A Secret of Ancient Geometry

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1 Introduction

The architecture of antiquity has a sense of harmony and proportion rarely equaled even in the greatest works of other eras. The quality of the work of an architect or designer is determined by how he or she comes to grips with the mathematical constraints on space inherent in all designs—"what is possible," in contrast with the designer's intention, "what ought to be." The history of architecture reflects the history of ideas in that "what ought to be" has changed from metaphysical "natural" world views to explorations of the individual "artist." Additionally, the history of technology is reflected in changes of "what is possible."

There are two kinds of constraints on space that the architect or designer must confront:

> constraints imposed on a design because of the geometrical properties of space;
>
> constraints imposed on a design by the designer who creates a geometrical foundation or scaffolding as an overlay to the design. The designer's choice is based on the context of the design and on the effect that he or she wishes to achieve.

Without constraints, a design is chaotic, irrelevant and lacking in focus. Where do the designer's constraints come from? In ancient times they were derived either from spiritual contexts or handed down from generation to generation by tradition. The results were cathedrals such as Chartres and Hagia Sophia or structures such as the Egyptian Pyramids and the Great Temple of Jerusalem or the temples of ancient Greece such as the Parthenon, the Theseum, and the temples of Poseidon and Ceres.

Modern architecture has replaced spiritual—and tradition-bound—contexts with the private vision of the designer or architect and substituted diversity for tradition. However, the designer is left with few tools to deal with such a lack of constraint. After all, what should the designer do when each design breaks new ground? In an effort to recover the principles of ancient architecture, many researchers have studied the geometric and spiritual bases of ancient structures [7, 16, 17, 19].

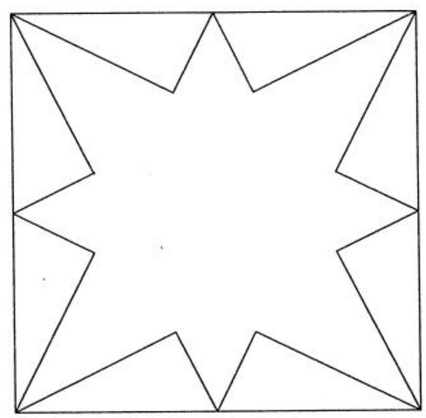

Figure 1: The Brunes star.

This paper will discuss the work of Tons Brunes, a Danish engineer, who hypothesized a system of ancient geometry that he believed lay at the basis of many of the temples of antiquity [4]. It was Brunes's belief that there existed until about 1400, a network of temples and a brotherhood of priests originating in ancient Egypt which had a secret system of geometry. At the basis of Brunes's theory is the eight-pointed star illustrated in Figure 1. Brunes claimed to have seen this star in ancient temples (but gave no references) where he stated that it was often mistaken for ornamentation. From the geometry of this star he was able to reconstruct reasonably close facsimiles to the plans and elevations of the ruins of ancient temples such as the Pantheon, the Theseum, and the Temples of Ceres and Poseidon, noting that certain intersections coincide with features of these temples [10]. Unfortunately, although the examples he uses to illustrate his theories are cleverly rendered, there is no historical record to support his claims. As a result his research has met considerable skepticism. Nevertheless, as we shall see, the Brunes star reveals a geometry consistent with ancient architecture, folk art, and the musical scale. Even though it is unlikely to have played the all-pervasive role for temple construction that Brunes conjectured, it may well have been one of the organizing tools. At any rate, the beauty of its geometry is reason enough to study it.

2 The Concept of Measure in Ancient Architecture

One thing that we can say for sure about the thought processes of antiquity is that they differed markedly from our own. Until Aristotle introduced observation and measurement as the only way to arrive at truth, it appears that reality was best described by numbers, music, and poetry.

R. A. Schwaller di Lubicz [5] felt that the combination of myth and symbol conveyed by ancient writings was the only way information about the workings of the universe could be conveyed without reducing its true meaning. According to Di Lubicz [5] the ancient Egyptians felt that:

> Measure was an expression of Knowledge that is to say that measure has for them a universal meaning linking the things of here below with things Above and not solely an immediate practical meaning—quantity is unstable: only function has a value durable enough to serve as a basis [for description]. Thus the Egyptians' unit of measurement was always variable—measure and proportions were adapted to the purpose and the symbolic meaning of the idea to be expressed. [For example] the cubit will not necessarily be the same from one temple to another, since these temples are in different places and their purposes are different.

Before a standard unit of measure can be introduced, there must be a sophisticated means of transportation in order for its users to be able to travel to a central location to retrieve the standard measure for their own purposes.

In societies without access to standard measures, other methods were developed to enable the craftsman to build or the architect to create structures without need of standard measure. Even when measuring rods were available, they may have been used only as an adjunct to the use of pure geometry in the design of sacred structures. In place of numbers to describe a measurement, a kind of applied geometry was developed in which lengths were constructed without the need to measure them. All that was needed was a length of rope and a straightedge (the equivalent of our compass and straightedge). Methods were then devised to subdivide any length into sublengths, always by construction. Evidence of construction lines have been discovered on the base of the unfinished Temple of Sardis in Turkey and also in the courtyard of the Temple of Zeus of Jerash in Jordan [18]. Artmann [2] has shown how such methods were used to construct the Gothic cathedrals. The geometry needed to build these cathedrals was learned from boiled-down versions of the first books of Euclid, known as pseudo-Boethius [3] which highlighted the constructive methods while eliminating the proofs of the theorems. The knowledge to implement this geometry was taught to the guilds of masons, other artisans, and builders and then passed on from generation to generation by oral tradition. One can imagine easily learned constructive techniques based on the Brunes star being transmitted by this tradition and applied to the construction of ancient sacred structures. We shall describe Brunes's hypothetical reconstruction of this geometry.

