1D Ising Model: Domain Argument

- **1D Ising:** Role of domain walls at $T > 0$.
  - ✓ Create a "domain wall" by flipping all the spins to the right of some site,
  - ✓ The energy increases by $2J$,
  - ✓ But the entropy increase by $kln(N\text{-}1)$: we have N-1 choices to place the wall.
  - ✓ Free energy associated with the creation of one domain wall is

$$\Delta F = E - TS = 2J - kTln(N-1)$$

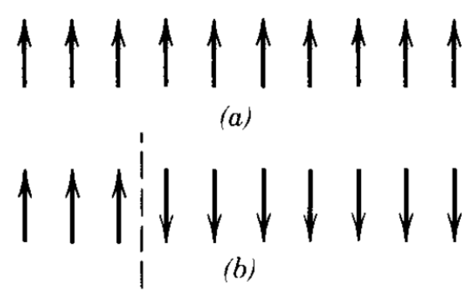
  - ✓ For $T > 0$ and $N \rightarrow \infty$, the 1D Ising model favors domain walls.

  More domain walls will be created until the spins are randomized: no spontaneous magnetization.

  - ✓ Q: How about antiferromagnetic 1-D Ising model?

  The sublattice is ferromagnetic, no spontaneous magnetization, so no antiferromagnetic ordering.

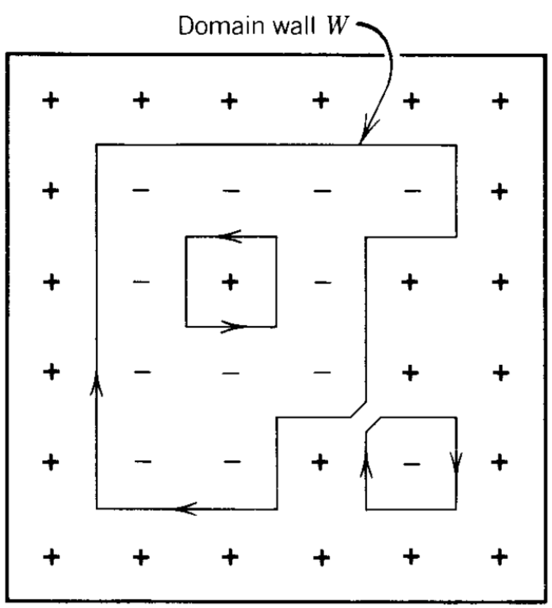
$$E_I = -J\sum_{k=1}^{N} s_k s_{k+1}$$



2D and 3D Ising Model (optional)

- **How about 2D and 3D Ising?**

  Existence of spontaneous magnetization at $T > 0$.
  - ✓ We can also use the domain walls to prove the existence of spontaneous magnetization at $T > 0$ (Griffiths proof) for 2D Ising model. (Skip…)
  - ✓ 2D Ising has exact solutions, but much more complicated. (Skip again…We will learn/use the numerical simulation method.)
  - ✓ For 3D, only numerical simulations or **approximative calculations** are possible.
  - ✓ One frequently used approximative method is the so-called ***mean field approximation***.
  - ✓ We have used mean field approximation to solve the interacting gas.
  - ✓ Now let's try magnetic system.



Numerical simulations: Monte Carlo

Magnetism

[1] Pauli's paramagnetism (no ordering)
[2] Landau diamagnetism (no ordering)
[3] 1D Ising mode (no ordering with $T > 0$)

Magnetic Phase Transition:
[4] Mean Field Approximation.
[5] Landau theory

Magnetic phase transition

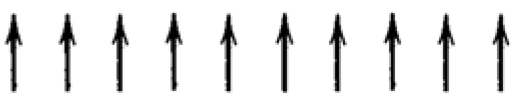
❑ **Mean field approximation.**

✓ Most difficult part is the interactions, even only consider the nearest neighbors…

✓ We can consider each spin s_i as being coupled to a reservoir consisting of its neighboring spins. These neighboring spins act then like a *fluctuating field* on the spin, and they form a net directed field on s_i, namely a mean field.

✓ The spin for each site $s_i = \langle s_i \rangle + \left(s_i - \langle s_i \rangle \right) = m + \left(s_i - m \right) = m + \delta s_i$

$\langle s_i \rangle = m$ uniformly.



✓ Hamiltonian ?

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - \sum_i s_i H = -J \sum_{\langle i,j \rangle} \left\{ m + \delta s_i \right\} \left\{ m + \delta s_j \right\} - \sum_i s_i H$$

$$= -J \sum_{\langle i,j \rangle} \left\{ m^2 + m\left(s_i - m \right) + m\left(s_j - m \right) + \delta s_i \delta s_j \right\} - \sum_i s_i H$$

$$= -J \sum_i \left(z m s_i - \frac{z}{2} m^2 \right) - \sum_i s_i H - J \sum_{\langle i,j \rangle} \delta s_i \delta s_j .$$

Here z is the number of nearest neighbors (for a hypercubic lattice in d dimensions $z = 2d$).

Pair-interactions are reduced to single particle energy.

Magnetic phase transition

- **Mean field approximation.** $$\mathcal{H} = -J\sum_i \left(zms_i - \frac{z}{2}m^2 \right) - \sum_i s_i H - J\sum_{\langle i,j\rangle} \delta s_i \delta s_j$$

In the mean field approximation we neglect the last term assuming that it is small. This means that the fluctuations around the mean value would be small,

$$E_{ij} = \frac{\langle \delta s_i \delta s_j \rangle}{\langle s_i \rangle \langle s_j \rangle} = \frac{\langle \delta s_i \delta s_j \rangle}{m^2} \ll 1$$

✓ the mean field Hamiltonian $$\mathcal{H}_{mf} = -\sum_i s_i h_{\text{eff}} + NJ\frac{z}{2}m^2 \text{ with } h_{\text{eff}} = Jzm + H$$

It has the form of an ideal paramagnet in a magnetic field h_{eff}.

✓ Now we can calculate the partition function, as a function of parameter m,

$$Z_N = \prod_i \sum_{s_i=\pm s} \exp\left(\frac{-\beta Jzm^2}{2} + \beta h_{\text{eff}} s_i \right) = \left\{ \sum_{s_i=\pm s} \exp\left(\frac{-\beta Jzm^2}{2} + \beta h_{\text{eff}} s_i \right) \right\}^N$$

Single particle partition function

$$= \exp\left(\frac{-\beta Jzm^2 N}{2} \right) \left\{ 2\cosh\left(\beta s h_{\text{eff}} \right) \right\}^N$$

✓ Free energy

$$F(T,H,m) = -k_B T \ln Z_N = NJ\frac{z}{2}m^2 - Nk_B T \ln\left\{ 2\cosh\left(\beta s h_{\text{eff}} \right) \right\}$$

Magnetic phase transition

- **Mean field approximation.**
- ✓ Free energy $$F(T,H,m) = -k_B T \ln Z_N = NJ\frac{z}{2}m^2 - Nk_B T \ln\{2\cosh(\beta s h_{\text{eff}})\}$$
- ✓ The equilibrium condition is reached when we find the minimum of F for given T and H. So we minimize F with respect to m as the only free variable,

$$0 = \frac{\partial F}{\partial m} = NJzm - NJzs\tanh(\beta s h_{\text{eff}}) \qquad m = s\tanh(\beta s(Jzm + H)) \qquad h_{\text{eff}} = Jzm + H$$

