CHAPTER 10

PROPERTIES OF DERIVATIVES AND DIFFERENTIATION BY THE " EASY WAY."

10.1 INTRODUCTION

In the last chapter you were introduced to a complex procedure that could be used to find the derivative of any function that by a method I refer to as the Derivative Machine. Although this method is completely general, each new function presents you with a different set of algebraic challenges in order to find the derivative. However, once the derivative of a function is determined, these functions can be put together in different ways by adding or subtracting them, multiplying them by constants, or multiplying and dividing them to create more complex functions. Computational methods can be derived to compute the derivatives of these more complex functions if you have knowledge of a few basic derivatives without having to use the derivative machine. It is these simple computational methods which has made calculus a useful subject. If we had to use the methods of the last chapter for each new function, calculus would not have its present utility. In this chapter we will begin to show you how easy it is to compute derivatives. Additional simplifications will be introduced in Chapter 13.

10.2 THE POWER LAW

From the Examples and Problems of the previous Chapter you discovered the following derivatives:

$$\frac{dx^2}{dx} = 2x$$

$$\frac{dx^3}{dx} = 3x^2$$

$$\frac{dx^{-1}}{dx} = (-1)x^{-2}$$

$$\frac{dx^1}{dx} = 1x^0 = 1$$

$$\frac{dx^{\frac{1}{2}}}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

Do you notice a pattern to these derivatives? In fact it can be shown that ,

$$\frac{dx^r}{dx} = rx^{r-1} \qquad (1)$$

where r is any real number. For example, $\frac{dx^5}{dx} = 5x^4$, $\frac{dx^{1/3}}{dx} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$, and $\frac{dx^{-2}}{dx} = (-2)x^{-3}$.

10.3 NEW FUNCTIONS AND THEIR DERIVATIVES FROM OLD ONES

Consider the functions f and g where f(x) = x and g(x) = x^2. We can define "new" functions 3f , 5g and $(f \pm g)$ as follows:

(3f)(x) = 3f(x) = 3x and (5g)(x) = 5g(x)=$5x^2$.

$(f \pm g)(x) = f(x) \pm g(x) = x \pm x^2$.

Similarly, $(3f \pm 5g)(x) = 3f(x) \pm 5g(x) = 3x \pm 5x^2$.

If h(x) = 1 for all x (i.e., the constant function) then we could also define,

$$(5g - 3f + 2h)(x) = 5g(x) - 3f(x) + 2h(x) = 5x^2 - 3x + 2$$

which is a *second degree polynomial* or *quadratic* function.

In general, given two functions f and g we can define multiplication of a function by a scalar (number) kf and the sum and difference functions, $(f \pm g)$ as follows,

a) (kf)(x) = k f(x) (2a)
b) $(f \pm g)(x) = f(x) \pm g(x)$. (2b)

10.4 DERIVATIVES OF THE "NEW" FUNCTIONS

Now we would like to show how to differentiate these functions.

We have seen that a function f(x) with a smooth graph can be written near point x_0 as,

$$f(x) = f(x_0) + f'(x_0)h + o_1(h) \qquad (3a)$$

for $h = x - x_0$ and $\lim_{h \to 0} \frac{o_1(h)}{h} = 0$
where the tangent line to the graph of the function, y = f(x), is,

$$y_{line} = f(x_0) + f'(x_0)h$$

with the slope m of the line at x_0 being "the derivative of f(x)" or f'(x_0)

Consider a second function, g(x). In a similar manner,

$$g(x) = g(x_0) + g'(x_0)h + o_2(h). \qquad (3b)$$

The derivatives of the slopes of the function in 2a and 2b are given as followed:

a) kf: Let $(kf)(x)=(kf)(x_0)+(kf)'(x_0)h+o(h)$ where $m=(kf)'(x_0)$ (4a)

Using Eq. 2a and 3a,

$$(kf)(x)=kf(x)=kf(x_0)+kf'(x_0)h+o(h) \qquad (4b)$$

Comparing Eq. 4a and 4b,

$$(kf_0)'(x)=kf'(x_0) \qquad (6)$$

Example 1:

