Week 9

Bose system and BEC

Bose gas: Discussion

- ☐ **Bose Gas.** In terms of particle number *N*, there are two situations for Bosons:
  - ✓ Bosons which results as modes of harmonic oscillators (particle number not fixed), e.g. photons, phonons, etc. chemical potentiel $\mu = 0$.
  - ✓ a system with well-defined particle number, e.g. bosonic atoms, such as the ^{4}He; Let's look at this now. chemical potentiel $\mu = ?$

- ☐ **Bosonic Atoms.** We consider Bosons without spin ($S = 0$) for which ^{4}He is a good example. Analogously to the Fermions we introduce functions of *z* to express the equation of state and the particle number. The bose occupation number

$$\langle n_{\vec{k}} \rangle_{+} = \frac{1}{\exp[\beta(\varepsilon(\vec{k}) - \mu)] - 1}, \qquad \varepsilon = \hbar^2 k^2 / 2m$$

Questions:

[1] Any requirements on the the value of $\langle n_{\vec{k}} \rangle_{+}$ we must consider?

[2] Any requirements on the chemical potential we have to consider?
Positive/negative at high temperature? Positive/negative at low temperature?

[3] How does chemical potential change with decreasing temperature?

[4] How does $\langle n_{\vec{k}} \rangle_{+}$ change with decreasing temperature?

Bose gas

- **Bosonic Atoms.** We consider Bosons without spin ($S = 0$) for which ^{4}He is a good example. The average bose occupation number

$$\langle n_{\vec{k}} \rangle_+ = \frac{1}{\exp[\beta(\varepsilon(\vec{k}) - \mu)] - 1}, \qquad \varepsilon = \hbar^2 k^2 / 2m$$

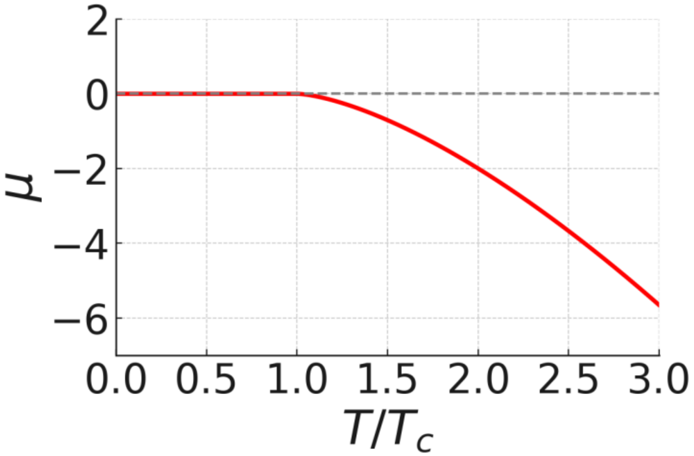
$\langle n_{\vec{k}} \rangle_+$ must be positive. This requires $\varepsilon - \mu$ to be positive for all $\vec{k}$, and hence $\mu < \min[\varepsilon]$, $\mu \leq 0$.

At high T, $\mu < 0$, classical ideal gas.

If T decreases, more particles will fall onto the ground state ($\varepsilon = 0$), so $\langle n_{\vec{k}} \rangle_+$ for the ground state increases. This requires μ to increase (less negative).

At low T, μ approaches zero.

At as special T_c, μ changes to zero. Phase transition?



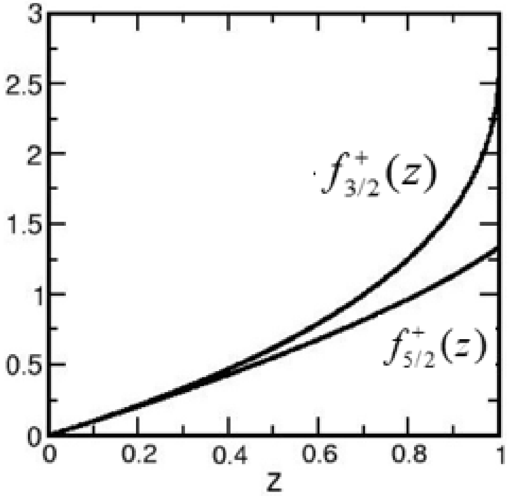
Bose gas

❑ **Bose Gas:** $n_+ = \frac{1}{\lambda^3} f^+_{3/2}(z)$ $\quad \beta P_+ = \frac{1}{\lambda^3} f^+_{5/2}(z)$ $\quad f^+_m(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{\mathrm{d}x x^{m-1}}{z^{-1}\mathrm{e}^x - 1}$

In the degenerate quantum limit (low T), μ approaches its limiting value of zero.

