CHAPTER 13

PRODUCT, QUOTIENT, CHAIN RULE, AND TRIG FUNCTIONS

13.1 NEW FUNCTIONS FROM OLD ONES

Given two functions f(x) and g(x) we can define three new functions in terms of these old ones: (fg)(x) = f(x)g(x) ; $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$; and $(f \circ g)(x) = f(g(x))$ (see Section 2.6. These new functions are called the *product, quotient, composition* of f and g. If the derivatives of f and g, f'(x) and g'(x), are known, then we can compute the derivatives of fg, $\frac{f}{g}$, and $f \circ g$ in terms of f'(x) and g'(x).

13.2 THE DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS

To find the derivative of (fg)(x) use the derivative machine:

$$(fg)'(x) = f(x)g(x) = f(x_0)g_0(x) + mh + o(h) \qquad (1)$$

where,

$$f(x) = f(x_0) + f'(x_0)h + o(h) \qquad (2a)$$
$$g(x) = g(x_0) + g'(x_0)h + o(h) \qquad (2b)$$

To solve for m, insert Eq. 2a and b in Eq. 1 to get,

$$(fg)(x) = f(x)g(x) = ((f(x_0) + (f'(x_0)h + o_1(h))(\ g(x_0) + g'(x_0)h + o_2(h))$$
$$= f(x_0)g(x_0) + ((f(x_0)g'(x_0) + f'(x_0)g(x_0))h + \text{(little o(h) terms)} \qquad (3)$$

Eliminate the o(h) terms as usual and, after some algebra,

$$(fg)'(x) = m = f(x)g'(x) + f'(x)g(x) \qquad (4)$$

Eq. 4 is known as the *product rule*.

Example:1

Use the product rule to find the derivative of (fg)(x) = $(x^2 - 2x)(x^3 + 1)$. where $f(x) = x^2 - 2x$, $g(x) = x^3 + 1$ and , $f'(x) = 2x - 2$, $g'(x) = 3x^2$. .From Eq. 4,
$(fg)'(x) = (x^2 - 2x)(3x^2) + (2x - 2)(x^3 + 1)$

Example 2

Find . $\frac{d(x^2e^x)}{dx}$.

Let f(x) = x^2 and g(x) = e^x where f'(x) = 2x and g'(x) = e^x . From Eq. 4,

$$\frac{d(x^2e^x)}{dx} = x^2e^x + 2xe^x$$

Problems:

a) y= $(x^2 - 2x + 5)(x^2 - 3)$ Find: dy/dx

b) We showed in Sec. 9.6 that $\frac{d(\cos x)}{dx} = -\sin x$. Use this with the help of the product rule to compute: $\frac{d(x\cos x)}{dx}$

13.3 THE DERIVATIVE OF THE QUOTIENT OF TWO FUNCTIONS

The following rule enables you to compute the derivative of the quotient of two functions if you know the derivatives of the two functions.. The proof follows in a similar manner to the Product Rule shown above. I leave the details to the student.

$$(\frac{f}{g})'(x) = (\frac{f(x)}{g(x)})' = \frac{g(x)f(x)' - f(x)g(x)'}{(g(x))^2} \qquad (5)$$

Eq. 5 is known as the *quotient rule.*

Example 3

Find, $\frac{d(\frac{x^2}{x^2-1})}{dx}$.

Let f(x) = x^2 and g(x) = $x^2 - 1$ where f '(x) = 2x and g '(x) = 2x. Replacing these functions in Eq. 5,

$$\frac{d(\frac{x^2}{x^2-1})}{dx} = \frac{(x^2-1)(2x) - 2x(x^2)}{(x^2-1)^2}$$

$$= \frac{-2x}{(x^2-1)^2}$$

13.4 THE DERIVATIVE OF A FUNCTION OF A FUNCTION – THE CHAIN RULE.

Consider $(f \circ g)(x) = f(g(x))$. I will show by using the Derivative machine that its derivative at x_0 is:

$$(f \circ g)'(x) = f'(g(x_0))g(x_0)$$

Using the Derivative machine:

$$g(x_0 + h) = g(x_0) + g'(x_0)h + o_1(h)$$
$$f(x_0 + h) = f(x_0) + f'(x_0)h + o_2(h)$$
$$(f \circ g)(x_0 + h) = f(g(x_0 + h)) = f(g(x_0) + g(x_0)'h + o_2(h))$$