If $f(x)=x^2$ then $f'(x)=2x$. Therefore $(3f)(x)=3f(x)$ and $(3f)'(x)=3f'(x)=6x$

Example 2:
If $f(x)=\frac{1}{x}$ then $f'(x)=-\frac{1}{x^2}$. Therefore, $(3f)(x)=3$ f(x) and
$(3f)'(x)=3f'(x)=3(-\frac{1}{x^2})=-3\frac{1}{x^2}$

b) $f\pm g$: $(f\pm g)(x)=(f\pm g)(x_0)+(f+g)'(x_0)h+o(h)$ (8a)

Using Eq. 1b, 3a, and 3b,

$$(f\pm g)(x)=f(x)\pm g(x)=(f(x_0)\pm g_0(x))+(f'(x_0)+g'(x_0))h+o_1(h)+o_2(h) \quad (8b)$$

From the fact that $o_1(h)+o_2(h)=o(h)$ and comparing Eq. 8a with 8b,

$$(f+g)'(x_0)=f'(x_0)\pm g'(x_0) \qquad (9)$$

Example 3: Consider $f(x)=x^2$ and $g(x)=\frac{1}{x}$ where $f'(x)=2x$ and $g'(x)=-\frac{1}{x^2}$

then $(f+g)'(x)=f'(x)+g'(x)=2x-\frac{1}{x^2}$

Example 4: $f(x)=x^2$ and $g(x)=x^3$ where $f'(x)=2x$ and $g'(x)=3x^2$
then $(4f+5g)'(x)=4f'(x)+5g)'(x)=4(2x)+5(3x^2)=8x+15x^2$ or
$\frac{d(4x^2+5x^3)}{dx}=8x+15x^2$

Example 5: $f(x)=x$, $g(x)=x^2$, and $h(x)=1$ where where f ’(x) = 1, g ’(x) = 2x, and h'(x) = 0.   The derivative of the quadratic function of Eq. 1 is: $\frac{d(5x^2-3x+2)}{dx}=10x-3$

Summary:

The work of computing derivatives can be greatly simplified by using the following three rules:

a) $\frac{dx^r}{dx}=rx^{r-1}$

b) $(kf)'(x)=kf'(x)$

c) $(f+g)'(x)=f'(x)\pm g'(x)$

Additional rules will be introduced in Chapter 13

10.5 ADDING A CONSTANT FUNCTION

Consider a function f(x) with derivative f '(x). If the constant function h(x), where h(x) = c for all x is added to f(x), the value of the derivative does not change because h'(x)=0 as we showed in Section 9.4, i.e.,

$$(f+h)'(x)=f'(x)\pm h'(x)=f'(x)+0=f'(x)$$

saying this in another way,

$$\frac{d(f(x)+c)}{dx}=\frac{df(x)}{dx}=f'(x)$$

10.6 ANTIDERIVATIES:

If f(x) is the derivative of F(x), i.e., $\frac{dF(x)}{dx}=f(x)$ then we can say that F(x) is the antiderivative of f(x). The notation of this is :

$$F(x)=\int f(x)dx$$

The logic of this notation will become evident in Chaptes 15 and 16.

In the last section we showed that when adding a constant to any function the derivative of the function is unchanged. Therefore, if F(x) is an anti-derivative of f(x) (i.e., F'(x) = f(x)) where $F(x)=\int f(x)dx$), then another anti-derivative is $\int f(x)dx+c$ for any value of c.

Here are some simple antiderivatives:

$$\frac{dx^2/2}{dx} = x \quad \text{or} \quad \int x dx = \frac{x^2}{2} + \text{c}$$

$$\frac{dx^3/3}{dx} = x^2 \quad \text{or} \quad \int x^2 dx = \frac{x^3}{3} + \text{c}$$

$$\frac{dx^{-1}}{dx} = -x^{-2} \quad \text{or} \quad \int x^{-2} dx = \frac{x^{-1}}{(-1)} + \text{c}$$

Do you notice a pattern among in these antiderivatives:

a) $\int x^r dx = \frac{x^{r+1}}{r+1} + c$ for r$\neq$ -1.