To find out about the behavior of bose gas at degenerate quantum limit, we have to examine the limiting behavior of $f^+_m(z)$ as $z=\exp(\beta\mu)$ goes to unity.

The functions $f^+_m(z)$ are monotonically increasing with z in the interval $0 \le z \le 1$. The maximum value attained at $z = 1$ (μ approaches 0 at low T)



$$n_+ = \frac{1}{\lambda^3} f^+_{3/2}(z)$$

If T is lowered, $\lambda \propto T^{-1/2}$ increases, such that z has to increase to keep the particle density n unchanged.

For $z = 1$, $\quad \zeta_m \equiv f^+_m(1) = \frac{1}{(m-1)!}\int_0^\infty \frac{\mathrm{d}x x^{m-1}}{\mathrm{e}^x - 1}$ $\quad$ Riemann's ζ_m function

The integrand has a pole as $x \to 0$, where it behaves as $\int dx x^{m-2}$. Therefore, ζ_m is finite for $m > 1$ and infinite for $m \le 1$. (Something (What?) is instable?)

Bose gas

$$f_m^+(z)=\frac{1}{(m-1)!}\int_0^\infty \frac{\mathrm{d}x\,x^{m-1}}{z^{-1}\mathrm{e}^x-1}$$

- **Instability of compressibility for Bose gas at low-*T*.**

Let’s exam compressibility (indicator of phase transition)

$$\kappa_T=-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T=\frac{1}{(N/V)}\left(\frac{\partial(N/V)}{\partial P}\right)_T=\frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_T=\frac{1}{n}\left(\frac{dn/dz}{dP/dz}\right)_T$$

$$\beta P=\frac{1}{\lambda^3}f_{5/2}^+(z)$$

$$n=\frac{1}{\lambda^3}f_{3/2}^+(z)$$

A useful recursive property of $f_m^\pm(z)$ is (for $m>1$)

$$\frac{\mathrm{d}}{\mathrm{d}z}f_m^\pm(z)=\int_0^\infty \mathrm{d}x\,\frac{x^{m-1}}{(m-1)!}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z^{-1}\mathrm{e}^x\mp1}\right)$$

$$\left(\text{use}\ \frac{\mathrm{d}}{\mathrm{d}z}f\left(z^{-1}\mathrm{e}^x\right)=-\frac{\mathrm{e}^x}{z^2}f'=-\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}x}f\left(z^{-1}\mathrm{e}^x\right)\right)$$

$$=-\frac{1}{z}\int_0^\infty \mathrm{d}x\,\frac{x^{m-1}}{(m-1)!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{z^{-1}\mathrm{e}^x\mp1}\right)$$

Integrate by parts

$$=\frac{1}{z}\int_0^\infty \mathrm{d}x\,\frac{x^{m-2}}{(m-2)!}\frac{1}{z^{-1}\mathrm{e}^x\mp1}=\frac{1}{z}f_{m-1}^\pm(z)$$

$$\begin{cases}\dfrac{\mathrm{d}P}{\mathrm{d}z}=\dfrac{k_BT}{\lambda^3}\dfrac{1}{z}f_{3/2}^+(z)\\[2ex]\dfrac{\mathrm{d}n}{\mathrm{d}z}=\dfrac{1}{\lambda^3}\dfrac{1}{z}f_{1/2}^+(z)\end{cases}$$

$$\kappa_T=\frac{f_{1/2}^+(z)}{nk_BTf_{3/2}^+(z)}$$

$$\lim_{z\to1}f_{1/2}^+(z)\to\infty$$

Infinite for $m\le1$: diverges at z=1, indicates an instability at low T.