3 The Ancient Geometry of Tons Brunes

In ancient times it was an important problem to find a way to create a square or rectangle with the same area or circumference as a given circle—"squaring the circle," as it was known. Since the circle symbolized the celestial sphere while a square or rectangle oriented with its sides perpendicular to the compass directions of north, east, south, and west symbolized the Earth, the squaring of the circle could be thought to symbolically bring "heaven down to earth." Brunes demonstrates one way in which ancient geometers may have attempted to solve this problem using only compass and straightedge (we now know that this cannot be done exactly). To square the circle with respect to circumference Brunes first considers a geometric construction which he refers to as a "sacred cut."

To create the sacred cut of a side of a unit square, place the point of your compass at a vertex and draw an arc through the center of the square as shown in Figure 2. This cuts the side down by a factor of $1/\sqrt{2}$. In Figure 3 arc AB and the diagonal CD of the half square are approximately equal. In

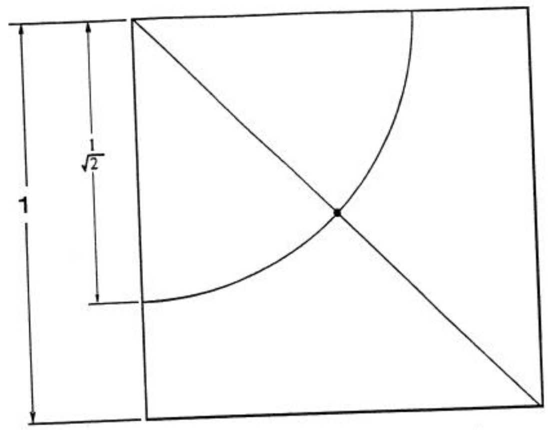

Figure 2: The sacred cut.

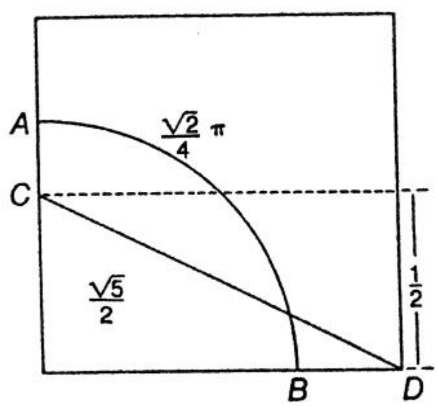

Figure 3: Comparison of lengths.

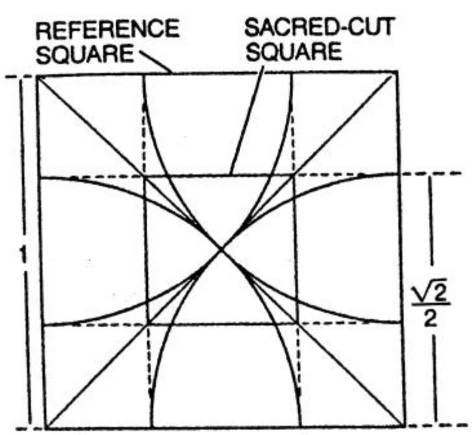

Figure 4: Four sacred cuts within a square.

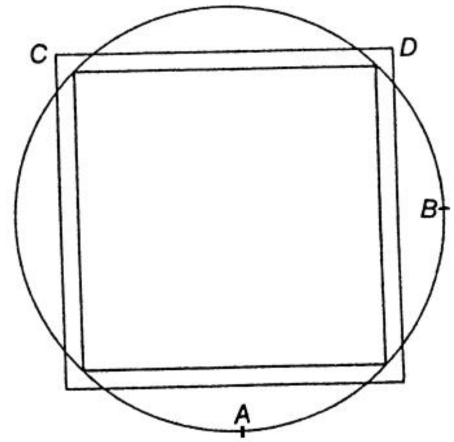

Figure 5: The circumference of the circle is approximately equal to the perimeter of the outer square.

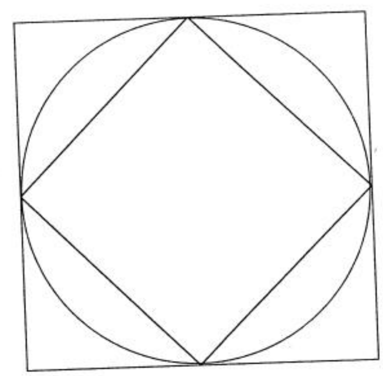

Figure 6: The ad quadratum square.

fact,

$$AB = \pi\sqrt{2}/4 = 1.1107\ldots \qquad \text{while}$$
$$CD = \sqrt{5}/2 = 1.1118\ldots.$$

In Figure 4 four sacred cuts AB are placed into a square. In Figure 5 the four sacred cuts form a circle whose circumference is equal to the perimeter of a square with edge CD to within 1.6%.

