CHAPTER 12

MAXIMA AND MINIMA OF FUNCTIONS AND OPTIMIZATION PROBLEMS I

12.1 INTRODUCTION

Derivatives can be used to find the value in the domain of a function at which the function takes on its largest or smallest value. If the function is smooth, i.e., it is continuous everywhere and is differentiable (you can calculate its derivative) everywhere except at all but a finite number of points then calculus can help to find the largest and smallest point directly without having to use trial and error. In the next Section we will find the points in any closed interval (intervals which include their endpoints), signified by square brackets [a,b], where the function takes on its largest and smallest values referred to as the *absolute maximum* and *absolute minimum* We will then extend these ideas to finding maxima and minima on intervals that do not include their endpoints. These ideas will be applied to solving problems with realistic contexts known as "word problems."

12.2 ABSOLUTE MAXIMUM AND MINIMUM

We assume that the function is continuous over the closed interval [a,b] for which mathematical theory guarantees there to be a largest and smallest value on [a,b]. Fig. 1 shows the graph of a function with several values of x identified. In general, we need only look for the absolute maximum or minimum at three locations: a) a critical point (where $\frac{dy}{dx} = 0$), e.g., points x = c,d, and f in Fig. 1; b) an endpoint of the interval, x = a or x = b; or c) a place where the derivative explodes to infinity (a cusp), illustrated by x = e, or the place where the derivative is finite but discontinuous (a kink), illustrated by x = g.. Looking at Fig. 1, we see that x = c is the absolute maximum while x = b is the absolute minimum.

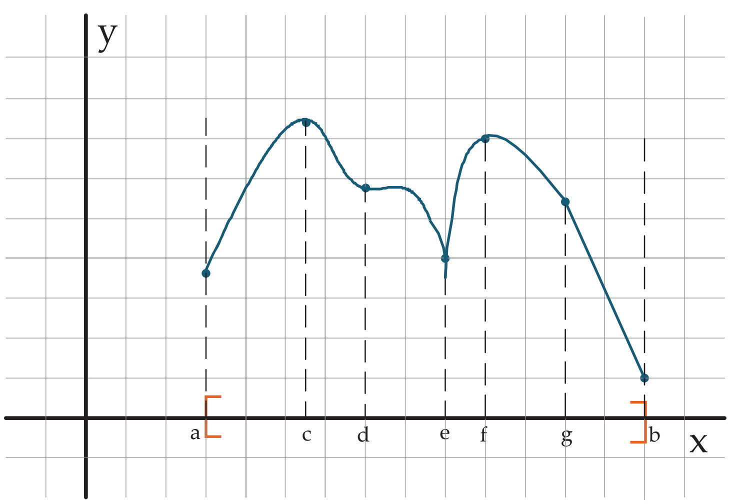

Fig. 1

For the function shown in Fig. 1, we had the luxury of a graph to help find max and min. In fact we can fly blind by computing $\frac{dy}{dx}$ for the function, and then finding the critical points, i.e., solving $\frac{dy}{dx} = 0$, finding the function values at the endpoints and places where there is no derivative such a cusp. A cusp can only occur at places where the derivative "blows up." . For example, if $y = x^{2/3}$ then $\frac{dy}{dx} = \frac{2}{3x^{1/3}}$. When x = 0 the function is continuous but the derivative has an infinite value, i.e., x = 0 is a cusp.

Having found these special values of x we can make a table and chose the largest and smallest values of y = f(x). .

Table 1

x	a	b	c	d	e	f	g
y = f(x)	f(a)	f(b)	f(c)	f(d)	f(e)	f(f)	f(g)

f(c) will be the largest and f(b) the smallest values.

Example 1:

Find the absolute maximum and minimum of $f(x) = x^2 + 4x + 7$ on [-3, 0]

Solution: $\frac{dy}{dx}$ = 2x+4 = 0 or x = - 2, a critical point.
The endpoints are: x = -3 and x = 0

Table 2

x	-3	-2	0
f(x)	4	3	7

Therefore the absolute Max is at x = 0 and min is at x = -3

Example 2:

$$f(x) = x^2 + \frac{16}{x} \quad ; \quad [1, 3]$$

$\frac{dy}{dx} = 2x - \frac{16}{x^2} = 0$ or $x^3 = 8$ or x = 2, a critical point.