Bose gas

❑ Critical temperature/volume of Bose gas at low-*T*.

- ✓ We let $z = 1$, to determine the critical temperature for the instability

$$n = \frac{1}{\lambda^3} f^+_{3/2}(z=1) = \frac{1}{\lambda^3}\zeta_{3/2} \qquad \zeta_m \equiv f^+_m(z=1) \qquad \zeta(3/2)\ (\sim 2.612)$$

$$= \frac{1}{h^3}(2\pi m k_B T)^{3/2}\zeta_{3/2} \qquad T_c(n) = \frac{h^2}{2\pi m k_B}\left(\frac{n}{\zeta_{3/2}}\right)^{2/3}$$

- ✓ Or μ approaches the singular point 0 ($z = 1$) by compressing. Critical volume can be determined by $V_c = \frac{h^3}{(2\pi m k_B T)^{3/2}} \frac{N}{\zeta_{3/2}}$

- ✓ It is useful to express the $\zeta(3/2)$ (~ 2.612) as a function of T and T_c.

$$\zeta\left(\frac{3}{2}\right) = \frac{N}{V}\left(\frac{h}{\sqrt{2\pi m k_B T_c}}\right)^3 = \frac{N}{V}\left[\frac{h}{\sqrt{2\pi m k_B T}}\cdot\frac{\sqrt{T}}{\sqrt{T_c}}\right]^3 = \frac{N}{V}\lambda^3\left(\frac{T}{T_c}\right)^{3/2}$$

Then, the questions is what happens to the Bose gas at $T < T_c$ or $V < V_c$.

Bose gas: Discussion

❑ Bose-Einstein condensation.

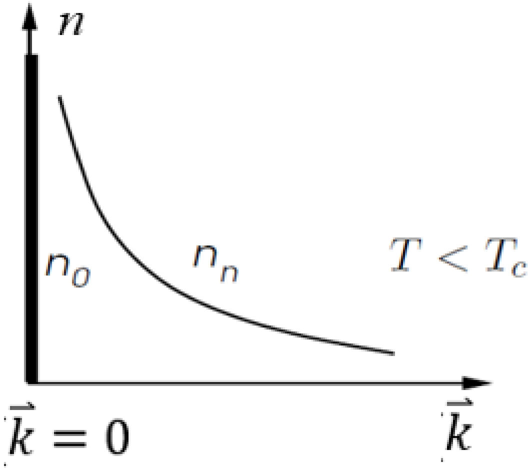
$$n = \frac{1}{\lambda^3} f^+_{3/2}(z) \qquad f^+_m(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{\mathrm{d}x x^{m-1}}{z^{-1}\mathrm{e}^x - 1} \qquad x = \frac{\beta\hbar^2 k^2}{2m}$$

✓ We did not count the occupation of the state $\vec{k} = 0$ in the integral, and we approximately treat it as zero at $n\left(\vec{k} = 0\right) = 0$.

✓ This is fine as long as the occupation of the $\vec{k} = 0$ state is vanishingly small compared to N.

✓ However, for $T < T_c$ or $V < V_c$, $\lim_{\vec{k}\to 0} n(\vec{k}) \neq 0$, since both numerator and denominator in the integrand go to zero.

✓ The occupation becomes macroscopic. We can not neglect this contribution in the calculation of n.



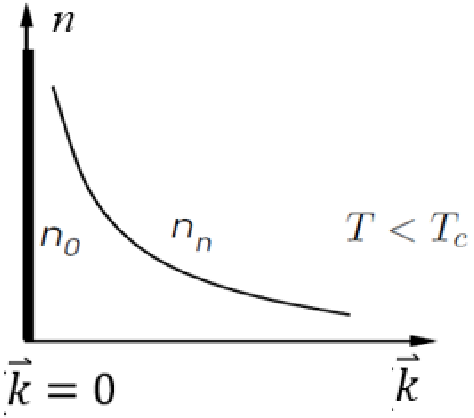
Q: How can we solve this $\vec{k} = 0$ problem?