In Figure 6, we see that a circle is drawn that is tangent to an outer square (inscribed circle) and touching the vertices of an inner square (circumscribed circle). This square-within-a-square, called an "ad quadratum" square, was much used in ancient geometry and architecture [19]. The area of the inner square is obviously half the area of the outer square. In a sequence of circles and squares inscribed within each other each square is 1/2 the area of the preceding square. Figure 7 shows a sequence of ad quadratum squares which are shaded to form a logarithmic spiral known as the Baravelle

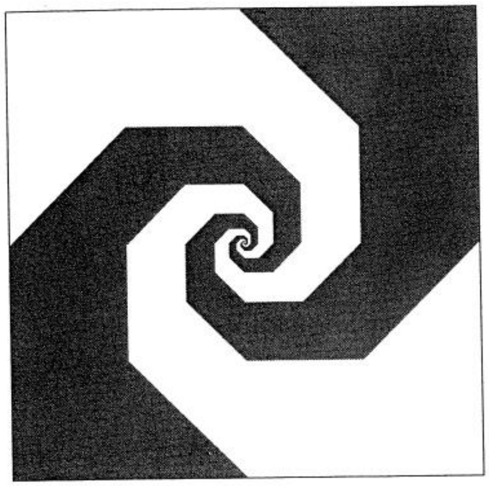

Figure 7: The Baravelle spiral.

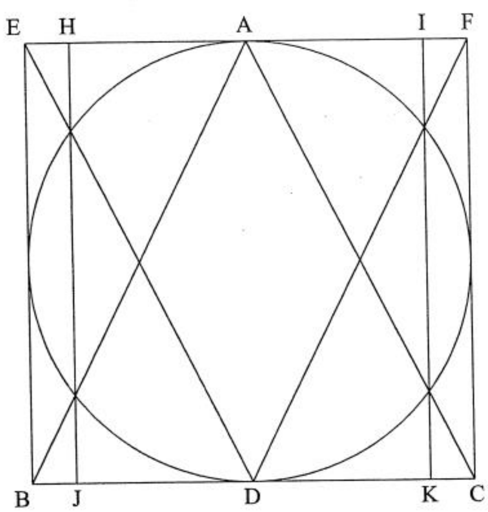

Figure 8: Approximate squaring of the circle.

spiral. It is easy to construct and with color makes an interesting design.

Another geometric structure used by ancient geometers was the upward-pointing triangle ABC in Figure 8 which also has half the area of the circumscribing square $BCFE$. If the downward-pointing triangle DEF is constructed, then rectangle $JKIH$, formed by the vertical lines through the intersection points of the upward- and downward-pointing triangles and the circle, has approximately the same area as the circle. It can be determined (not shown here) that the width of this rectangle is 4/5 of the diameter of the circle. Taking the square to have length equal to 1 unit, then the radius of

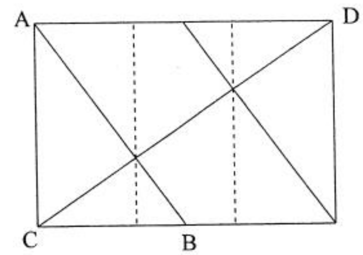

Figure 9: Trisection of the diagonal of a rectangle.

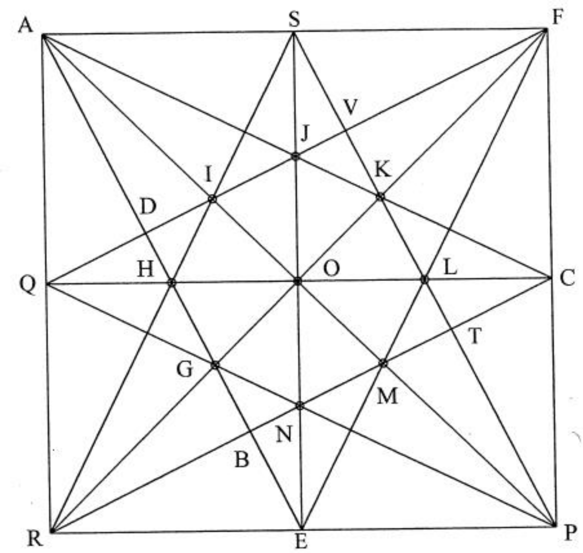

Figure 10: Construction of the Brunes star.

the circle equals 1/2 and

$$\text{Area of circle} = \pi(1/2)^2 = .7853\ldots$$
$$\text{Area of rectangle} = 4/5 = .80,$$

an error of 1.8%.

In Figure 9 we show that for an arbitrary rectangle the line AB from a vertex to the center of the opposite side cuts the diagonal CD at the 1/3 point. We now use this geometrical property to describe the structure of the Brunes star of Figure 1. Take the circumscribing square and subdivide it by placing perpendicular axes within it, as shown in Figure 10. This divides the outer square into four overlapping rectangular half-squares. Place two diagonals into each of the four half squares and add the two diagonals of the outer square. Notice that the resulting diagram (shown in Figure 10) is the Brunes star.