Endpoints: x = 1, x = 3

Table 3

x	1	2	3
f(x)	17	12	14 1/3

Max is at x = 1 , Min is at x = 2

In Examples 1,2, and 3 the interval is closed, i.e., it contains its endpoints. We can also consider intervals signified by curved brackets, (a,b), which do not include their endpoints as Examples 3 and 4 demonstrate..

Example 3

$$f(x) = x^3 - 3x + 1 \ ; \quad [0, \infty)$$

Because the function does not exist at the infinite endpoint , we must look at the behavior of the function as x approaches infinity, As we saw in Chapter 11, the behavior at infinity is determined by approximating the polynomial by its highest degree term, i.e.,

$$x^3 - 3x + 1 \approx x^3$$

so that $f(x) \to \infty$ as $x \to \infty$., therefore the function does not attain its largest value; it gets large without bound.. To find the absolute minimum,

$$\frac{dy}{dx} = 3x^2 - 3.$$

$$\frac{dy}{dx} = 0 \text{ when x = -1 and 1 (critical points).}$$

Since x = - 1 is not in the interval being considered we need only consider x = 1. The endpoint is x = 0 so that the table is,

Table 4

x	0	1
f(x)	1	-1

Therefore, the absolute minimum is at x = 1 and has the value, f(1) = -1.

Example 4

$$f(x) = x + \frac{4}{x^2} \ , \quad (0, \infty)$$

Note that 0 and ∞ are excluded from the search for maximum and minimum because the function does not exist at those points. First look at the behavior as x approaches infinity. As x approaches infinity,

$$x + \frac{4}{x^2} \approx x$$

and $f(x) \to \infty$ as $x \to \infty$.

Also $f(x) \to \infty$ as $x \to 0$. So the function does not have a maximum value. You can always find a larger value as you approach either 0 or ∞.

To find the absolute minimum,

$$\frac{dy}{dx} = 1 - \frac{8}{x^3} = 0$$

$x^3 = 8$ or x = 2, a critical point..

Since neither of the endpoints are being considered, the table reduces to a single point,

Table 5

x	2
f(x)	3

Therefore this function has no absolute maximum, but its absolute minimum at x = 2.

Problem 1:

For the following functions over the given intervals, find the absolute maximum and minimum values,

a) $f(x) = x^3 - 3x^2$, [-1, 3]
b) $f(x) = x^3 - 3x^2 - 9x + 15$, [-5,4]
c) $f(x) = x + 1/x$, x >0

12.3 FINDING THE MAXIMUM AND MINIMUM OF WORD PROBLEMS.

We can apply these simple ideas to finding the absolute maximum and minimum of problems with some realistic context. These are often referred to as word problems. In the process of solving these problems we will develop a procedure in which all “word problems” can be effectively solved. We consider three examples.

Example 5

A piece of land 36 feet long borders a river.. One hundred feet of fencing is used to enclose a rectangular plot of land in which the land is surrounded by the fence on three sides leaving the river unfenced. Find the dimensions of the rectangular fence that encloses the largest area. What is the maximum area?

We solve this problem by a series of steps. I refer to Steps 1,2, and 3 as the "Entry Phase," Steps 3, 4, and 5 as the "Attack Phase," and Step 6 as the "Review Phase."

Step 1: Read the problem and place the given information on a diagram or table as shown in Fig. 2. State any assumptions being made. Define the variables and their units used..

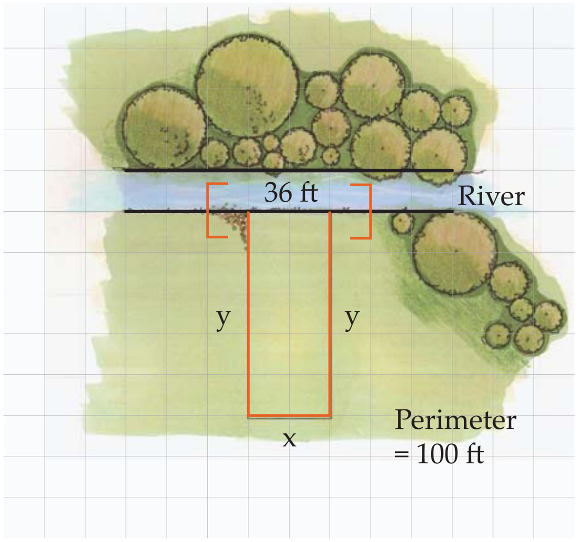

Fig. 2

Assume that the river is straight. and the plot of land is flat.
Let A = the Area in square feet,
x = the length of the rectangle in feet, and
y = the width of the rectangle in feet.