- ✓ In zero magnetic field this equation is given as $$m = s\tanh\left(\frac{Jzsm}{k_B T}\right)$$
- ✓ Determine the critical temperature T_c, we consider the magnetic susceptibility $\chi(T)$ at zero magnetic field,

$$\chi(T) = N\frac{dm}{dH}\bigg|_{H=0} = Ns\frac{d}{dH}\tanh[\beta(Jzsm(H) + sH)]\bigg|_{H=0} \approx \frac{Ns}{k_B T}\left\{Jzs\frac{dm}{dH}\bigg|_{H=0} + s\right\} = \frac{s}{k_B T}Jzs\chi(T) + \frac{Ns^2}{k_B T}$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x \approx 1 \qquad \chi(T) = \frac{Ns^2}{k_B T - Jzs^2}$$

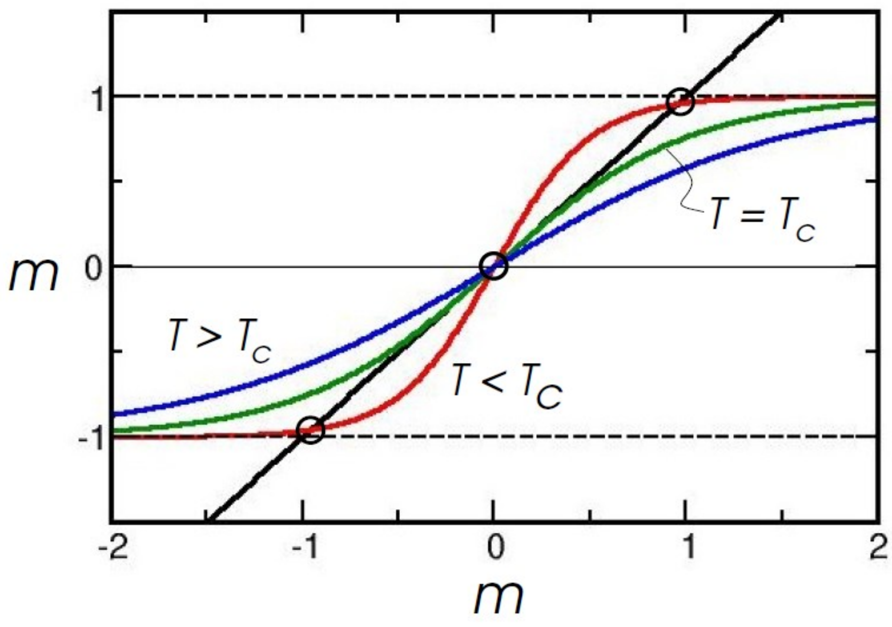
$\chi(T)$ diverges at the critical temperature $$T_c = Jzs^2 / k_B$$

Magnetic phase transition

- ❑ **Mean field approximation.**
- ✓ Looking at graphical solution of magnetization

$$m = s\tanh\left(\frac{Jzsm}{k_B T}\right) \qquad T_c = Jzs^2 / k_B$$

- ✓ T_c plays an important role in separating two types of solutions.
  For $T \geq T_c$ there is one single solution at $m = 0$
  For $T < T_c$ there are 3 solutions including $m = 0$, $m(T)$, $-m(T)$.



Magnetic phase transition

- **Mean field approximation: Free energy**
- ✓ Now we want to study the F: expand F in m assuming that m and H are small.

$$F(T,H,m) = -k_B T \ln Z_N = NJ\frac{z}{2}m^2 - Nk_BT\ln\left\{2\cosh\left(\beta s h_{\text{eff}}\right)\right\}$$

$$\ln(1+x) \approx x, \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\approx NJz\left[\frac{m^2}{2} - \frac{k_BT}{Jz}\left\{\frac{\left(\beta s h_{\text{eff}}\right)^2}{2} - \frac{\left(\beta s h_{\text{eff}}\right)^4}{12}\right\}\right] - Nk_BT\ln 2$$

Set $H = 0$, so $h_{\text{eff}} = Jzm + H = Jzm = \frac{T_c k_B}{s^2}m$, where $T_c = Jzs^2 / k_B$

$$F(T,H=0,m) \approx F_0(T) + NJz\left[\left(1 - \frac{T_c}{T}\right)\frac{m^2}{2} + \frac{1}{12s^2}\left(\frac{T_c}{T}\right)^3 m^4\right]$$

for $T \approx T_c$, $x = (T - T_c)/T_c$

$$\approx F_0(T) + NJz\left[\left(1 - \frac{1}{1+x}\right)\frac{m^2}{2} + \frac{1}{12s^2}m^4\right]$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\approx F_0(T) + \frac{NJz}{2}\left(\frac{T}{T_c} - 1\right)m^2 + \frac{NJz}{12s^2}m^4$$

The last step we took into account that our expansion is only valid for $T \approx T_c$. This form of the free energy is the famous **Landau theory of a continuous phase transition**.

Coefficient of m^2 term is temperature dependent, and m^4 term is temperature independent.

Magnetic phase transition

- ❑ **Mean field approximation.**

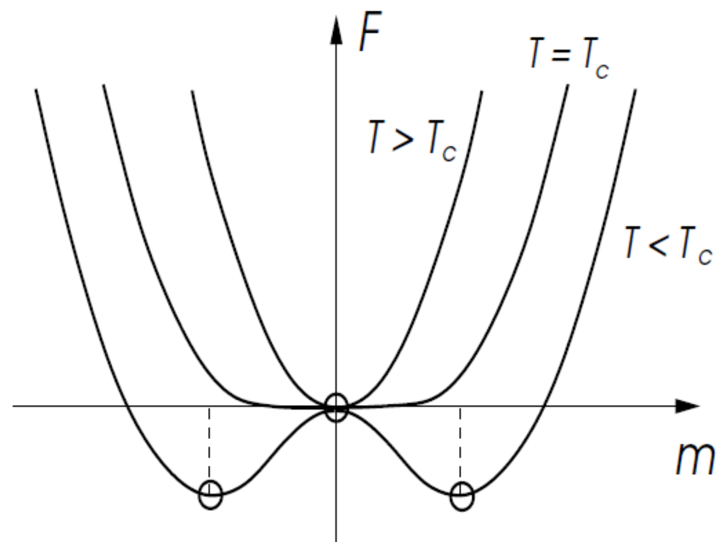
✓ For $H = 0$, we obtain

$$F(T) \approx F_0(T) + \frac{NJz}{2}\left(\frac{T}{T_c} - 1\right)m^2 + \frac{NJz}{12s^2}m^4$$

Graph of F vs m?

✓ For $T > T_c$ the minimum lies at $m = 0$.
For $T < T_c$ the coefficient of the m^2 term (2nd order) changes sign and a finite value of m minimizes F. The minimization are

$$m(T) = \begin{cases} \pm s\sqrt{3\tau} & T < T_c \\ 0 & T \geq T_c \end{cases} \qquad \tau = 1 - T/T_c$$



✓ There are two degenerate minima and the system chooses spontaneously one of the two (**spontaneous symmetry breaking**).