Set $o_2(h)$ to zero and let $g'(x_0)h = k$

$$(f \circ g)(x_0 + h) = f(g(x_0) + k)$$
$$= f(g(x_0)) + f'(g(x_0))k$$
$$f(g(x_0 + h)) = f(g(x_0)) + f'(g(x_0))g'(x_0)h + o(h)$$

Therefore, ‘

$$(f \circ g)'(x) = f'(g(x_0))g(x_0) \qquad (6)$$

Example 4

Find $\dfrac{d\sqrt{1+x^2}}{dx}$

Let $f(x) = x^{1/2}$ and $g(x) = x^2 + 1$, $f'(x) = \dfrac{1}{2}x^{-1/2}$ and $g'(x) = 2x$..

From Eq. 6,

$$\frac{d\sqrt{1+x^2}}{dx} = \frac{1}{2}(1+x^2)^{-1/2}(2x)$$
$$= \frac{x}{\sqrt{1+x^2}}$$

Successfully finding the derivative of the function of a function often presents the student with considerable difficulty. There is another approach that I call the "*box method*" which often simplifies this computation. We illustrate this for the case of Example 4. One first sketches a diagram of the compound function by representing it as a box with an outside function and an inside function as shown in Fig. 1. The function is gotten by placing the inside function, $x^2 + 1$, into the open parenthesis of the outside function, $(\ )^{1/2}$

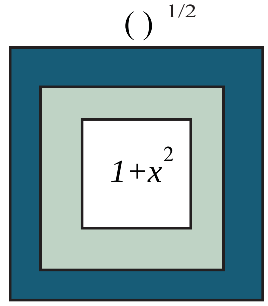

Fig. 1

In order to differentiate the compound function, we use the following step by step procedure: a) Begin with the outside function. Its derivative is: $\frac{1}{2}(\ )^{-1/2}$.; b) replace the open parenthesis by whatever is in the box, i.e., $\frac{1}{2}(x^2 + 1)^{-1/2}$; c) Differentiate the inside function, i.e., 2x; d) multiply the result of b) and c) to get, $\frac{d\sqrt{1+x^2}}{dx} = \frac{1}{2}(1+x^2)^{-1/2}(2x)$.

Example 5

Sometimes you need several boxes in order to define the function . Use the "box method" to find,

$$\frac{d(\cos 3x)^3}{dx}$$

First sketch the function. We need three boxes to do this. The outermost function is the square function, $(\ )^2$, the next function is the cosine function, i.e., cos ( ), while the innermost function is the 3x function (see Fig. 2).

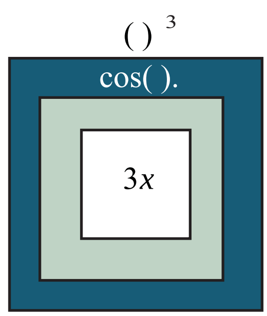

Fig.2

Using a slightly enhanced version of the procedure,

a) $d(\ )^3/dx = 3(\ )^2$; b) Insert everything inside the outer box into the open parenthesis, i.e., $3(\cos(3x))^2$; c) d cos ( ) /dx = -sin ( ); d) Insert everything within the second box into the open parenthesis, i.e., - sin (3x); e) differentiate the innermost function, i.e., d (3x)/dx = 3 ; f) multiply the results of b, d, and e to get,

$$\frac{d(\cos 3x)^3}{dx} = -3(3)(\cos 3x)^2 \sin(3x)$$

$$= -9\cos^2(3x)\sin(3x)$$

13.5 THE CHAIN RULE

Computation of the derivative of compound functions is often referred to as the chain rule. Why do we use this terminology? Let us illustrate the chain rule for Example 4. We can define the compound function by a "chain" of functions.