We now see that this star contains all the information needed to get good approximations to squaring the circle in both circumference and area. Also hidden within the Brunes star are numerous

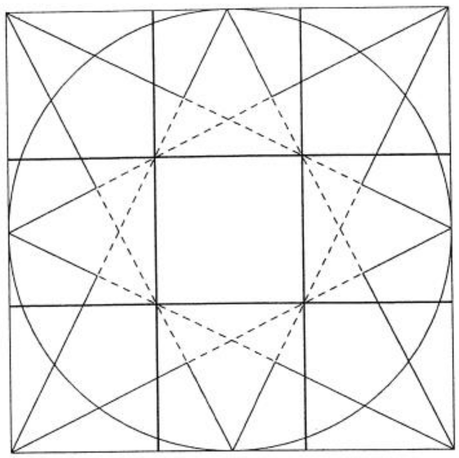

Figure 11: The 3×3 grid of subsquares determined by the Brunes star.

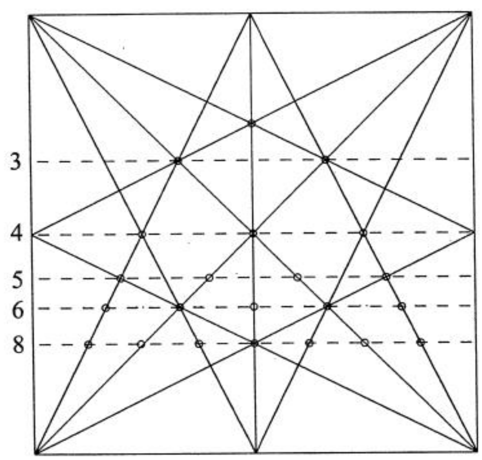

Figure 12: Division of a line into 3, 4, 5, 6, or 8 equal parts.

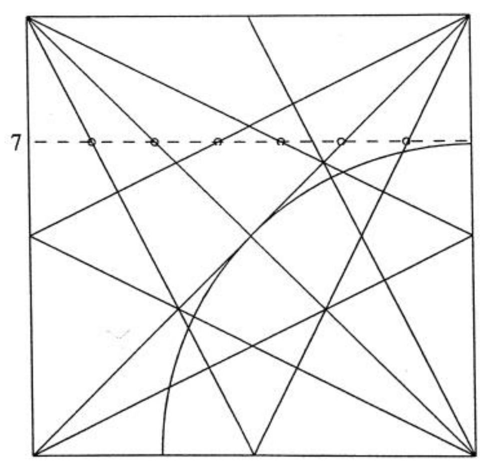

Figure 13: Division of a line into seven approximately equal parts.

3,4,5-right triangles. For example, triangle ABC is a 3,4,5-right triangle because,

$$\tan \frac{1}{2}C = \frac{AQ}{QC} = \frac{1}{2}.$$

Therefore using the trigonometry identity,

$$\tan C = \frac{2 \tan \frac{1}{2}C}{(1 - \tan^2 \frac{1}{2}C)},$$

it follows that,

$$\frac{AB}{BC} = \tan C = \frac{1}{(1 - \frac{1}{4})} = \frac{4}{3}. \qquad (1)$$

If the Brunes star with all of its construction lines depicted in Figure 10 is placed on each face of a cube, it can be shown that the vertices of all of the six Archimedean solids and two Platonic solids (cube and octahedron) related to the cubic system of symmetry as well as the tetrahedron coincide with the points of intersection of the construction lines [10]. The Brunes star also succeeds in providing the geometrical basis for dividing an arbitrary length into any number of equal sublengths without the use of measure.

4 Equidivision of Lengths: A Study in Perspective

Figure 10 contains the construction points with which to subdivide lengths into 3 and 4 equal parts without the need of a standard measure, i.e., points I and M divide diagonal AP into thirds (see Figure 9) while H, O, and L divide QC in quarters. The central cross of Figure 10 is therefore subdivided by the central irregular octagon $GHIJKLMN$ into four equal parts and the diagonals into three equal parts in a similar way. Points I, K, G, and M then provide the points that subdivide the outer square into a 3×3 grid of subsquares, as shown in Figure 11. Figure 12 indicates how the Brunes star divides a line segment into 3, 4, 5, and 8 equal parts, while a sacred cut drawn from a vertex of the outer square in Figure 13 defines the level that partitions a line into seven parts which are approximately the same; the error is within 2%. In similar ways, Brunes has shown that the Brunes star can be used to equipartition a line into between 1 and 12 parts in a manner which does not require a standard measure, but only a length of stretched rope.

This equipartitioning property of the Brunes star has its roots in another ancient geometric construction [15] which was first related to me by Michael

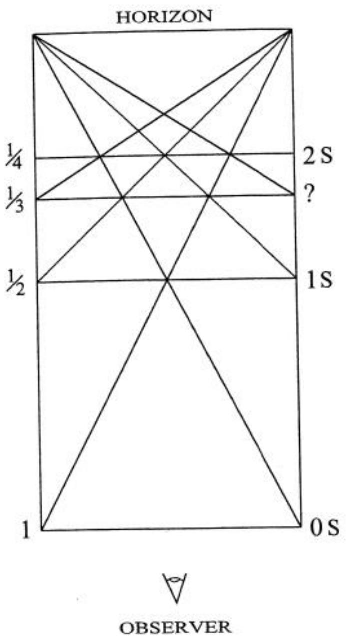

Figure 14: Double square with diagonals.