What is the value of $n\left(\vec{k} = 0\right)$?

What is the physical meaning of $n = \frac{1}{\lambda^3} f^+_{3/2}(1)$?

Bose gas

❑ Bose-Einstein condensation.



- ✓ The occupation becomes macroscopic. We can not neglect this contribution in the calculation of N. Thus, we need to correct density is

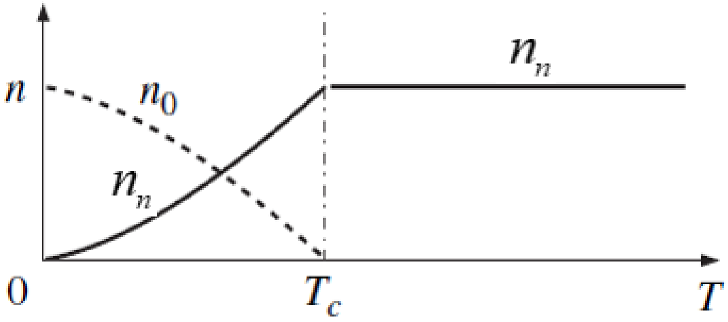
$$N/V = \frac{1}{\lambda^3} f^{+}_{3/2}(z=1) + n_0(T) = n_n(T) + n_0(T)$$

$n_0(T)$ denotes the density of Bosons at $\vec{k} = 0$ state

- ✓ We encounter here a "two-fluid" system: the total particle density split into a condensed fraction n_0 and a normal fraction n_n. The normal fraction n_n is just "normal". The condensed fraction n_0 is the key.
- ✓ We can find the temperature dependence of n_0 ($z = 1$ when $T < T_c$)

$$N/V = \frac{1}{\lambda^3} f^{+}_{3/2}(1) + n_0(T) = \frac{1}{\lambda^3} \zeta_{3/2} + n_0(T)$$

$$\zeta\left(\frac{3}{2}\right) = \frac{N}{V} \lambda^3 \left(\frac{T}{T_c}\right)^{3/2} \qquad n_0(T) = \frac{N}{V}\left[1 - \left(\frac{T}{T_C}\right)^{3/2}\right]$$



Bose gas: Discussion

- ☐ **Pressure.** Next we determine the pressure below T_c

$$\beta P = \frac{1}{\lambda^3} f^+_{5/2}(z=1) = \frac{1}{\lambda^3}\zeta_{5/2} \approx 1.341\frac{1}{\lambda^3} \qquad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

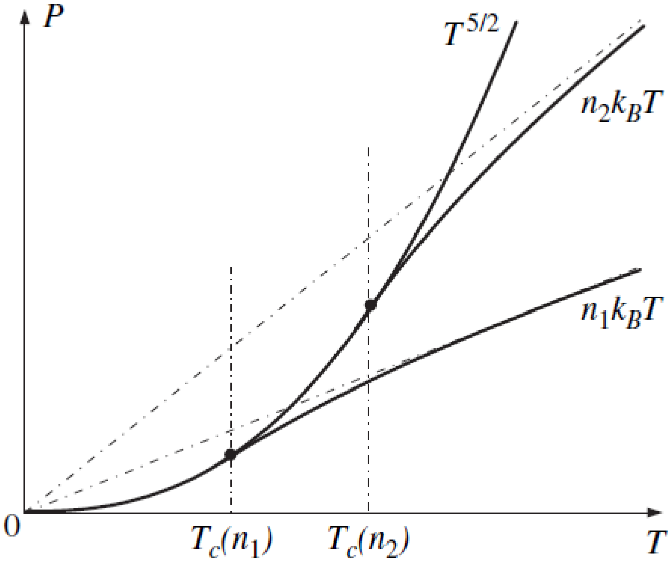
$$P \approx \frac{1.341(2\pi m)^{3/2}}{h^3}(kT)^{5/2}$$

Questions:

[1] How does the pressure of BEC condensed phase distinguish from that of the normal phase?