Example 6

Define $y = (x^2 + 1)^{1/2}$ as follows,

$$y = (u)^{1/2}$$
$$u = x^2 + 1$$

Therefore we have a chain of dependencies,

$$x \to u \to y$$

In other words, knowing x you can find u and then knowing u you can find y. To find the derivative of y with respect to x just differentiate the function down the chain as follows:

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = \frac{1}{2}u^{-1/2}(2x)$$
$$= \frac{x}{\sqrt{1+x^2}}$$

where we have inserted the value of u in terms of x.

Remark 1:

The chain rule merely makes the box method more formal.

Let us apply the chain rule to Example 5 to find the derivative of . $(\cos 3x)^3$

Example 7

Define $y = \cos^3(3x)$ as,

$$y = (u)^3$$

u = cos (w)
w = 3x

The chain for this function is:

$$x \to w \to u \to y$$

where,

$$\frac{dy}{dx} = (\frac{dy}{du})(\frac{du}{dw})(\frac{dw}{du}) = 3u^2(-\sin w)(3)$$

$$= 3\cos^2 w(-\sin w)(3)$$

$$= -9\cos^2(3x)\sin(3x)$$

where u is expressed in terms of w and then w is expressed in terms of x.

13.6 THE INVERSE OF THE CHAIN RULE – THE SUBSTITUTION METHOD OF FINDING ANTIDERIVATIVES

In Sections 13.4 and 13.5 we have computed the derivative of compound functions. In this section we reverse the chain rule and find antiderivatives of certain compound functions using the *substitution method*. For example,.

Example 8

Find,

$$\int (1+3x)^{1/2} dx \qquad (7a)$$

We know how to find.,

$$\int x^{1/2} dx\,, \qquad (7b)$$

and the following procedure that will reduce Expression 7a to 7b. Let's see how it works.

a) Let u = 1 + 3x (8)

b) Define, $du = \frac{du}{dx} dx$, or du = 3 dx

c) Solve for dx, i.e., $dx = \frac{1}{3} du$

d) Replace the result of a) and c) into Eq. 7a,

$$\int \frac{1}{3}(u)^{1/2} du$$

d) Take the constant out of the antiderivate sign using the property b) in Section 10.5, i.e.,

$$\frac{1}{3}\int u^{1/2} du \qquad (9)$$

e) Use the power law (property a) of Section 10.5 to anti differentiate Eq. 9,

$$\frac{1}{3}(\frac{2}{3})u^{3/2} + c \qquad (10)$$

f) Express u in Expression 10 by x using Eq. 8,

$$\int (1+3x)^{1/2} dx = \frac{2}{9}(1+3x)^{1/2} + c \qquad .$$

Remark 2:

This substitution method will work whenever the derivative of the substitution, is sitting under the integral sign differing only by a constant, e.g., in *Example 8*, $\frac{d(1+3x)}{dx} = 3$. In other words, the function to be antidifferentiated is off by the constant 3.

Example 9

Use this procedure to compute,

$$\int xe^{x^2} dx \qquad (11)$$

a) Let u = x^2

b) $du = \frac{du}{dx} dx$ or du = 2x dx

c) $dx = \frac{1}{2x} du$

d) $\int x \frac{1}{2x} e^{u^2} du$ or $\frac{1}{2}\int e^u du$ (cancelling x)

e) $\frac{1}{2}e^u + c$

f) $\int xe^{x^2} dx = \frac{1}{2}e^{x^2} + c$

Remark 3: Since $\frac{dx^2}{dx} = 2x$ and there is an x under the antiderivative sign in expression 11, so the function to be antidifferentiated differs only by the constant, 2. Therefore the method will work.

Remark 4: We cannot use the substitution method to find the antiderivative of $\int e^{x^2} dx$. (why?)