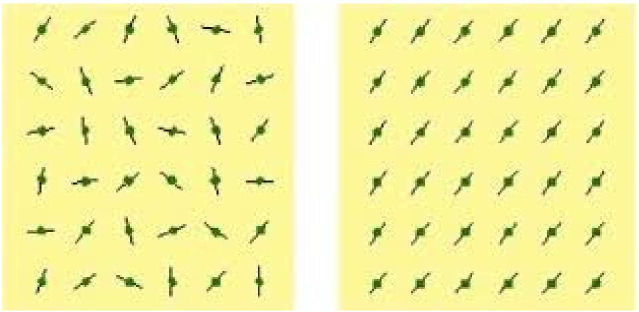
✓ This a very general feature for many phase transitions.

Phase transition: Landau theory

- **Symmetry:**

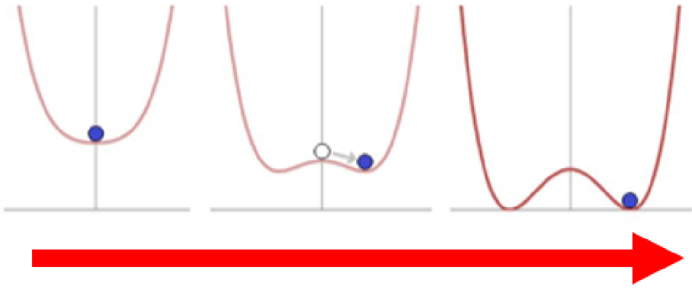
✓ Crossing the critical temperature, a magnetic system goes from paramagnetic (disordered spins) to ferromagnetic (ordered spins).

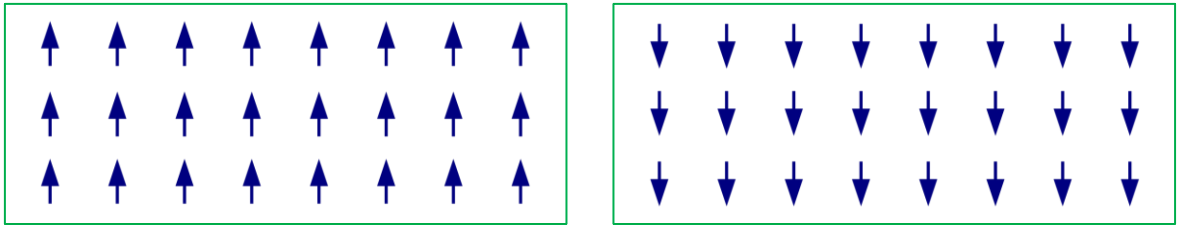
✓ The two phases have different spatial symmetries. Above the critical temperature, there is no magnetization, the system is rotationally invariant. Below the critical temperature, the non-zero magnetization vector defines a preferred direction in space, destroying the rotational invariance.



Disordered   Ordered phase

Spontaneous symmetry breaking

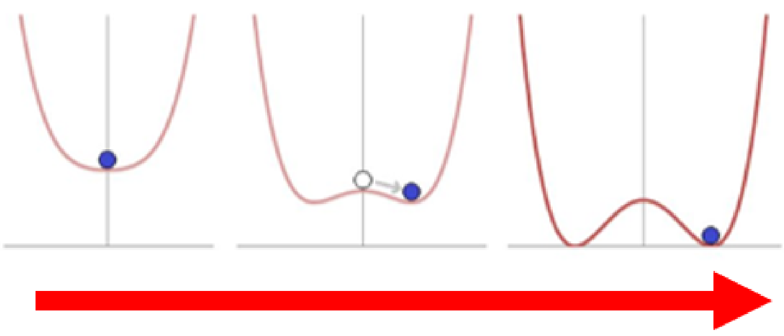




Phase transition: Landau theory

- ❑ **Order parameter:**
- ✓ Because of a reduction of the symmetry, an extra parameter is needed to describe the thermodynamics of the low temperature phase. The extra parameter is called the "order parameter".
- ✓ For ferromagnetic system, magnetic moment M is the order parameter.
- ✓ Near the critical point, the order parameter is the only important thermodynamics quantity.
- ✓ Order parameter density: $m(x)$. Such as the magnetic moment density, averaged over atomic distances.

Spontaneous symmetry breaking



Transition	*Order Parameter*
Ferromagnetic	M
Antiferromagnetic	Staggered M
gas-liquid	$V_G - V_L$
Superfluidity	$\int d^3r \langle \psi \rangle$
Superconductivity	$\int d^3r \langle \psi_\uparrow \psi_\downarrow \rangle$

Phase transition: Landau theory

- **Landau free energy:** Landau defined a "free energy" (density) using an expansion in powers of the order parameter density and its derivatives.

$$\psi(m(x), H(x)) = \frac{1}{2}|\nabla m(x)|^2 - \frac{1}{kT} m(x)H(x) + \frac{1}{2} r_0 m^2(x) + s_0 m^3(x) + u_0 m^4(x) + \cdots$$

- ✓ For $H = 0$, the free energy should be invariant under the transformation of $m \to -m$. So only the even power terms are allowed in the free energy.
- ✓ Second linear term comes from the Zeeman energy: the energy of magnetic moments in magnetic field.
- ✓ The first describes the energy fluctuations. A positive coefficient ensures that the spatially uniform state gives the lowest value of the free energy.
- ✓ All other coefficients are phenomenological parameters that may depend on the temperature and cutoff. In practice we simply choose them to suit our purpose. It is customary to assume that u_0 is independent of T, while $r_0 = a_0 t$, $t = (T - T_c)/T_c$, where a_0 is a positive constant.
- ✓ Using the **mean field approximation** (fluctuation is very small),

$$\psi(m(x), H(x)) = \frac{1}{2} r_0 m^2(x) + u_0 m^4(x) - \frac{1}{kT} m(x)H(x)$$

Phase transition: Landau theory

- **Mean field approximation:**

✓ We guess the free energy has the following form, and expect the state that minimizes the free energy to be the physically realized state.

$$\psi(m,H) = \frac{1}{2} r_0 m^2 + u_0 m^4 - \frac{mH}{kT}, \qquad \left.\frac{\partial \psi(m,H)}{\partial m}\right|_{m=\bar{m}} = 0, \quad \left.\frac{\partial^2 \psi(m,H)}{\partial m^2}\right|_{m=\bar{m}} \geq 0$$

✓ The mean field approximation is very crude. It assumes that the only important configuration is the uniform density near the critical point. We know from experiments that the density fluctuations are actually strong near the critical point. However, it is more convenient and of great value in exploring the qualitative phase structures of a system – Landau's original goal.