Step 2: Clearly state what the problem asks you to find.

Find : a) x and y such that A is a maximum.
b) Find A_{max}.

Step 3: Find an equation involving the quantity that you wish to maximize, and state other equations involving the variables of the problem.

$$A = xy \quad (1)$$
$$100 = x + 2y \quad (2)$$

Step 4: If the quantity that you wish to maximize has more than one variable (as in Eq. 1), eliminate one of the variables by solving the other equation for it (as in Eqs. 2 and 3). Be sure to express the domain of the function to be maximized. (as in Eq. 4).

$$y = 50 - \frac{x}{2} \qquad (3)$$

$$A = 50x - \frac{x^2}{2}, \quad 0 \le x \le 36 \qquad (4)$$

Step 5: Forget that you are dealing with a word problem and use the ideas from Section 2 to find the maximum value of A, i.e., the maximum will occur at either the critical point or the endpoint..

$$\frac{dA}{dx} = 50 - x = 0$$

$x = 50$,

Since $x = 50$ is not in the interval of interest, we need only consider the endpoints, as shown in Table 6.

Table 6

x	0	36
y	50	32
A	0	1152

Conclusion: $x = 36$ ft., $y = 32$ ft. , $A_{max} = 1152$ ft^2.

Step 6: We check t he results and see if they make physical sense, i.e., are the magnitudes reasonable and are the signs correct?

We will solve all max-mim problems by the same procedure. We demonstrate this in Examples 2 and 3 without comment.

Example 6

A rectangular parallelopiped box with square base has a volume of 108 ft^3. Find the dimensions of the box that has the least exposed surface area. We assume that the bottom of the box is not exposed, only the top and sides.

The given information is shown in Fig. 3.

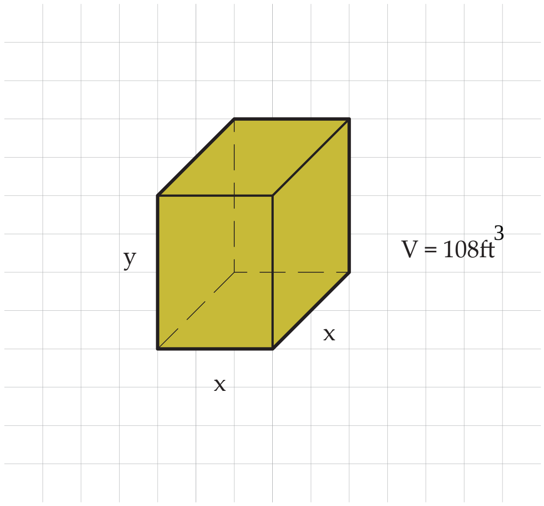

Fig. 3

Assume that the dimensions of t he box can be any positive number.
Let x = the dimension of the square base in ft.
y = the height in ft.
V = the volume in ft^3.
A = the exposed area ft^2.

Find: x and y such that A is a minimum. Find A_{min}.

$$A = x^2 + 4xy$$
$$V = x^2 y = 108$$

$$y = \frac{108}{x^2}$$

$$A = x^2 + \frac{(4)(108)}{x}, \qquad 0 < x < \infty$$

Note that there are no endpoints to this interval and that $A \to \infty$ as $x \to 0$ and $x \to \infty$
Therefore there is no maximum value. To find the minimum value of A we need only consider the critical point.

$$\frac{dA}{dx} = 2x - \frac{432}{x^2} = 0$$

$x^3 = 216$ or x = 6, y = 3.

Table 7

x	6
y	3
A	108

Conclusion: x = 6 ft, y = 3 ft, A = 108 ft^2

Example 3

Four square corners are cut out of a 5 x 8 rectangular piece of paper. The paper is then folded into a box with open top. Find the dimension of the cut that maximizes the volume of the box.

Solution:

The given information is placed in Fig. 4.