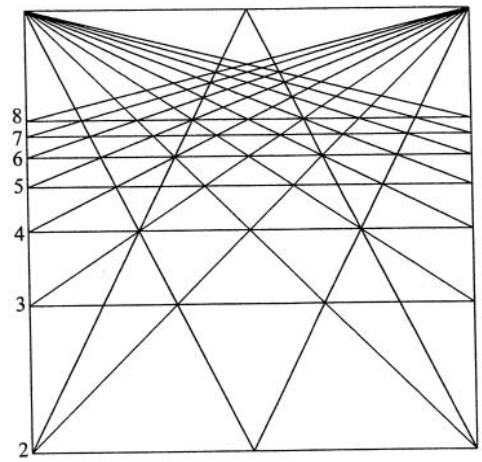

Figure 15: Square with diagonals.

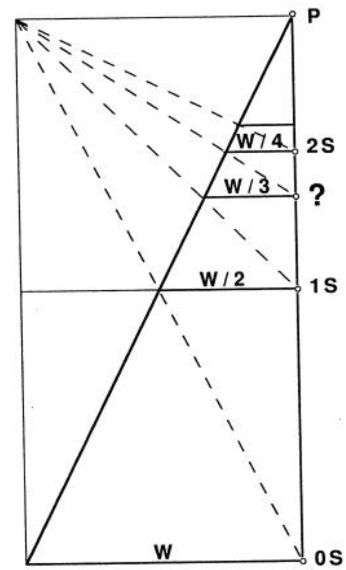

Figure 16: Railroad tracks in perspective. The point P is the central vanishing point.

Porter, a Professor of Architecture at Pratt Institute. In Figure 14 the outer square of the Brunes star has been extended to a double square. The principal diagonals of the double square divide the width of the upper square into two equal parts. The principal diagonals intersect the two diagonals of the upper square at the trisection points of the width. At the same time, the trisected width intersects the long side of the double square at the 1/3 point. Continuing one more step, the two diagonals of the 1/3-rectangle intersect the principal diagonal at points which divide the width into four equal parts. This width also divides the long side of the double square at the 1/4 point. This construction may be continued to subdivide a line segment into any number of equal parts, as in Figure 15 with eight subdivisions.

As is often the case with mathematics, a diagram set up to demonstrate one concept is shown to have a deeper structure. We could also view Figures 14 and 15 as a pair of railroad tracks in perspective receding to the horizon line. The diagonal and the right side of the double square play the role of the railroad tracks as shown in Figure 16. If the observer is at an arbitrary location in the foreground, then the distance between the tracks appears half as great as at the base of the double square at some measured distance in the direction of the horizon referred to as a "standard distance," or $1S$. At a distance from the observer of $2S$ the distance between the tracks appears to be 1/4 as large as the base width. In a similar manner, the tracks appear to be 1/8 as wide at $3S$ (not shown). How many standard units S make the tracks appear 1/3 as wide? To answer this question requires us to analyze the pattern in greater depth. Table 1 shows the relation between apparent width between the railroad tracks L and the receding distance D (in units of S) towards the horizon. The receding distance is also expressed in terms of logarithms to the base 2. In other words, the relation between D and L in Table 1 can be expressed by the formula:

$$D \text{ (in units of } S) = \log_2 1/L.$$

It is clear from Table 1 that for $L = 1/3$, $1/L = 3$ and

$$? = \log_2 3 = \log_{10} 3/\log_{10} 2 = 1.58S.$$

The author has further examined the projective transformation that gives rise to Figures 14, 15, and 16 and has related it to the series of overtones resulting from plucking the string of a monochord or other stringed instrument [14].

Apparent width (L)	Receding distance (D)
$1 = 1/2^0$	$0 = \log_2 1$
$1/2 = 1/2^1$	$1 = \log_2 2$
$1/3 = 1/2^?$	$? = \log_2 3$
$1/4 = 1/2^2$	$2 = \log_2 4$
$1/8 = 1/2^3$	$3 = \log_2 8$

Table 1

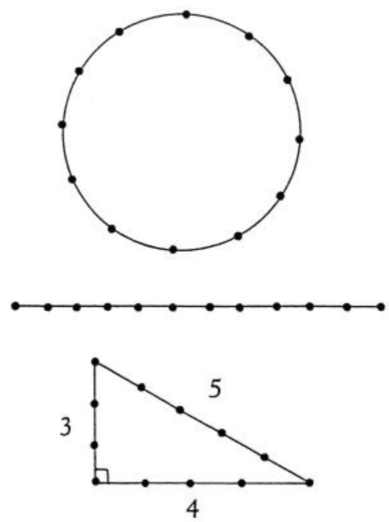

Figure 17: Ropes.

Figure 18: Rope used to create the Brunes star in Figure 19.

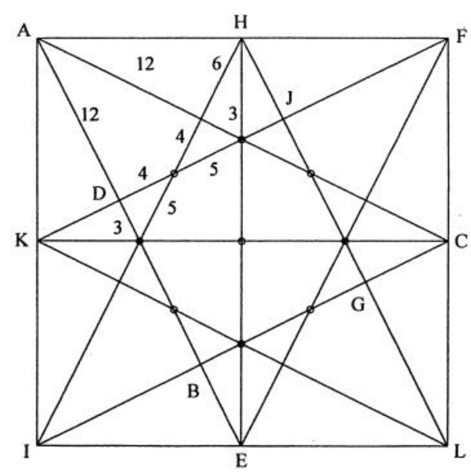

Figure 19: Segment lengths of the Brunes star.