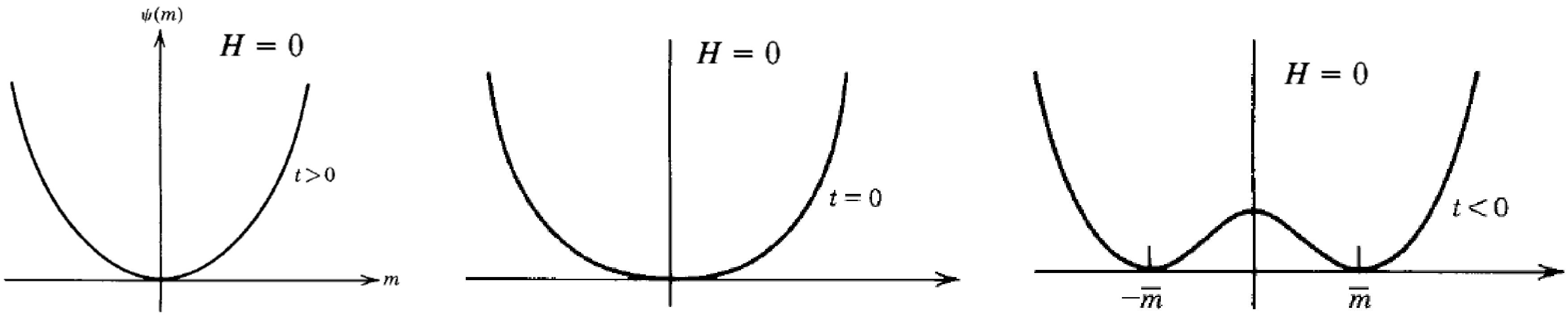
✓ It is a phenomenological approach that deals only with macroscopic quantities. The whole point is to avoid dealing with the underlying microscopic structure. (not easy to calculate partition function)

✓ It is meant to be used only in the neighborhood of a critical point, where the order parameter is small.

Phase transition: Landau theory

- Critical exponent :

$$\psi(m,H)=\frac{1}{2}r_0 m^2+u_0 m^4-\frac{mH}{kT},\quad u_0>0 \text{ and } r_0=a_0 t$$



The figure shows how a transition from a regime with $\bar{m}=0$ to one with $\bar{m}\neq 0$ can be described by an underlying continuous function ψ

$$\left.\frac{\partial\psi(m,H)}{\partial m}\right|_{m=\bar{m}}=0,\qquad r_0\bar{m}+4u_0\bar{m}^3-\frac{H}{kT}=0$$

$$H=0,\ \bar{m}=\begin{cases}0 & (t>0)\\ \pm\left(a_0/4u_0\right)^{1/2}|t|^{1/2} & (t<0)\end{cases}$$

$M\sim|t|^{\beta}$

critical exponent $\beta=\frac{1}{2}$

Phase transition: Landau theory

- **Critical exponent:**
  - ✓ To calculate the susceptibility we differentiate the $\bar{m}$ with respect to H,

$$\left.\frac{\partial \psi(m,H)}{\partial m}\right|_{m=\bar{m}} = 0, \qquad \chi = \frac{1}{kT\left(r_0 + 12u_0\bar{m}^2\right)} = \begin{cases} \left(kTa_0\right)^{-1} |t|^{-1} & (t>0) \\ \frac{1}{4}\left(kTa_0\right)^{-1} |t|^{-1} & (t<0) \end{cases}$$

$r_0 = a_0 t,\ t = (T - T_c)/T_c$

$\chi \sim |t|^{-\gamma}$, hence $\gamma = 1$

$$r_0\bar{m} + 4u_0\bar{m}^3 - \frac{H}{kT} = 0 \qquad \bar{m} = \begin{cases} 0 & (t>0) \\ \pm\left(a_0/4u_0\right)^{1/2} |t|^{1/2} & (t<0) \end{cases}$$

  - ✓ The equation of state (M-H) at $t = 0$ is obtained from $r_0\bar{m} + 4u_0\bar{m}^3 - H/kT = 0$ by setting $r_0 = 0$: $\bar{m}(0,H) = \left(4u_0 kT_c\right)^{-1/3} H^{1/3}$

$M \sim H^{1/\delta}$, hence $\delta = 3$

  - ✓ The free energy density

$$\psi(m,H) = \frac{1}{2}r_0 m^2 + u_0 m^4 - \frac{H}{kT} \qquad \psi(\bar{m},0) = \begin{cases} 0 & (t>0) \\ \left(3a_0^2/16u_0\right)t^2 & (t<0) \end{cases}$$

  - ✓ The heat capacity is $\frac{C}{V} = \begin{cases} 0 & (t>0) \\ 3a_0^2 kT_c^2/8u_0 & (t<0) \end{cases}$

$C \sim |t|^{-\alpha}$, $\alpha = 0.$

$$S = -\frac{\partial \psi}{\partial T} \qquad C_V = T\frac{\partial S}{\partial T} = -T\left(\frac{\partial^2 \psi}{\partial T^2}\right)$$

Phase transition: Landau theory

❑ Scaling law and Universality:

✓ Only two of the six critical exponents defined in the preceding discussion are independent, because of the following "scaling laws":

Fisher law: $\gamma = \nu(2-\eta)$ Widom law: $\gamma = \beta(\delta - 1)$

Rushbrooke law: $\alpha + 2\beta + \gamma = 2$ Josephson law: $\nu d = 2 - \alpha$

$$\text{Mean field: } \alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3$$

✓ The following table summarized the experimental values of the critical exponents as well as the results from some theoretical models. We can see that the scaling laws seem to be universal.

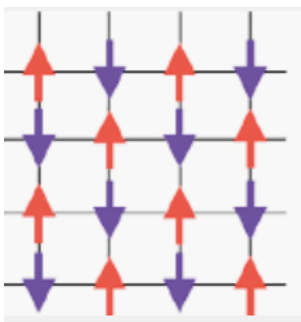
Exponent	*TH*	*EXPT*	*MFT*	*ISING2*	*ISING3*	*HEIS3*
α		0–0.14	0	0	0.12	−0.14
β		0.32–0.39	1/2	1/8	0.31	0.3
γ		1.3–1.4	1	7/4	1.25	1.4
δ		4–5	3	15	5	
ν		0.6–0.7	1/2	1	0.64	0.7
η		0.05	0	1/4	0.05	0.04
$\alpha + 2\beta + \gamma$	2	2.00 ±0.01	2	2	2	2
$(\beta\delta - \gamma)/\beta$	1	0.93 ±0.08	1	1	1	
$(2 - \eta)\nu/\gamma$	1	1.02 ±0.05	1	1	1	1
$(2 - \alpha)/\nu d$	1		$4/d$	1	1	1

Monte Carlo simulations

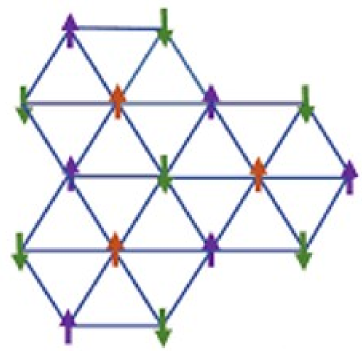
- **Spin model:**
- ✓ The energy for the spin model with isotropic interactions:

$$E\{s_i\} = -J\sum_{\langle ij \rangle} s_i s_j - H\sum_{i=1}^{N} s_i$$

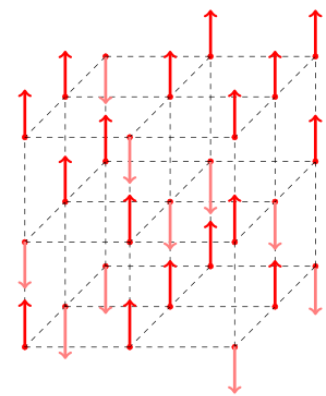
$J > 0$ *and* < 0 correspond to ferromagnetism and antiferromagnetism, respectively.