13.7 TRIG FUNCTIONS

The sine and cosine functions of trigonometry are defined in terms of a unit circle. For the unit circle in Fig. 3, if we consider the ray coming from the origin in the direction of the positive x-axis as $\theta = 0$ deg. Then $(\cos\theta, \sin\theta)$ is the point on the unit circle at a positive angle of θ deg. measured from the x-axis in a counterclockwise direction, and negative angle in the clockwise direction. You can see from Fig. 3, using the Pythagorean Theorem, that

$$\sin^2\theta + \cos^2\theta = 1. \qquad (12)$$

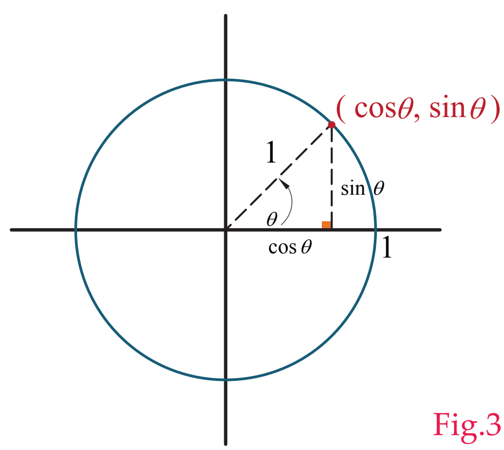

Fig.3

In Section 9. we found that,

$$\frac{d\cos\theta}{dx} = -\sin\theta$$

We can now use this to find the derivative of $\sin\theta$. From the right triangle in Fig. 4 we see that,

$$\cos(\frac{\pi}{2} - \theta) = \sin\theta, \qquad (13)$$

$$\sin(\frac{\pi}{2} - \theta) = \cos\theta. \qquad (14)$$

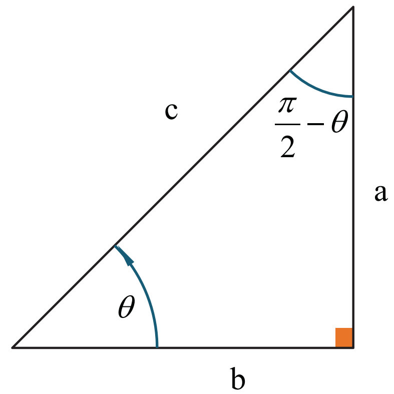

Fig.4

Example 10

Show that, $\frac{d\sin x}{dx} = \cos\theta$.

Using Eq. 13,

$$\frac{d\sin x}{dx} = \frac{d\cos(\frac{\pi}{2}-x)}{dx}$$

Using the chain rule,

$$\frac{d\sin x}{dx} = -(-1)\sin(\frac{\pi}{2}-x)$$
$$= \cos x$$

Example 11:

Show that $\frac{d\tan x}{dx} = \sec^2 x$

Let $\tan x = \frac{\sin x}{\cos x}$.

Using the quotient rule,

$$\frac{d\tan x}{dx} = \frac{d}{dx}(\frac{\sin x}{\cos x})$$
$$\frac{d\tan x}{dx} = \frac{\cos x(\cos x)-(-\sin x)\sin x}{\cos^2 x}$$

Using Eq. 12,

$$\frac{d\tan x}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

Example 12

Show that $\frac{d\tan^{-1}x}{dx} = \frac{1}{x^2+1}$ where $\tan^{-1}x$ is the inverse tangent function (see Sec. 2.1 and 2.7)

In Section 2.7 we saw that if $y = f(x)$ then $f^{-1}(y) = f^{-1}(f(x)) = \text{x}$, i.e.,

$$x = f^{-1}(y)$$

Applying this to $y = \tan^{-1} x$ we have that $x = \tan y$

In order to find the derivative of an inverse function we state without proof that,

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{15}$$

or,

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{d\tan y/dy} \tag{16}$$

From Example 11,

$$\frac{d\tan y}{dy} = \sec^2 y \tag{17}$$

Replacing Eq. 17 in 16,

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{\sec^2 y}$$

But, $y = \tan^{-1} x$ so that we have that,

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{(\sec(\tan^{-1}x))^2} \tag{18}$$

where $\sec(\tan^{-1} x)$ means "the secant of the angle whose tangent is x."