5 The 3,4,5-Triangle in Sacred Geometry and Architecture

We showed in Section 3 that triangle ABC in Figure 10 is a 3,4,5-right triangle. The 3,4,5-right triangle was called the Egyptian triangle by Vitruvius, the architect of the Emperor Augustus, and had great significance in the construction of the pyramid of Cheops [9, 17]. In cap. 56, Plutarch [17] described this triangle as the symbol of the Egyptian trinity, associated with the three significant Egyptian deities:

$$\begin{array}{lcl} 3 & \Longleftrightarrow & \text{Osiris} \\ 4 & \Longleftrightarrow & \text{Isis} \\ 5 & \Longleftrightarrow & \text{Horus} \end{array}$$

The key to understanding the geometry of the Brunes diagram lies in its construction. But how did ancient architects construct the star diagram? This diagram is easy to construct if one begins with a square, but it is not an easy matter to construct a large square if one has only a length of rope and some stakes to work with. However the entire diagram can equally well be constructed beginning with the 3,4,5-right triangle. The 3,4,5-right triangle can be constructed from a loop of rope with 12 knots, as shown in Figure 18. The 12 sectors of the circle shown in Figure 17 could also have represented the 12 regions of the zodiac visited by the sun during the course of the year, as viewed from a geocentric standpoint.

I have created a videotape of a group of students constructing this star on an open field using four lengths of 50-foot clothesline anchored by camping stakes [11]. To construct the Brunes star begin with four lengths of rope each length divided into 12 equal sections by 12 knots as shown in Figure 17. Although the rope is shown stretched out in a line, the ends are connected so that it forms a loop. Four such loops—$ADBGCA$ (see Figure 19), $FJDBEF$, $IBGJHI$, and $LGJDKL$—are stretched into four 3,4,5-right triangles, each providing one vertex of the outer square of Figure 10. The right angles of these 3,4,5-triangles are located at the vertices of the inner square $DBGJ$. We have succeeded in constructing the outer square $AILF$ along with the midpoints of its sides $HKEC$. Now that the outer square has been formed, we can stand back and observe the harmony of this figure. In order to better appreciate its geometry, we must make a brief digression and consider the geometry of the 3,4,5-right triangle. From Equation 1 it follows that triangle ABC is a 3,4,5-right triangle. All other right triangles in Figure 10 are either 3,4,5-right triangles or fragments of a 3,4,5-triangle obtained by bisecting its acute angles. In Figure 19 the dimensions of the sublengths are indicated. These may be gotten from Figure 10 by assigning each segment of the string a length of 6 units. The properties of 3,4,5-triangles given by Equation 1 can also be used to verify these lengths. Figure 19 shows the star diagram to have 3,4,5-right triangles at four different scales. Referring to vertex labels of Figure 10,

$\triangle ABC$	:	18	:	24	:	30
$\triangle ADJ$	:	9	:	12	:	15
$\triangle QDG$	:	6	:	8	:	10
$\triangle DHI$	:	3	:	4	:	5

So we see that the star diagram is entirely harmonized by the 3,4,5-right triangle.

As we previously mentioned, Brunes used these principles of geometry to show how many of the structures of antiquity might have been proportioned [10]. He subsumed the principles of this geometry into a series of 21 diagrams (not shown) related to the star diagram and the sacred cut [11]. He claims that each step in the creation of a plan for one of the ancient structures follows one or another of these diagrams. We illustrate the result of Brunes's analysis for the Temple of Ceres by the gulf of Salerno in Southern Italy built by Greek colonists during the period from 550–450 B.C. Brunes has reconstructed his analysis from the ruins of this temple. Although Brunes obtained close fits between key lines of the elevation and plan (not shown) of these structures, his constructions require an initial "reference circle" the choice of which is quite arbitrary as shown in Figures 21 and 22. Despite the close fits between Brunes's diagrams and the actual temple, one never knows the degree to which

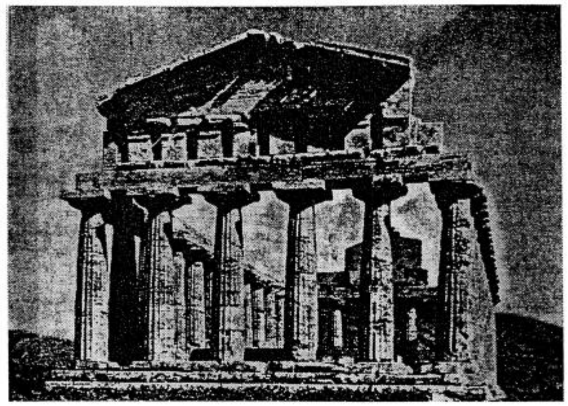

Figure 20: The Temple of Ceres.

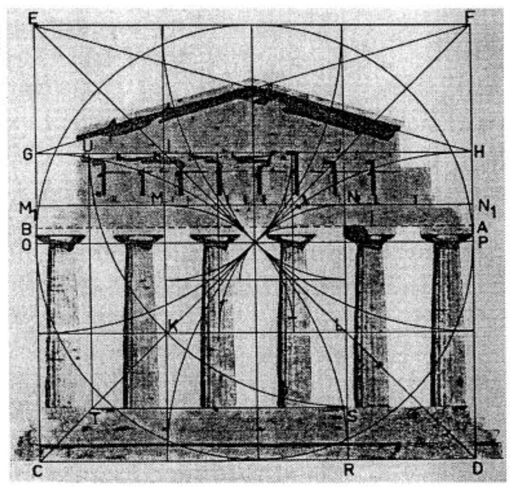

Figure 21: Analysis of the Temple of Ceres by Brunes using the sacred cut.