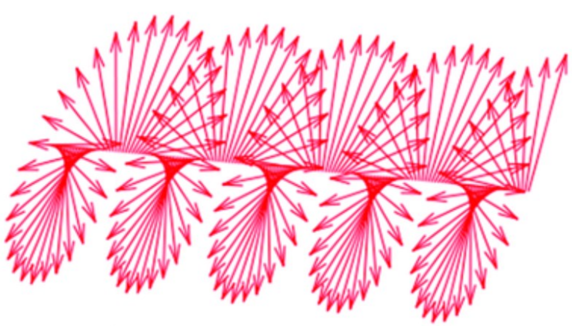
2-D Ising: Up/Down

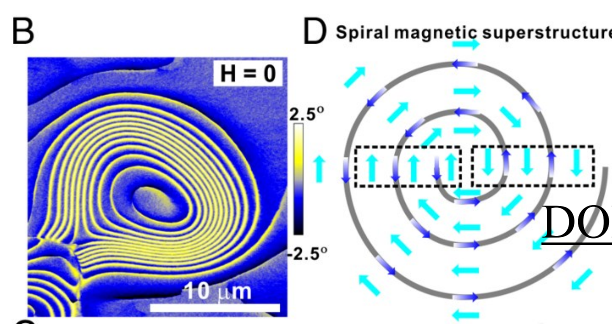


2-D XY: In-plane spin



3-D (cubic)





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DOI: 10.1073/pnas.2023337118

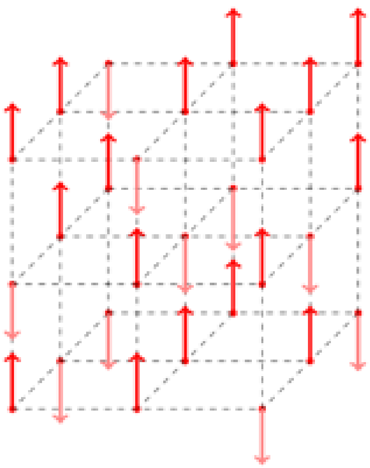
Citations ~ 89 in 5 years

Monte Carlo simulations

❑ Numerical simulation

✓ If the lattice is complicated, it is not easy to analytical solve the partition function.

✓ However, we know the probability of appearance of a given configuration is proportional to its **Boltzmann weight** $\exp(-E/kT)$.



3-D cubic lattice

✓ **Monte Carlo simulations** work by creating a stream of lattice configurations that together approximate an ensemble with **Boltzmann weighting**.

✓ Suppose we have an initial configuration $a = 0$, from which we will produce $a = 1, 2, \ldots$ sequentially, with each configuration determined by the previous one (how?) and some Monte Carlo algorithm.

Monte Carlo simulations

❑ Numerical simulation

✓ We consider the probability $P(a)$ and $P(b)$ for two configurations a *and* b:

$$\frac{P(a)}{P(b)} = \frac{\exp(-E^a / T)}{\exp(-E^b / T)} = \frac{w(a \leftarrow b)}{w(b \leftarrow a)}$$

where $w(b \leftarrow a)$ is the rate at which a configuration a becomes a configuration b.

✓ For rapid equilibration, the w is given by the Metropolis algorithm

$$w(b \leftarrow a) = \min\left[1, \exp\left(E^a / T - E^b / T\right)\right]$$

✓ So that $w(b \leftarrow a) = 1$ if $E^b < E^a$, but is less than one if $E^b > E^a$.

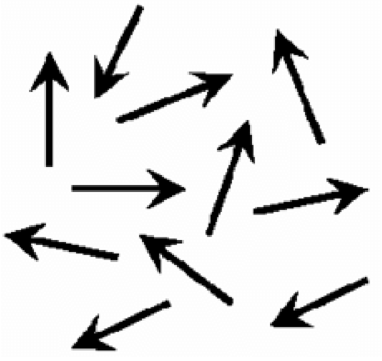
✓ The essential algorithm is this: consider a change from a to b:
(i) if the change lowers the energy, always make the change; **(ii)** if it raises the energy, make the change with a **probability** given by $p = \exp(E^a/T - E^b/T)$.
For (ii), p can be compared with a **random number *R*** between [0, 1]: if $p > R$, make the change; if $p < R$, no change.

XY model

- **A specific spin model: XY model**
  - ✓ Let's consider the so-called XY model which describes a system of 2-D spins $\boldsymbol{s} = (s\cdot\cos\varphi,\ s\cdot\sin\varphi)$ with ferromagnetic interaction i.e. with the Hamiltonian

$$\mathcal{H} = -J\sum_{i,j} \mathrm{s}(i)\cdot\mathrm{s}(j) = -Js^2\sum_{i,j}\cos\left(\theta_i - \theta_j\right)$$

$$\text{Set } s = 1, \quad \mathcal{H} = -J\sum_{i,j}\cos\left(\theta_i - \theta_j\right)$$



  - ✓ The partition function of this model is

$$Z = \int_0^{2\pi}\prod_i \frac{d\theta_i}{2\pi}\exp\left[\beta J\sum_{\langle ij\rangle}\cos\left(\theta_i - \theta_j\right)\right]$$

XY model

❑ **Python code and results:**

```python
def Metropolis(S, T):  # Define Metropolis function (S: initial configuration, T: temperature)
    delta_max = 0.5 * np.pi # the maximum change of angle
    k = 1  # Boltzmann constant
    for i in range(S.shape[0]):
        for j in range(S.shape[0]):
            delta = (2 * np.random.random() - 1) * delta_max   # angle change from -90 to 90
            newAngle = S[i, j] + delta  # new angle
            energyBefore = getEnergy(i=i, j=j, S=S, angle=None)  # get the energy
            energyLater = getEnergy(i=i, j=j, S=S, angle=newAngle)  # get the energy of new angle
            alpha = min(1.0, math.exp(-(energyLater - energyBefore)/(k * T)))  # Boltzmann distribution
            if random.uniform(0, 1) <= alpha:
                S[i, j] = newAngle   # Accept new state
            else:
                pass  # Keep the last state
    return S
```

$$\mathcal{H} = -Js^2 \sum_{i,j} \cos\left(\theta_i - \theta_j\right) \qquad w(b \leftarrow a) = \min\left[1, \exp\left(E^a/T - E^b/T\right)\right]$$