Also it is easy to show that corresponding to the two derivative rules are two rules for anti-derivatives;

b) $\int kf(x)dx = k\int f(x)dx$, and

c) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

With these three rules many anti-derivatives can easily be computed.

Example 6: $\int 5x^3 dx = 5\int x^3 dx = \frac{5}{4}x^4 + c$

Example 7: $\int (3x + \frac{5}{x^3})dx = \int 3xdx + \int 5x^{-3}dx = \frac{3}{2}x^2 + 5\frac{x^{-2}}{-2} = \frac{3}{2}x^2 - \frac{5}{2x^2} + c$

Example 8: Find the function y = f(x) with t he properties that $\frac{dy}{dx} = 3x - 2$ and y = 1 when x = 2.

Solution: $\frac{dy}{dx} = 3x - 2 \Rightarrow y = \int (3x-2)dx = \frac{3}{2}x^2 - 2x + c$,

Since y = 1 when x = 2, 1 = 6 - 4 + c

c = -1

Therefore, $y = \frac{3}{2}x^2 - 2x - 1$

10.7 OTHER ANTIERIVATIVES

Any time you know the derivative of a function you also know its antiderivative. For example, in Chapter 9 we discovered that,

$$\frac{de^x}{dx} = e^x \quad \text{therefore} \quad \int e^x dx = e^x + c$$

$$\frac{d\ln x}{dx} = \frac{1}{x} \quad \text{therefore} \quad \int \frac{1}{x} dx = \ln|x| + c$$

Note: Absolute value of x is used because logs are not defined for negative numbers.
Remark: This fills in the case of $r \neq -1$ not accounted for in property a of Section 10.6.

$$\frac{d\sin x}{dx} = \cos x \quad \text{therefore} \quad \int \cos x dx = \sin x + c$$

$$\frac{d\cos x}{dx} = -\sin x \quad \text{therefore} \quad \int \sin x dx = -\cos x + c$$

Rules b and c of Section 10.4 continue to hold as illustrated by the following examples:

Example 10: $\int (2e^x - 3x)dx = 2e^x - \frac{3}{2}x^2 + c$

Example 11: $\int (3\sin x + 5\cos x)dx = -3\cos x + 5\sin x + c$

Example 12: $\int (3x + \frac{5}{x})dx = \int 3x dx + \int \frac{5}{x} dx = \frac{3}{2}x^2 + 5\ln|x| + c$

10.8 DERIVATIVES AND ANTIDERIVATIVES FROM THE BEAM PROBLEMS

We have stated without proof in Chapters 7 and 8 that*: a) The value of the bending moment equals the slope of the shear force, i.e.,* $\frac{d\overline{M}}{dx} = \overline{V}$ making use of the interpretation of derivative as the slope of a tangent line; and b) *the rate of decrease of the shear force equals the force density,* $\frac{d\overline{V}}{dx} = -\rho(x)$, making use of the interpretation of derivate as a rate of change . Compute the derivatives of the following previously determined bending moments and shear stresses to check these hypotheses.

Example 13 : The cantilever problem, Problem 2 of Section 7.8:

$$\overline{M} = -500 + 50x$$
$$\overline{V} = 50 \text{ lb.}$$
$$\rho = 0 \quad \text{for } x \neq 5 \quad \text{(i.e., a weightless beam)}$$

Example 14: Example 3 of Section 8.5:

$$\overline{M} = \begin{cases} -10x^2 & \text{for } 0 \leq x \leq 4 \\ -720 + 180x - 10x^2 & \text{for } 4 \leq x \leq 12 \end{cases}$$

$$\overline{V} = \begin{cases} -20x & \text{for } 0 \leq x < 4 \\ 180 - 20x & \text{for } 4 < x \leq 12 \end{cases}$$

$$\rho = 20 \quad \text{for } 0 \leq x \leq 12$$

Example 15: Example 1 of Section 8.3:

$$\overline{M} = -5x^2 + 50x$$
$$\overline{V} = 50 - 10x$$
$$\rho = 10$$

Example 16 : Example 2 of Section 8.4:

$$\overline{M} = 144x - x^3$$
$$\overline{V} = 144 - 3x^2$$
$$\rho = 6x$$

Problem : For Examples 15 and 16, use antiderivatives to compute $\overline{M}$ given $\overline{V}$, i.e., $\overline{M} = \int \overline{V} dx + c$ and the fact that $\overline{M}$ =0 when x = 0. For Example 11 find $\overline{M}$ given $\overline{V}$ and the fact that $\overline{M}$ = - 500 when x = 0 (Hint: Follow the approach of Example 8).