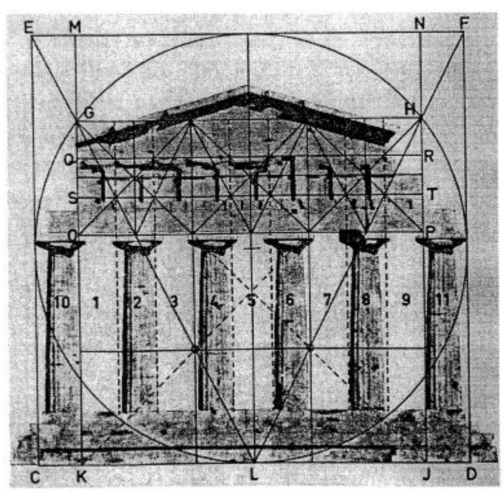

Figure 22: Analysis of the Temple of Ceres by Brunes using triangles.

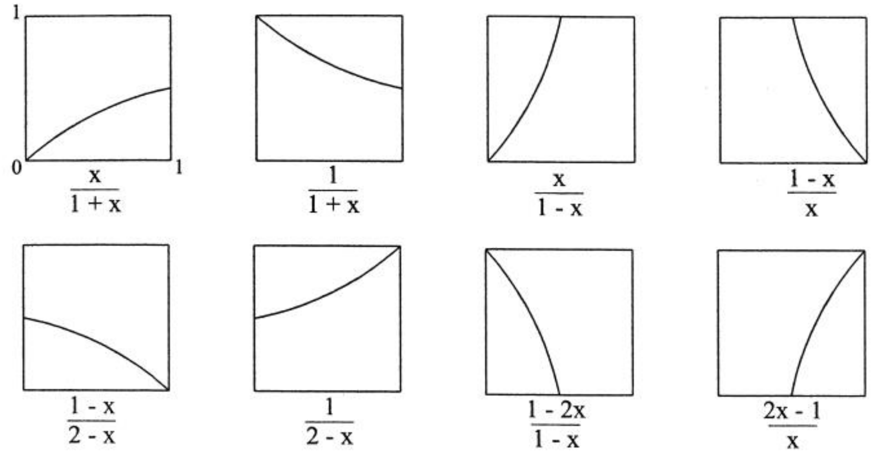

Figure 23: Line segments used to generate the generalized Brunes star.

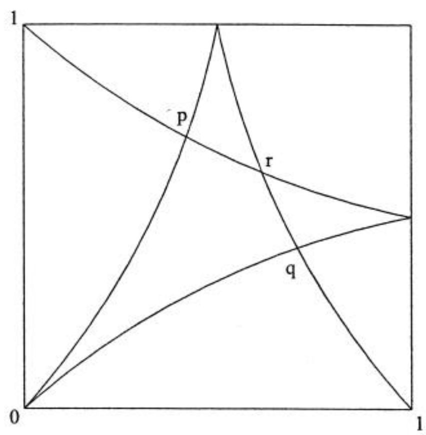

Figure 24: Intersection points: $p = (1/\theta, 1/\sqrt{2})$, $q = (1/\sqrt{2}, 1/\theta)$, $r = (1/\tau, 1/\tau)$.

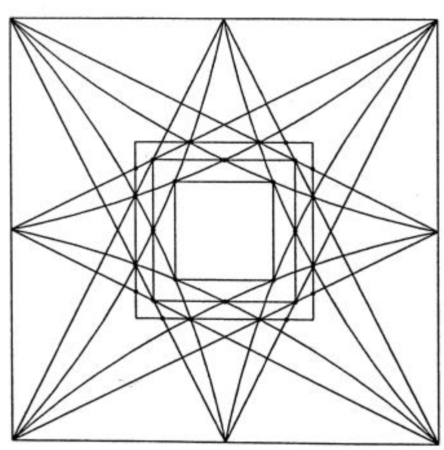

Figure 25: The generalized Brunes star.

they have been forced by his imagination. In my opinion, it is unlikely that this method was actually used as described by Brunes. Nevertheless, the simplicity and harmony of Brunes's diagrams make it plausible that they could have been used in some unspecified manner as a tool for temple design.

6 A Generalized Brunes Star

Gary Adamson [1] has generalized the Brunes star by replacing the eight line segments that make up the diagonals of the four half-squares by a segment of an hyperbola juxtaposed in eight different orientations within a unit square, as shown in Figure 23. Four of these hyperbolas intersect as shown in Figure 24 at three characteristic points p, q, r with coordinates:

$$p = (.414..., .707...) = \left(\frac{1}{\theta}, \frac{1}{\sqrt{2}}\right) \quad \text{where } \theta = 1 + \sqrt{2}$$

$$q = (.707..., .414...) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\theta}\right)$$

$$r = (.618..., .618...) = \left(\frac{1}{\tau}, \frac{1}{\tau}\right) \quad \text{where } \tau = (1 + \sqrt{5})/2$$

Therefore, the key numbers of the ancient Roman system of proportions $\sqrt{2}$ and θ (also referred to in modern dynamical systems theory as the "silver mean") [13], and the golden mean τ are represented in a single diagram. The generalized Brunes star is shown in Figure 25. The points of intersection lie on the edges of the three inner squares. The edge length of the innermost square is τ^{-3}, the middle square is $1/3$, and the outer square is $\theta^{-1} = \sqrt{2} - 1$.