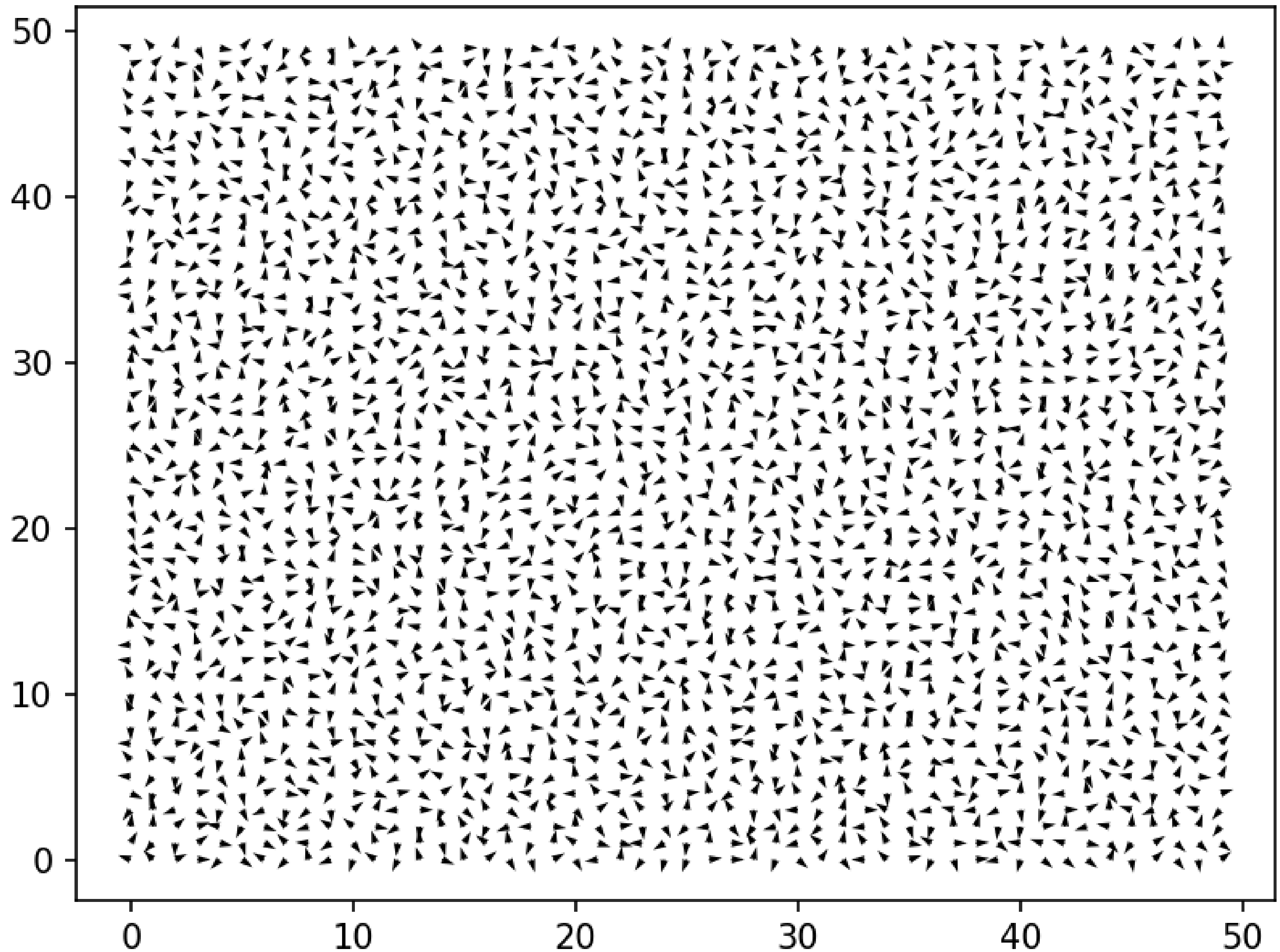
✓ The essential algorithm is this: consider a change from *a* to *b*:

(i) if the change lowers the energy, always make the change; **(ii)** if it raises the energy, make the change with a **probability** given by $p = \exp(E^a/T - E^b/T)$.

For (ii), p can be compared with a **random number *R*** between [0, 1]: if $p > R$, make the change; if $p < R$, no change.