[2] For normal phase, what is the relationship between pressure, temperature, and density?

[3] Does the pressure of BEC condensed phase depend on volume or n? Why?



$$P = \frac{3}{2V}(E_n + E_0)$$

In condensed phase pressure is independent of volume or n.

Bose gas

- **Entropy.** Next we consider the entropy S from the grand canonical potential

  Q: What is the entropy at $T < T_c$ where BEC occurs? Is the entropy zero?

$$\mathcal{G}(T,\mu,V) = E - TS - \mu N = -PV \qquad S = -\left.\frac{\partial \mathcal{G}}{\partial T}\right|_{\mu,V} = \left.\frac{\partial (PV)}{\partial T}\right|_{\mu,V},$$

For $T > T_c$, V and μ are fixed, $S = \frac{5Vk_B}{2\lambda^3} f^{+}_{5/2}(z) - Nk_B \ln z$

$$S = \frac{\partial}{\partial T}\frac{Vk_BT}{\lambda^3} f^{+}_{5/2}(z) = \frac{5Vk_B}{2\lambda^3} f^{+}_{5/2}(z) + \left\{ \frac{Vk_BT}{\lambda^3}\,\frac{f^{+}_{3/2}(z)}{z}\,\frac{\partial z}{\partial T} = -k_B \frac{V}{\lambda^3} f^{+}_{3/2}(z)\beta\mu = -Nk_B \ln z \right\}$$

$$P = \frac{k_BT}{\lambda^3} f^{+}_{5/2}(z) \qquad \frac{\mathrm{d}}{\mathrm{d}z} f^{+}_{m}(z) = \frac{1}{z} f^{+}_{m-1}(z) \qquad \frac{N}{V} = \frac{1}{\lambda^3} f^{+}_{3/2}(z)$$

For $T < T_c$, $z = 1$, $PV = V\frac{k_BT}{\lambda^3}\zeta_{5/2}$

$$S = \left.\frac{\partial (PV)}{\partial T}\right|_{V,\mu} = V\,\frac{5}{2}\,\frac{k_B}{\lambda^3}\zeta_{5/2} = \frac{5Nk_B}{2}\left(\frac{T}{T_c}\right)^{3/2}\frac{\zeta_{5/2}}{\zeta_{3/2}} \qquad \zeta\left(\frac{3}{2}\right) = \frac{N}{V}\lambda^3\left(\frac{T}{T_c}\right)^{3/2}$$

Bose gas

❑ **Heat capacity.**

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{3}{2} V \left(\frac{\partial P}{\partial T}\right)_{V,N}$$

$$E = (3/2)PV$$

$$P = \frac{k_B T}{\lambda^3} f^+_{5/2}(z)$$

$$\frac{N}{V} = \frac{1}{\lambda^3} f^+_{3/2}(z)$$

Fixed particle number, for $T > T_c$,

$$V \frac{\partial P}{\partial T} = \left[\frac{5}{2} k_B \lambda^{-3} f^+_{5/2}(z) + \frac{k_B T}{\lambda^3} f'^+_{5/2}(z)\right] \cdot V$$

$$= \left[\frac{5}{2} k_B \lambda^{-3} f^+_{5/2}(z) V - \frac{3}{2} \frac{f^+_{3/2}(z)}{f^+_{1/2}(z)} \frac{N\lambda^3}{VT} \frac{k_B T}{\lambda^3} V\right] = N k_B \left[\frac{5}{2} \frac{V}{N\lambda^3} f^+_{5/2}(z) - \frac{3}{2} \frac{f^+_{3/2}(z)}{f^+_{1/2}(z)}\right]$$

$$\frac{df^+_{5/2}(z)}{dT} = \frac{f^+_{3/2}(z)}{z} \frac{dz}{dT} = -\frac{3}{2} \frac{f^+_{3/2}(z)}{f^+_{1/2}(z)} \frac{N\lambda^3}{V}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} f^+_m(z) = \frac{1}{z} f^+_{m-1}(z)$$

$$\frac{f^+_{1/2}(z)}{z} \frac{dz}{dT} = \frac{d}{dT} f^+_{3/2}(z) = \frac{d}{dT} \frac{N}{V} \lambda^3 = -\frac{3}{2} \frac{N}{V} \frac{\lambda^3}{T} \Rightarrow \frac{1}{z} \frac{dz}{dT} = -\frac{3}{2} \frac{N}{V} \frac{\lambda^3}{T} \frac{1}{f^+_{1/2}(z)}$$