Problems

For Problems 1–36, find the derivative. Assume a, b, c, k are constants.

1. $y = 5$

2. $y = 3x$

3. $y = x^{12}$

4. $y = x^{-12}$

5. $y = x^{4/3}$

6. $y = 8t^3$

7. $y = 3t^4 - 2t^2$

8. $y = 5x + 13$

9. $f(x) = \dfrac{1}{x^4}$

10. $f(q) = q^3 + 10$

11. $y = x^2 + 5x + 9$

12. $y = 6x^3 + 4x^2 - 2x$

13. $y = 3x^2 + 7x - 9$

14. $y = 8t^3 - 4t^2 + 12t - 3$

15. $y = 4.2q^2 - 0.5q + 11.27$

16. $y = -3x^4 - 4x^3 - 6x + 2$

17. $g(t) = \dfrac{1}{t^5}$

18. $f(z) = -\dfrac{1}{z^{6.1}}$

19. $y = \dfrac{1}{r^{7/2}}$

20. $y = \sqrt{x}$

21. $h(\theta) = \dfrac{1}{\sqrt[3]{\theta}}$

22. $f(x) = \sqrt{\dfrac{1}{x^3}}$

23. $y = 3t^5 - 5\sqrt{t} + \dfrac{7}{t}$

24. $y = z^2 + \dfrac{1}{2z}$

25. $y = 3t^2 + \dfrac{12}{\sqrt{t}} - \dfrac{1}{t^2}$

26. $h(t) = \dfrac{3}{t} + \dfrac{4}{t^2}$

27. $y = \sqrt{x}(x + 1)$

28. $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2})$

29. $f(x) = kx^2$

30. $y = ax^2 + bx + c$

31. $Q = aP^2 + bP^3$

32. $v = at^2 + \dfrac{b}{t^2}$

33. $P = a + b\sqrt{t}$

34. $V = \frac{4}{3}\pi r^2 b$

35. $w = 3ab^2 q$

36. $h(x) = \dfrac{ax + b}{c}$

37. Let $f(t) = t^2 - 4t + 5$.

(a) Find $f'(t)$.
(b) Find $f'(1)$ and $f'(2)$.
(c) Use a graph of $f(t)$ to check that your answers to part (b) are reasonable. Explain.

38. Let $f(x) = x^2 + 1$. Compute the derivatives $f'(0)$, $f'(1)$, $f'(2)$, and $f'(-1)$. Check your answers graphically.

39. Let $f(x) = x^2 + 3x - 5$. Find $f'(0)$, $f'(3)$, $f'(-2)$.

40. The height of a sand dune (in centimeters) is represented by $f(t) = 700 - 3t^2$, where t is measured in years since 2005. Find $f(5)$ and $f'(5)$. Using units, explain what each means in terms of the sand dune.

41. Find the rate of change of a population of size $P(t) = t^3 + 4t + 1$ at time $t = 2$.

42. If $f(t) = 2t^3 - 4t^2 + 3t - 1$, find $f'(t)$ and $f''(t)$.

43. If $f(t) = t^4 - 3t^2 + 5t$, find $f'(t)$ and $f''(t)$.

44. Zebra mussels are freshwater shellfish that first appeared in the St. Lawrence River in the early 1980s and have spread throughout the Great Lakes. Suppose that t months after they appeared in a small bay, the number of zebra mussels is given by $Z(t) = 300t^2$. How many zebra mussels are in the bay after four months? At what rate is the population growing at that time? Give units.

45. **(a)** Find the equation of the tangent line to $f(x) = x^3$ at the point where $x = 2$.
(b) Graph the tangent line and the function on the same axes. If the tangent line is used to estimate values of the function, will the estimates be overestimates or underestimates?