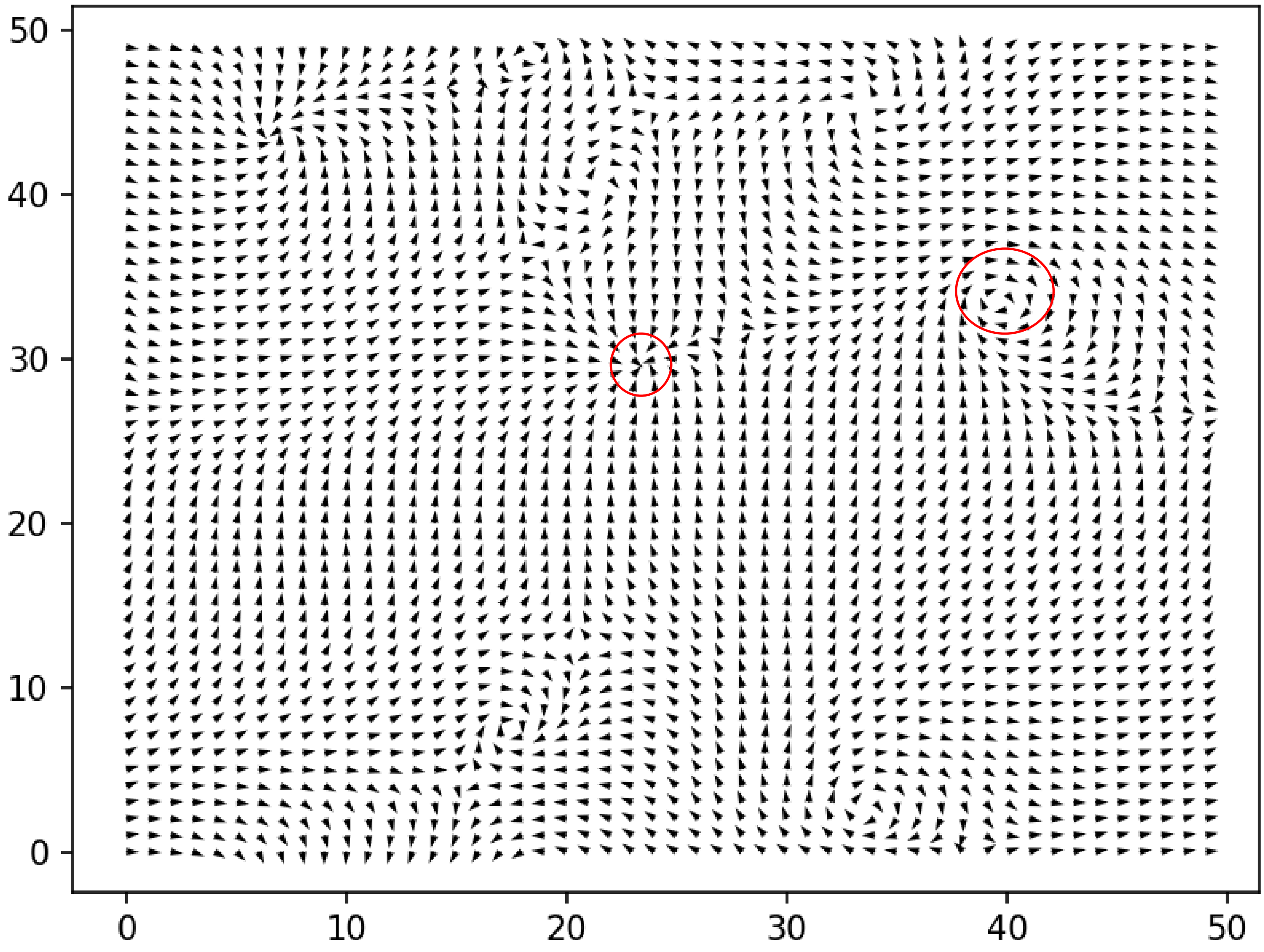
XY model

- **Python code and results:** 50 × 50, $T = 10$ K, thermal energy $k_B T >> J$



XY model

- **Python code and results:** 50 × 50, $T = 1$ K, thermal energy $k_B T << J$

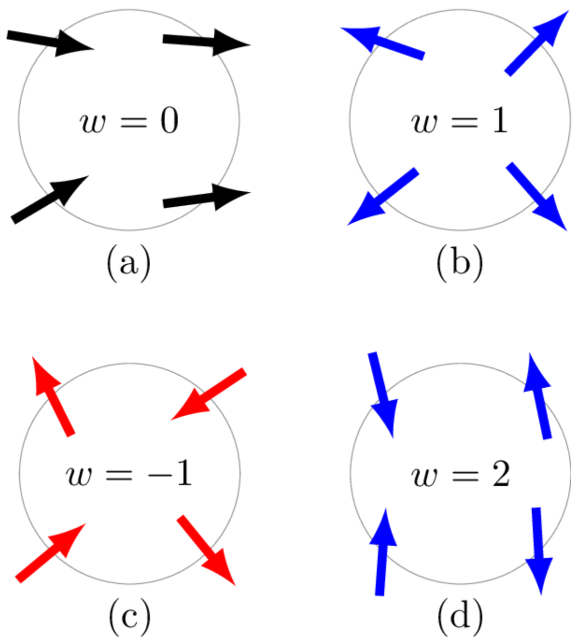


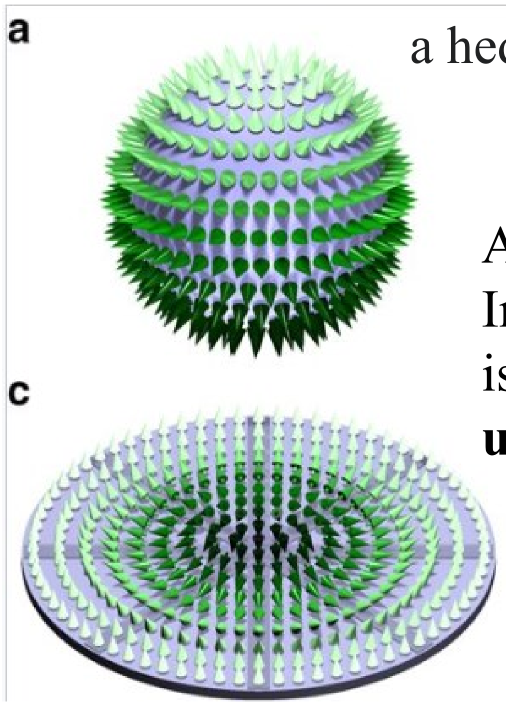
XY model

❑ **Berezinskii-Kosterlitz-Thouless (BKT) phase transition:**

✓ The BKT transition is a topological phase transition, in the sense that it involves the winding of topological vortex. BTK theory is also important for understanding many other condensed matter systems such as superfluid and superconductivity.

✓ Work on the transition led to the 2016 Nobel Prize in Physics being awarded to Thouless, Kosterlitz and Duncan Haldane.

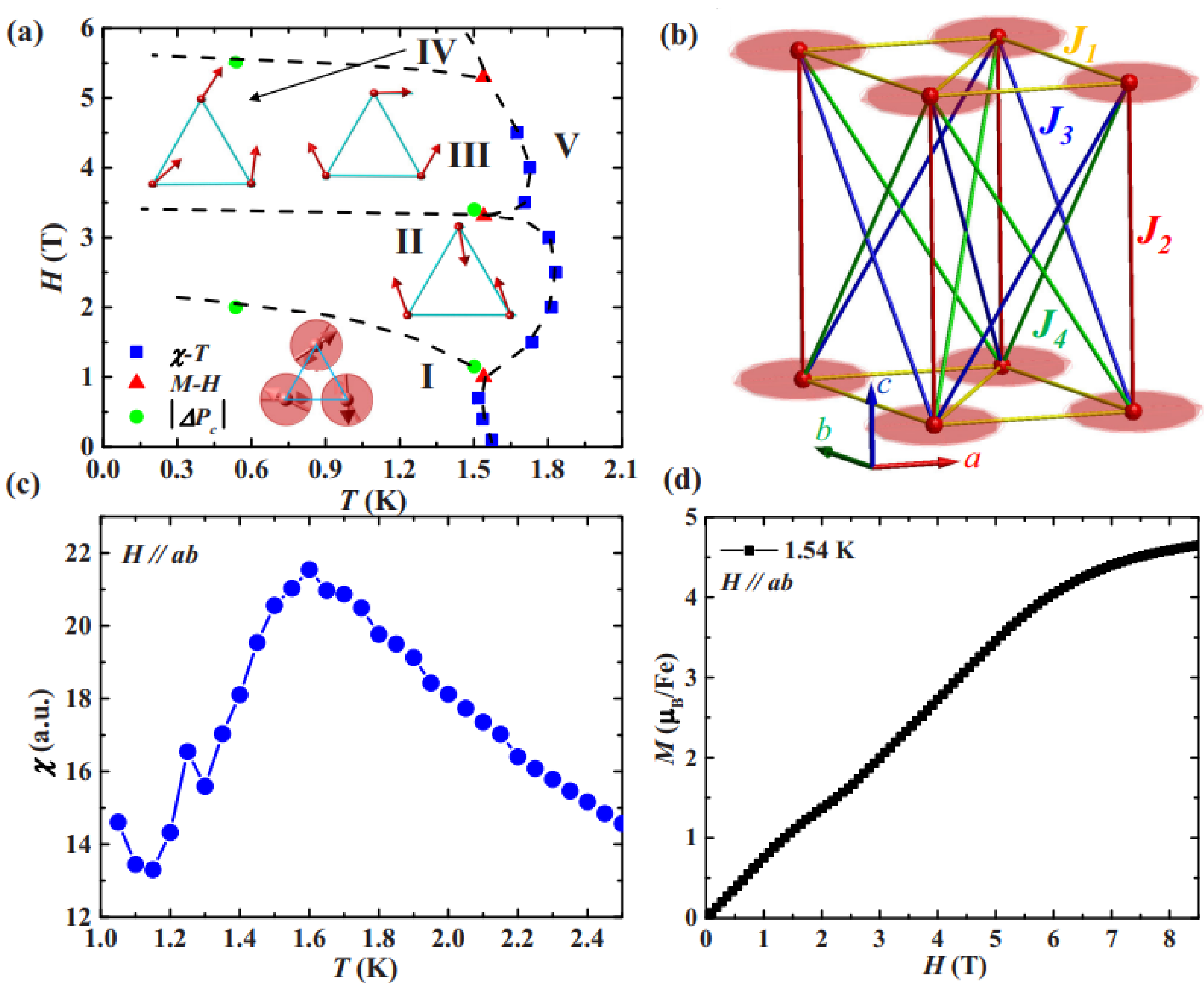




a hedgehog skyrmion

A skyrmion:
In high-energy physics, it is a **theoretical soliton used to model baryons.**

More Examples



Junjie Yang et al, *Phys. Rev. Materials* 9, 114405(2025).

