}{2}\right)\hbar\omega_c,\quad \omega_c=\frac{eB}{m},\quad g=\frac{eBA}{2\pi\hbar},\ A=L^2$$

$$\mathcal{G}\approx -k_BT2g\sum_{j=0}^{\infty} ze^{-\beta\varepsilon_j} = -(k_BTgz)e^{-\frac{\beta\hbar\omega_c}{2}}\frac{2}{1-e^{-\beta\hbar\omega_c}} = -(k_BTgz)\operatorname{csch}\left(\frac{\beta\hbar\omega_c}{2}\right)$$

For $\beta\to 0$ (high T limit), $\frac{\beta\hbar\omega_c}{2}\to 0$, using $\operatorname{csch}(x)\approx x^{-1}-\frac{1}{6}x$, we get

$$\mathcal{G}\approx -(k_BTgz)\left(\frac{2}{\beta\hbar\omega_c}-\frac{\beta\hbar\omega_c}{12}\right) = -z\frac{Am}{\pi}\left(\frac{k_BT}{\hbar}\right)^2\left[1-\frac{1}{24}\left(\frac{\hbar\omega_c}{k_BT}\right)^2\right]$$

Magnetization $M=-\frac{1}{V}\frac{\partial\mathcal{G}}{\partial B} = -\frac{mz}{12\pi}\left(\frac{k_BT}{\hbar}\right)^2\left(\frac{\hbar e}{k_BTm}\right)^2 B = -\frac{ze^2}{12\pi m}B,$

Magneticsusceptibility $\chi=\frac{\partial M}{\partial B}=-\frac{ze^2}{12\pi m}$ Need to solve z using $N=-\frac{\partial\mathcal{G}}{\partial\mu}$

First-order approximation $\mathcal{G}\approx -z\frac{Am}{\pi}\left(\frac{k_BT}{\hbar}\right)^2$, we get $N=-\frac{\partial\mathcal{G}}{\partial\mu}=\frac{zAm}{\pi\hbar^2}(k_BT)$,

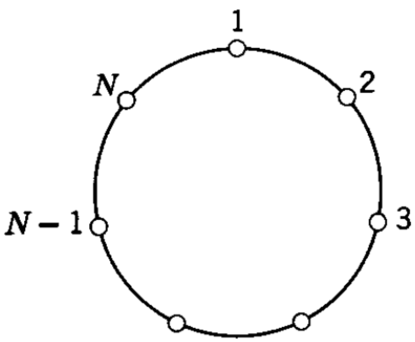
So $z=\frac{N\pi\hbar^2}{Amk_BT}$, then we get $\chi=-\frac{N}{3A}\left(\frac{e\hbar}{2m}\right)^2\frac{1}{k_BT}$, $M=-\frac{N}{3A}\left(\frac{e\hbar}{2m}\right)^2\frac{1}{k_BT}B$

Interaction Spins: 1D Ising Model

- **1D Ising model:** Analytical method
- ✓ A chain of N spins, each spin interacting only with its two nearest neighbors and with an external magnetic field. The energy for a configuration of $\{s_1, s_2, \dots, s_N\}$

$$E_I = -J\sum_{k=1}^{N} s_k s_{k+1} - H\sum_{k=1}^{N} s_k$$

with the periodic boundary condition $s_{N+1} \equiv s_1$



- ✓ The partition function is

$$Q_I(H,T) = \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N} \exp\left[\beta\sum_{k=1}^{N}\left(Js_k s_{k+1} + Hs_k\right)\right]$$

$$= \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N} \exp\left\{\beta\sum_{k=1}^{N}\left[Js_k s_{k+1} + \frac{1}{2}H\left(s_k + s_{k+1}\right)\right]\right\}$$ = ? Using matrix **P?**

- ✓ Let's define a 2×2 matrix **P:** $\langle s\,|\,\mathbf{P}\,|\,s'\rangle = \exp\left[\beta\left(Jss' + \frac{1}{2}H\left(s+s'\right)\right)\right]$

$$\langle -1\,|\,\mathbf{P}\,|\,-1\rangle = e^{\beta(J-H)},\ \langle +1\,|\,\mathbf{P}\,|\,+1\rangle = e^{\beta(J+H)}$$

$$\langle +1\,|\,\mathbf{P}\,|\,-1\rangle = \langle -1\,|\,\mathbf{P}\,|\,+1\rangle = e^{-\beta J}$$

$$\mathbf{P} = \begin{bmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{bmatrix}$$

Interaction Spins: 1D Ising Model

- ❑ **1D Ising model:** Analytical method
- ✓ The partition function could be rewritten

$$\langle s \mid \mathbf{P} \mid s' \rangle = \exp\left[\beta J s s' + \frac{1}{2} H\left(s + s'\right)\right]$$

$$\mathbf{P} = \begin{bmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{bmatrix}$$

$$Q_I(H,T) = \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N} \exp\left\{\beta \sum_{k=1}^{N}\left[J s_k s_{k+1} + \frac{1}{2} H\left(s_k + s_{k+1}\right)\right]\right\}$$

$$= \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N} \langle s_1 \mid \mathbf{P} \mid s_2\rangle\langle s_2 \mid \mathbf{P} \mid s_3\rangle\cdots\langle s_N \mid \mathbf{P} \mid s_1\rangle$$

P is called transfer matrix.

$$= \sum_{s_1}\left\langle s_1 \left| \mathbf{P}^N \right| s_1 \right\rangle = \mathrm{Tr}\,\mathbf{P}^N = \lambda_+^N + \lambda_-^N$$

$$\mathrm{tr}\left(\mathbf{A}^k\right) = \sum_i \lambda_i^k$$

λ_+ and λ_- are the two eigenvalues of $\mathbf{P}$, with $\lambda_+ \geq \lambda_-$

- ✓ The eigenvalues of **P** is $\lambda_\pm = e^{\beta J}\left[\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right]$

- ✓ The Helmholtz free energy (per spin) is

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$F_I(H,T) = -kT \ln Q_I(H,T) = -kT \ln\left(\lambda_+^N + \lambda_-^N\right) = ?$$

$\lambda_+ > \lambda_-$ for all H

$$= -kT \ln\left\{\lambda_+^N\left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right]\right\} = -kTN\left\{\ln\lambda_+ + \frac{1}{N}\ln\left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right]\right\} \underset{N\to\infty}{\longrightarrow} -NkT\ln\lambda_+$$

Interaction Spins: 1D Ising Model

- **1D Ising model:** Analytical method

$$\lambda_\pm = e^{\beta J}\left[\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right]$$

- ✓ The magnetization (per spin) is

$$\frac{1}{N} F_I(H,T) = -kT \ln \lambda_+ = -J - kT \ln\left[\cosh(\beta H) + \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right]$$

$$M_I(H,T) = -\frac{\partial}{\partial H}\left(\frac{F_I}{kT}\right) = N \frac{\sinh(\beta H)}{\sqrt{\sinh^2(\beta H) + \exp(-4\beta J)}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

For $T > 0$, $\frac{M_I(H=0,T)}{N} = 0$;

for $T = 0$, $\frac{M_I(H=0,T)}{N} = \pm 1$.

(1) No spontaneous magnetization at non-zero temperature.
(2) Non-zero spontaneous magnetization at zero temperature.