Let's call this angle α referring to Fig 5,

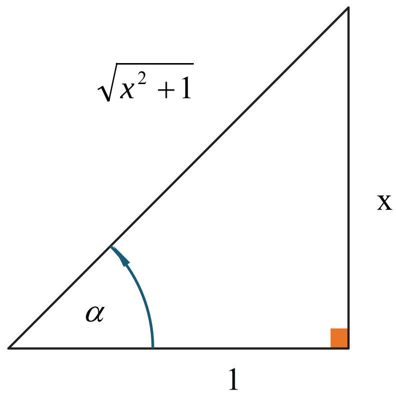

Fig.5

We see that $\sec(\tan^{-1} x) = \sqrt{x^2+1}$

And replacing this in Eq. 18 gives,

$$\frac{d\tan^{-1} x}{dx} = \frac{1}{x^2+1}$$

Once we have a new derivative we also have a new antiderivative so that,

$$\int (\frac{1}{x^2+1})dx = \tan^{-1} x + c$$

Problems

Find derivatives for the functions in Problems 1–63. Assume a, b, c, and k are constants.

1. $f(t) = t^2 + t^4$
2. $g(x) = 5x^4$
3. $y = 5x^3 + 7x^2 - 3x + 1$
4. $s(t) = 6t^{-2} + 3t^3 - 4t^{1/2}$
5. $f(x) = \frac{1}{x^2} + 5\sqrt{x} - 7$
6. $P(t) = 100e^{0.05t}$
7. $f(x) = 5e^{2x} - 2 \cdot 3^x$
8. $P(t) = 1{,}000(1.07)^t$
9. $D(p) = e^{p^2} + 5p^2$
10. $y = t^2e^{5t}$
11. $y = x^2\sqrt{x^2+1}$
12. $f(x) = \ln\left(x^2 + 1\right)$
13. $s(t) = 8\ln(2t+1)$
14. $g(w) = w^2\ln(w)$
15. $f(x) = 2^x + x^2 + 1$
16. $P(t) = \sqrt{t^2+4}$
17. $C(q) = (2q+1)^3$
18. $g(x) = 5x(x+3)^2$
19. $P(t) = be^{kt}$
20. $f(x) = ax^2 + bx + c$
21. $y = x^2\ln(2x+1)$
22. $f(t) = \left(e^t + 4\right)^3$
23. $f(x) = 5\sin(2x)$
24. $W(r) = r^2\cos r$
25. $g(t) = 3\sin(5t) + 4$
26. $y = e^{3t}\sin(2t)$
27. $y = 2e^x + 3\sin x + 5$
28. $f(t) = 3t^2 - 4t + 1$
29. $y = 17x + 24x^{1/2}$
30. $g(x) = -\frac{1}{2}(x^5 + 2x - 9)$
31. $f(x) = 5x^4 + \frac{1}{x^2}$
32. $y = \frac{e^{2x}}{x^2+1}$
33. $f(x) = \frac{x^2+3x+2}{x+1}$
34. $y = \left(\frac{x^2+2}{3}\right)^2$
35. $g(x) = \sin(2-3x)$
36. $f(z) = \frac{z^2+1}{3z}$
37. $q(r) = \frac{3r}{5r+2}$
38. $y = x\ln x - x + 2$
39. $j(x) = \ln(e^{ax} + b)$
40. $g(t) = \frac{t-4}{t+4}$
41. $h(w) = (w^4 - 2w)^5$
42. $h(w) = w^3\ln(10w)$
43. $f(x) = \ln(\sin x + \cos x)$
44. $w(r) = \sqrt{r^4+1}$
45. $h(w) = -2w^{-3} + 3\sqrt{w}$
46. $h(x) = \sqrt{\frac{x^2+9}{x+3}}$
47. $v(t) = t^2e^{-ct}$
48. $f(x) = \frac{x}{1+\ln x}$
49. $g(\theta) = e^{\sin\theta}$
50. $p(t) = e^{4t+2}$
51. $j(x) = \frac{x^3}{a} + \frac{a}{b}x^2 - cx$
52. $f(z) = \frac{z^2+1}{\sqrt{z}}$
53. $h(r) = \frac{r^2}{2r+1}$
54. $g(x) = 2x - \frac{1}{\sqrt[3]{x}} + 3^x - e$
55. $f(t) = 2te^t - \frac{1}{\sqrt{t}}$
56. $w = \frac{5-3z}{5+3z}$
57. $f(x) = \frac{x^3}{9}(3\ln x - 1)$
58. $g(x) = \frac{x^2+\sqrt{x}+1}{x^{3/2}}$
59. $y = \left(x^2+5\right)^3\left(3x^3-2\right)^2$
60. $f(x) = \frac{a^2-x^2}{a^2+x^2}$
61. $w(r) = \frac{ar^2}{b+r^3}$
62. $H(t) = (at^2+b)e^{-ct}$
63. $g(w) = \frac{5}{(a^2-w^2)^2}$