Bose gas

☐ **Heat capacity.**

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{3}{2}V\left(\frac{\partial P}{\partial T}\right)_{V,N}$$

$$E = (3/2)PV$$

$$P = \frac{k_B T}{\lambda^3} f^+_{5/2}(z)$$

$$\frac{N}{V} = \frac{1}{\lambda^3} f^+_{3/2}(z)$$

For $T < T_c$, we set $z = 1$ in C_V ($T > T_c$), we get

$$V\frac{\partial P}{\partial T} = Nk_B\left[\frac{5}{2}\frac{V}{N\lambda^3} f^+_{5/2}(z) - \frac{3}{2}\frac{f^+_{3/2}(z)}{f^+_{1/2}(z)}\right] = Nk_B\left[\frac{5}{2}\frac{V}{N\lambda^3}\zeta_{5/2} - \cancel{\frac{3}{2}\frac{\zeta_{3/2}}{\zeta_{1/2}}}\right]$$

$$\zeta_{1/2} = \infty$$

Or $PV = V\dfrac{k_B T}{\lambda^3}\zeta_{5/2}$ $\qquad \left.\dfrac{\partial(PV)}{\partial T}\right|_{V,\mu} = V\dfrac{5}{2}\dfrac{k_B}{\lambda^3}\zeta_{5/2}$

Bose gas

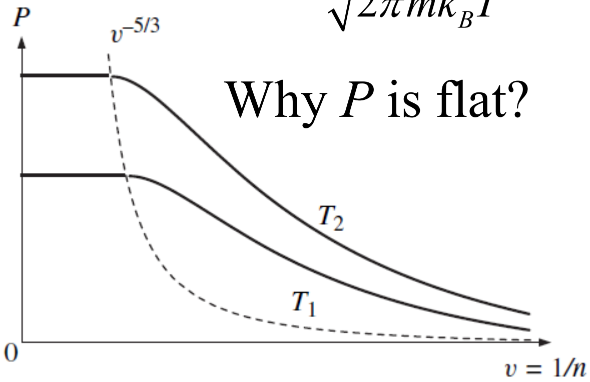
- ***P-V* Phase diagram:** How does the *P*-*V* curve look like?

$$\lambda=\frac{h}{\sqrt{2\pi m k_B T}}$$

$$P=\frac{k_BT}{\lambda^3}f^+_{5/2}(z) \qquad \frac{N}{V}=\frac{1}{\lambda^3}f^+_{3/2}(z)$$

Set $z = 1$,
cancel out T.

$$P_0V_0^{5/3}=\frac{k_BT}{\lambda^3}\zeta_{5/2}\left(\frac{N\lambda^3}{\zeta_{3/2}}\right)^{5/3}=k_BT\lambda^2\frac{N^{5/3}\zeta_{5/2}}{(\zeta_{3/2})^{5/3}}=\frac{h^2N^{5/3}}{2\pi m}\frac{\zeta_{5/2}}{(\zeta_{3/2})^{5/3}}$$



Why P is flat?

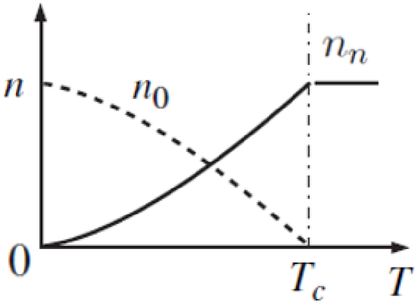
At $T < T_c$, two phases coexist, pressure flat: compressibility diverges.