Phase transition: Landau theory

- **Tricritical point: (optional)**
  - ✓ Within the mean field approximation, consider

$$\psi(m,H) = \frac{1}{2}am^2 + \frac{1}{4}bm^4 + \frac{1}{6}cm^6 - mH/kT$$

For simplicity we take c to be a positive constant; but a and b are variable.

  - ✓ Introducing $t = (T - T_c)/T_c$ and $p = (P - P_c)/P_c$, we shall take a and b to be linear in t and p, so $a = At + Bp$, and $b = Ct + Dp$
  - ✓ First put $H = 0$. To find the minima of ψ, we need

$$\psi'(m) = am + bm^3 + cm^5$$

$$\psi''(m) = a + 3bm^2 + 5cm^4$$

  - ✓ Setting $\partial\psi/\partial m = 0$, we obtain five roots, as befits a quantic polynomial:

$$\bar{m} = 0, \quad \pm m_+, \quad \pm m_- \qquad\qquad m_\pm^2 = \frac{1}{2c}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

  - ✓ Noting that $\psi''(0) = a$, we conclude that $\bar{m} = 0$ is a maximum if $a < 0$. This means that for $a < 0$, the only possible minima are $\pm m_\pm$. Substituting the $m_\pm$ into ψ'' we obtain

$$\psi''(m_\pm) = \pm\frac{1}{c}\sqrt{b^2 - 4ac}\,m_\pm^2$$

  - ✓ Therefore m_+ corresponds to a minimum, while m_- corresponds to a maximum

Phase transition: Landau theory

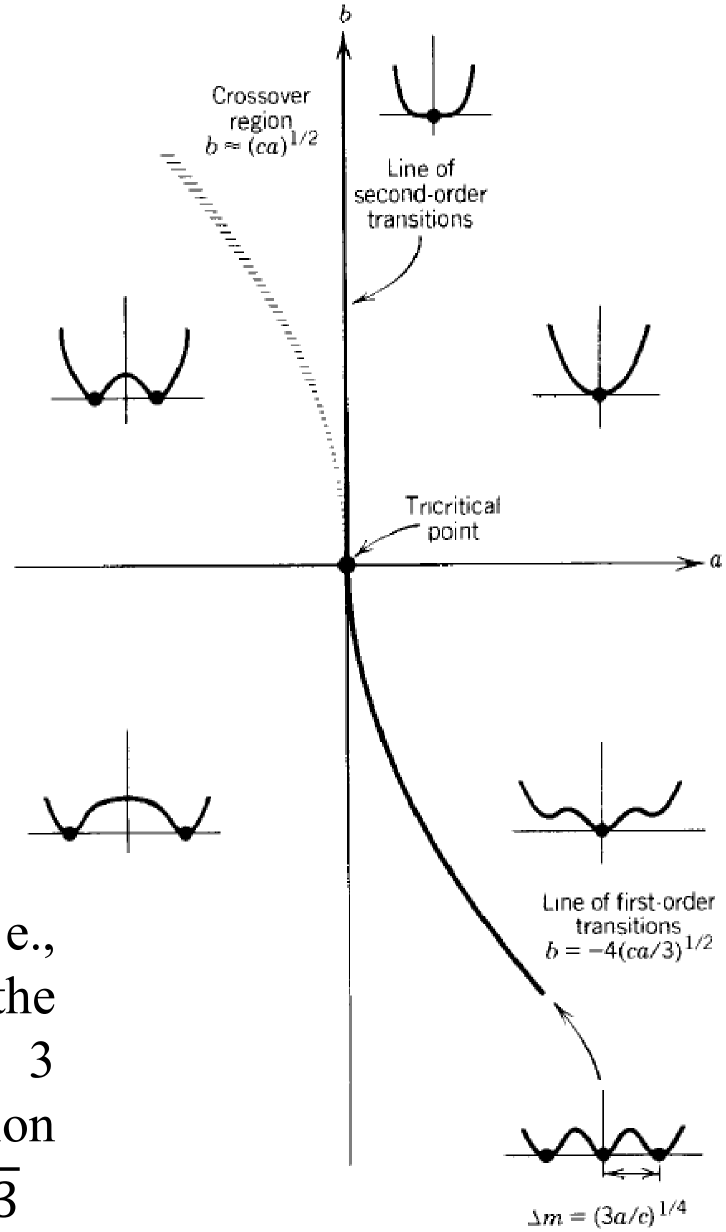
- **Tricritical point:** **(optional)**
  - ✓ For $a < 0$, m_+ corresponds to a minimum, while m_- corresponds to a maximum

$$\bar{m}^2 = \frac{1}{2c}\left(\sqrt{b^2 - 4ac} - b\right), \quad (a < 0)$$

$$\psi'(m) = am + bm^3 + cm^5, \quad \psi''(m) = a + 3bm^2 + 5cm^4$$

$$\psi''(m_\pm) = \pm\frac{1}{c}\sqrt{b^2 - 4ac}\,m_\pm^2 \qquad \bar{m} = 0, \quad \pm m_+, \quad \pm m_-$$

  - ✓ Consider now $a > 0$. In this region $\bar{m} = 0$ is a minimum. The question is whether it is a lowest minimum.
  - ✓ If $a > 0$ and $b > 0$, the only minimum there is $\bar{m} = 0$.
  - ✓ If $a > 0$ and $b < 0$, then there are 3 minima, i. e., $\bar{m} = 0$ and $\bar{m} = \pm m_+$. Since $\psi(0) = 0$, the locus of points in the ab plane where the 3 minima are equal is determined by the condition $\psi(m_+) = 0$, so $m_+^2 = -\frac{4a}{b}$ and $b = -4\sqrt{ca/3}$



Phase transition: Landau theory

- **Tricritical point:** **(optional)**

✓ If $a > 0$ and $b < 0$, then there are 3 minima, i. e., $\bar{m} = 0$ and $\bar{m} = \pm m_+$. Since $\psi(0) = 0$, the locus of points in the ab plane where the 3 minima are equal is determined by the condition $\psi(m_+) = 0$, so $m_+^2 = -\frac{4a}{b}$ and $b = -4\sqrt{ca/3}$

✓ This is a line of first-order transition, across which $\bar{m}$ changes discontinuously by the amount

$$\Delta m = \left(-\frac{4a}{b}\right)^{1/2} = \left(\frac{3a}{c}\right)^{1/4}$$

✓ The conclusions are summarized in left figure. The point a=b=0 marks the termination of a 2nd phase transition, and the ordinary critical point of a 1st phase transition. We call it "tricritical point".