Find the integrals in Problems 1–40. Check your answers by differentiation.

1. $\int 2x(x^2+1)^5 dx$

2. $\int \frac{4x^3}{x^4+1} dx$

3. $\int (x+10)^3 dx$

4. $\int 5e^{5t+2} dt$

5. $\int \frac{2x}{\sqrt{x^2+1}} dx$

6. $\int e^{-x}\, dx$

7. $\int xe^{-x^2}\, dx$

8. $\int y(y^2+5)^8\, dy$

9. $\int t^2(t^3-3)^{10}\, dt$

10. $\int x^2(1+2x^3)^2\, dx$

11. $\int x(x^2-4)^{7/2}\, dx$

12. $\int x(x^2+3)^2\, dx$

13. $\int \frac{1}{\sqrt{4-x}} dx$

14. $\int \frac{dy}{y+5}$

15. $\int t\cos(t^2)\, dt$

16. $\int (2t-7)^{73}\, dt$

17. $\int (x^2+3)^2\, dx$

18. $\int \sin(3-t)\, dt$

19. $\int y^2(1+y)^2\, dy$

20. $\int \sin\theta(\cos\theta+5)^7\, d\theta$

21. $\int \sqrt{\cos 3t}\sin 3t\, dt$

22. $\int \frac{t}{1+3t^2} dt$

23. $\int \sin^6\theta\cos\theta\, d\theta$

24. $\int x^2 e^{x^3+1}\, dx$

25. $\int \sin^6(5\theta)\cos(5\theta)\, d\theta$

26. $\int \sin^3\alpha\cos\alpha\, d\alpha$

27. $\int x\sin(x^2) dx$

28. $\int e^{3x-4} dx$

29. $\int xe^{3x^2}\, dx$

30. $\int x\sqrt{x^2+1} dx$

31. $\int \frac{q}{5q^2+8} dq$

32. $\int \frac{(\ln z)^2}{z} dz$

33. $\int \frac{e^t+1}{e^t+t} dt$

34. $\int \frac{y}{y^2+4} dy$

35. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

36. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}} dy$

37. $\int \frac{1+e^x}{\sqrt{x+e^x}} dx$

38. $\int \frac{e^x}{2+e^x} dx$

39. $\int \frac{x+1}{x^2+2x+19} dx$

40. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}} dx$

41. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

(a) $\int x\sin(x^2)\, dx$

(b) $\int x^2\sin x\, dx$

(c) $\int \frac{x^2}{1+x^2} dx$

(d) $\int \frac{x}{(1+x^2)^2} dx$

(e) $\int x^3 e^{x^2}\, dx$

(f) $\int \frac{\sin x}{2+\cos x} dx$

42. Find $\int 4x(x^2+1)\, dx$ using two methods:

(a) Do the multiplication first, and then antidifferentiate.
(b) Use the substitution $w = x^2+1$.
(c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?

43. (a) Find $\int (x+5)^2\, dx$ in two ways:
(i) By multiplying out
(ii) By substituting $w = x+5$
(b) Are the results the same? Explain.