- **Justify BEC:** We can also exam the particle number at excited/normal states:

$$N=\sum_{\vec{k}}\frac{1}{\exp[\beta(\varepsilon(\vec{k})-\mu)]-1} \Rightarrow \frac{V}{h^3}\int_0 d^3k\frac{1}{\exp[\beta(\varepsilon(k)-\mu)]-1}=N-N_0=\frac{V}{h^3}\int_0 d\varepsilon g(\varepsilon)\frac{1}{\exp[\beta(\varepsilon-\mu)]-1}$$

Integral only count excited/normal states

Check if $N - N_0$ diverges or not when $\mu = 0$ (for BEC emerges).
(a) If it is a finite number, BEC will occur.
(b) If it diverges, no BEC.



Bose gas

☐ BEC: Superfluid of HeII

- ✓ HeII has unusual hydrodynamic properties. It flows through the finest capillaries without any resistance.
- ✓ For ordinary fluids a finite pressure difference between the containers, proportional to viscosity, is necessary to maintain the flow. HeII flows even in the limit of zero pressure difference and acts as if it has zero viscosity. For this reason it is referred to as a *superfluid*.

In 1938, Fritz London suggested that a good starting hypothesis is that the transition to the superfluid state is related to the Bose–Einstein condensation.

Bose gas

Bose -
Einstein
Condensation

Qualifying exam problem

Consider a gas of non-interacting, identical, spin-0 bosons confined to a two-dimensional square area $A = L^2$. Assume that the bosons have a well-defined particle number of N, and they have the single-particle energy dispersion relation $\epsilon = c|p| = c\hbar|k|$, where c is a positive real constant, p is the momentum and k is the wave vector

Demonstrate whether there is or is not Bose-Einstein condensation in this system at nonzero temperature. If yes, find the T_c and N_0.

$$f_m^+(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\,x^{m-1}}{z^{-1}e^x - 1} \qquad \langle n_\epsilon \rangle_+ = \frac{1}{z^{-1}e^{\beta\epsilon} - 1}$$

In a small volume $d^2\vec{k}$, the total number of states is $\rho(\vec{k})d^2\vec{k} = \frac{A}{(2\pi)^2}d^2\vec{k}$

$|k| = \varepsilon/c\hbar$, then $\mathrm{d}k = d\varepsilon/c\hbar$, and $d^2\vec{k} = 2\pi k dk$

$\rho(\vec{k})d^2\vec{k} = \frac{A}{(2\pi)^2}d^2\vec{k} = \frac{A}{(2\pi)^2}2\pi k dk = \frac{A}{(2\pi)^2}2\pi(\varepsilon/c\hbar)d\varepsilon/c\hbar = g(\varepsilon)d\varepsilon,$

Therefore, the energy density of states $g(\varepsilon) = \frac{A\varepsilon}{2\pi(c\hbar)^2}$,

$N_n = \int_0^\infty d\epsilon g(\epsilon)n(\epsilon) = \frac{A}{2\pi(c\hbar)^2}\int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}d\epsilon = \frac{A}{2\pi(c\hbar)^2}\int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}d\epsilon,$

Fugacity $z = e^{\beta\mu}$, set $x = \beta\epsilon$, we can rewrite $N_n = \frac{A}{2\pi(c\hbar\beta)^2}\int_0^\infty \frac{x}{z^{-1}e^x-1}dx,$

Using the function $f_2^+(z) = \int_0^\infty \frac{x}{z^{-1}e^x-1}dx$, we obtain $N_n = \frac{A}{2\pi(c\hbar\beta)^2}f_2^+(z)$

Qualifying exam problem

Consider a gas of non-interacting, identical, spin-0 bosons confined to a two-dimensional square area $A = L^2$. Assume that the bosons have a well-defined particle number of N, and they have the single-particle energy dispersion relation $\epsilon = c|p| = c\hbar|k|$, where c is a positive real constant, p is the momentum and k is the wave vector

Demonstrate whether there is or is not Bose-Einstein condensation in this system at nonzero temperature. If yes, find the T_c and N_0.