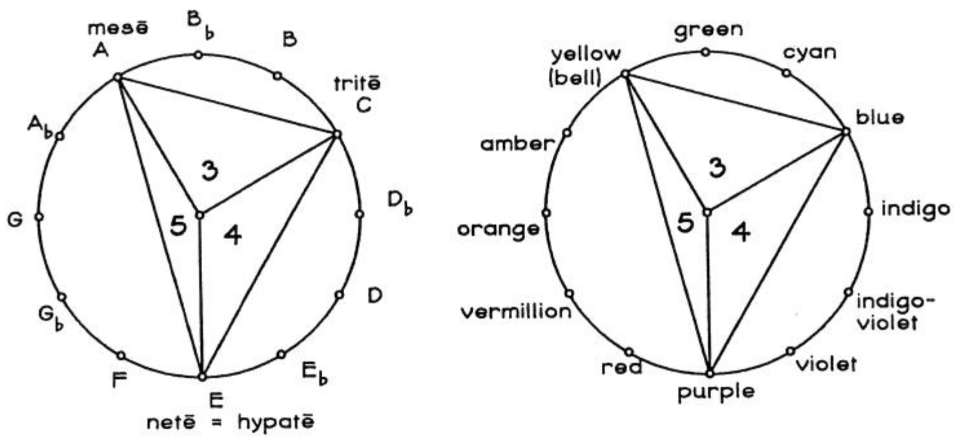

Figure 26: The equal-tempered chromatic scale and the color wheel.

7 What Pleases the Ear Should Please the Eye

We have seen that 3,4,5-triangles pervade the Brunes star. Not all relationships involving the numbers 3, 4, and 5 refer to the 3,4,5-right triangle. Such relationships play a major role in the structure of the musical scale and make a surprise appearance in the structure of the color spectrum of light which could be thought of as a kind of "musical scale" for the eye. Elsewhere I have shown that the ancient musical scale, in which tones are associated with the ratio of string lengths, is organized by a 3,4,5-relationship between the tones [14]. Also, if we regard the 12 sectors of the circle as tones of the equal-tempered chromatic scale, we see in Figure 26 that a subdivision of the tonal circle into 3, 4, and 5 semitones gives rise to the tones A,C,E of the musical A minor triad [6]. The association between tones and number ratios led the architects of the Italian Renaissance to build a system of architectural proportions based on the musical scale [14].

According to the leading architect of that period, Leon Battista Alberti [20]: "The numbers by which the agreement of sounds affect our ears with delight are the very same which please our eyes and our minds. We shall therefore borrow all our rules for harmonic relations from the musicians to whom this kind of numbers is well known and wherein Nature shows herself most excellent and complete." Eberhart [6] has made the observation that the wavelengths of visible light occur over a range between 380 mμ (millimicrons; 1 mμ = 10^{-7}cm) in the ultraviolet range to about twice that amount in the infrared, or a visual "octave." He states,

> When the colors of visible light are spread out in such a way that equal differences in wavelength take equal amounts of space, it stands out that blue and yellow occupy relatively narrow bands while violet, green, and red are broad [see Figure 27]. Observe that the distance from the ultraviolet threshold to blue to yellow to the infrared threshold is very closely 4 : 3 : 5 of that spectral "octave", i.e., $383.333\ldots \times 2^{4/12} = 483$ mμ (mid blue) and $383.333... \times 2^{7/12} = 574.333\ldots$ mμ (mid yellow). This means that if we subjectively identify the two thresholds of ultraviolet and infrared, as is commonly done in making color wheels, calling both extremes simply "purple," then the narrow bands of "blue" and "yellow" have approximate centers lying at corners of the same triangle with 3,4,5 proportions as the A minor triad [see Figure 26].

Eberhart's observation adds some additional substance to the Renaissance credo that what pleases the ear also pleases the eye.

8 Conclusion

According to Plato, the nature of things and the structure of the universe lay in the study of music, astronomy, geometry and numbers, the so-called quadrivium. Built into sacred structures would be not only a coherent geometrical order but also a sense of the cosmic order in terms of the cycles of

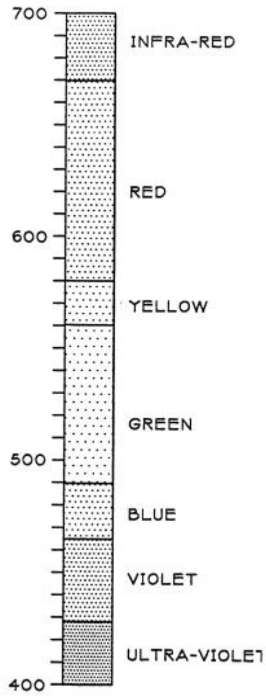

Figure 27: Color Scale.

the sun and the moon and the harmonies of the musical scale. The Brunes star with its ability to approximately square the circle, its equipartioning properties, its relationship to 3,4,5-triangles, and its ability to be generalized to a geometrical figure exhibiting the golden and silver mean makes it a plausible tool for use of the builders of ancient sacred structures.

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