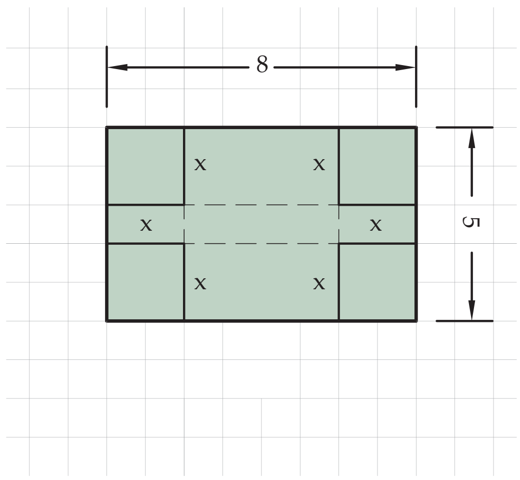

Fig. 4

Let x = the side of the square,
V = volume of the box

Find: x such that V is a maximum.

$$V = x(5-2x)(8-2x)\ , \qquad 0 \le x \le 2.5$$

$$V = 4x^3 - 26x^2 + 40x$$

$$\frac{dV}{dx} = 12x^2 - 52x^2 + 40 = 0$$

$$3x^2 - 13x + 10 = 0$$

$$(3x-10)(x-1) = 0$$

$$x = 3.33\ldots,\ x = 1$$

Since x = 3.33…is not in the interval of interest, the only critical point is x = 1 so the Table is,

Table 8

x	0	1	2.5
V	0	18	0

Conclusion: The maximum occurs for x = 1 unit and V = 18 units^3.

Problem 2

You must construct a closed rectangular box with volume 576 in^3 and with its bottom twice as long as it is wide (see Fig. 5). Find the dimensions of the box that will minimize its total surface area.

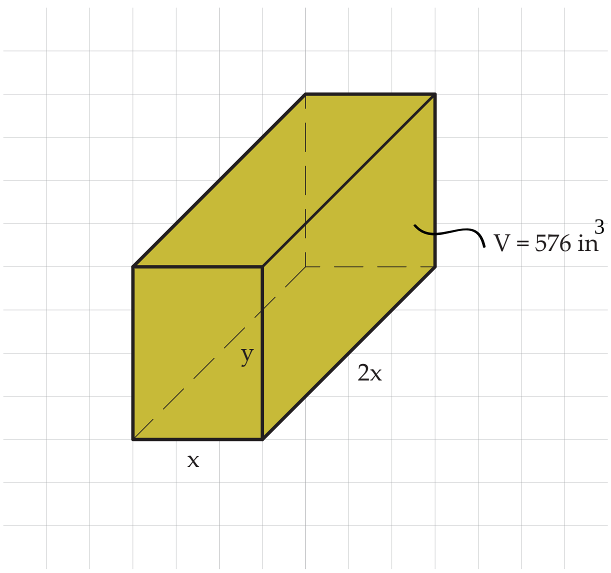

Fig. 5

Problem 3

A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing $25 per foot along three sides and fencing costing $10 per foot along the fourth side. Find the total cost and the dimensions of the rectangle.

Problem 4

Suppose that a post office can accept a package for mailing only if the sum of its length and its girth (the circumference of its cross section) is at most 100 in. What is the maximum volume of a rectangular box with square cross section that can be mailed?

Problem 5

A farmer has 600 yd. of fencing with which to build a rectangular corral. Some of the fencing will be used to construct two interval divider fences, both parallel to the same two sides of the corral (see Fig 6). What is the maximum possible total area of such a corral?

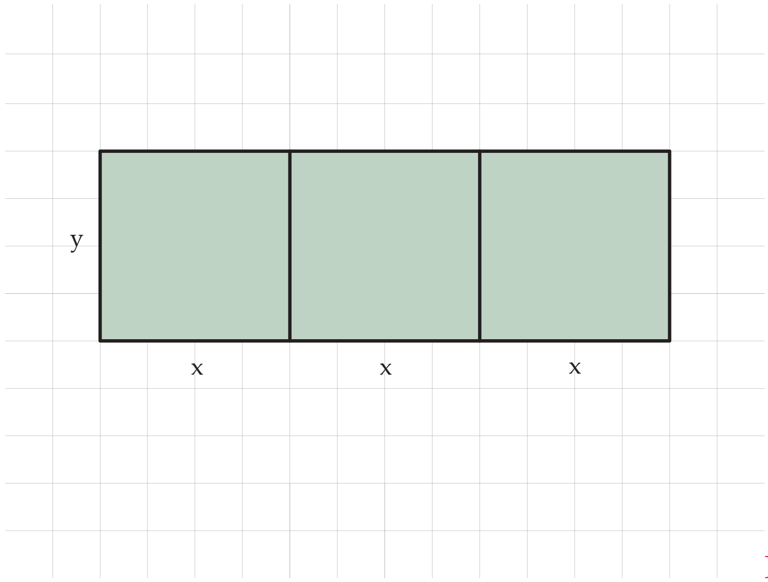

Fig. 6