For $\mu = 0$, $N_n = \frac{A}{2\pi(c\hbar\beta)^2} f_2^+(z=1) = \frac{A}{2\pi(c\hbar\beta)^2}\zeta(2) = \frac{A\pi}{12(c\hbar\beta)^2}$

We used the Riemann zeta-function $\zeta(2) = \pi^2/6$.

Since the particle number in excited states is a finite number when the chemical potential approaches zero, the bosons in 2-D do indeed form a Bose-Einstein condensation.

For $T = T_c$, $N_n = N$, so $N = \frac{A\pi}{12(c\hbar\beta)^2}$, we get $T_c = (\frac{12N}{A\pi})^{1/2}\frac{c\hbar}{k_B}$.

For $T \le T_c$, $N = N_0 + N_n$.

Since $\mu = 0$ for $T \le T_c$, $N_n = \frac{A\pi}{12(c\hbar\beta)^2} = \frac{A\pi {k_B}^2}{12(c\hbar)^2}T^2 = \frac{A\pi {k_B}^2}{12(c\hbar)^2}{T_c}^2\left(\frac{T}{T_c}\right)^2 = N\left(\frac{T}{T_c}\right)^2$.

Therefore, $N_0 = N - N_n = N[1 - \left(\frac{T}{T_c}\right)^2]$.

Summary of ideal of quantum gas

☐ **Fermi gas.**

Dimensions	**$\varepsilon(k)$ dispersion**	**Density of states**	**High-*T* limit (Bose similar)**	**Low-*T* limit $T \geq 0$**	**Magnetic**
1D	Quadratic	$g(\varepsilon) = ?$	P, E, C_v, et al	ε_F, p_F, T_F μ, P, E, C_v, et al	$M\,(T, H)$
1D	Linear	$g(\varepsilon) = ?$	P, E, C_v, et al	ε_F, p_F, T_F μ, P, E, C_v, et al	$M\,(T, H)$
2D	Quadratic	$g(\varepsilon) = ?$	P, E, C_v, et al	ε_F, p_F, T_F μ, P, E, C_v, et al	$M\,(T, H)$
2D	Linear	$g(\varepsilon) = ?$	P, E, C_v, et al	ε_F, p_F, T_F μ, P, E, C_v, et al	$M\,(T, H)$
3D	Quadratic	$g(\varepsilon) = ?$	P, E, C_v, et al	ε_F, p_F, T_F μ, P, E, C_v, et al	$M\,(T, H)$
3D	Linear	$g(\varepsilon) = ?$	P, E, C_v, et al	ε_F, p_F, T_F μ, P, E, C_v, et al	$M\,(T, H)$

Average occupying number $\langle n \rangle$: how to use it?

Equation of states, pressure, internal energy, chemical potential, heat capacity et al.

Various thermodynamic quantities at $T = 0$. General dimensional d?

Summary of ideal of quantum gas

❑ **Bose gas**

Phonons/photons (N is not conserved), He atom (conserved N)

Dimensions	**$\varepsilon(k)$dispersion**	**Density of states**	**Low-T limit $T \geq 0$**	**Low-T limit (conserved N)**
1D	Quadratic	$g(\varepsilon) = ?$	T_c, μ, P, S, E, C_v	BEC: Yes or No
1D	Linear	$g(\varepsilon) = ?$	T_c, μ, P, S, E, C_v	BEC: Yes or No
2D	Quadratic	$g(\varepsilon) = ?$	T_c, μ, P, S, E, C_v	BEC: Yes or No
2D	Linear	$g(\varepsilon) = ?$	T_c, μ, P, S, E, C_v	BEC: Yes or No
3D	Quadratic	$g(\varepsilon) = ?$	T_c, μ, P, S, E, C_v	BEC: Yes or No
3D	Linear	$g(\varepsilon) = ?$	T_c, μ, P, S, E, C_v	BEC: Yes or No

Occupation number $<n>$: how to use it?
Equation of states, pressure, internal energy, chemical potential, heat capacity et al.
General dimensional d?

Phonons/photons (no conserved N): T^4, low temperature heat capacity T^3