CHAPTER 14

CURVE SKETCHING AND MAX-MIN II

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14.1 INTRODUCTION

In Chapter 11, we developed a procedure for graphing polynomial functions. In this Chapter we take this one step further and expand the procedure to include functions of any kind. In Chapter 12 we found the maximum and minimum for word problems that involved only polynomials. The same ideas can be applied to functions that are more complex than polynomials. In order to extend these results we will make use of our new ability to differentiate functions using the product, quotient, and chain rules.

14.2 CURVE SKETCHING

Example 1:

Sketch the graph of ,

$$y = \frac{4}{x^2+1}$$

a) Find a simple point: x = 0, y = 4, and place this point on Fig. 1

b) Find the behavior of the function for large and small values of x: For large and small values of x,

$$f(x) = \frac{4}{x^2+1} \approx \frac{4}{x^2}$$

$y \to 0$, as $x \to \pm\infty$ or using the language of *limits* , $\lim_{x\to\infty} f(x) = 0$. Place marks on the graph in Fig. 1 to illustrate this behavior. We say that f(x) has a *horizontal asympote* at y = 0 as x $\longrightarrow \pm\infty$

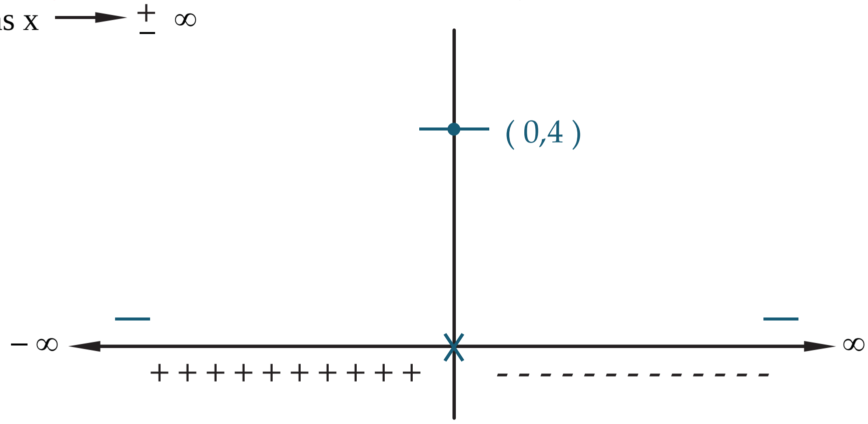

Fig. 1

c) Find critical points. Rewrite Eq. 1 as,

$$y = 4(x^2 + 1)^{-1} \qquad (2)$$

Use the chain rule to differentiate Eq. 2 and set the result equal to 0.

$$\frac{dy}{dx} = \frac{-2x}{(x^2+1)^2} = 0 \qquad \text{or } x = 0.$$

Therefore, x = 0, y = 4 is the only critical point. Place a short horizontal line through this point in Fig. 1. Place an mark at x = 0 on the x-axis. This divides the x- axis as follows:

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d) Determine the regions over which the function increases and decreases: Let x = 1, $\frac{dy}{dx} < 0$. Let x = -1, $\frac{dy}{dx} > 0$. The curve increases until it hits the horizontal line at (0,4) and then decreases. It is also *asymptotic* to y = 0 for large and small values of x.

e) Putting together this information, the resulting curve is shown in Fig. 2.

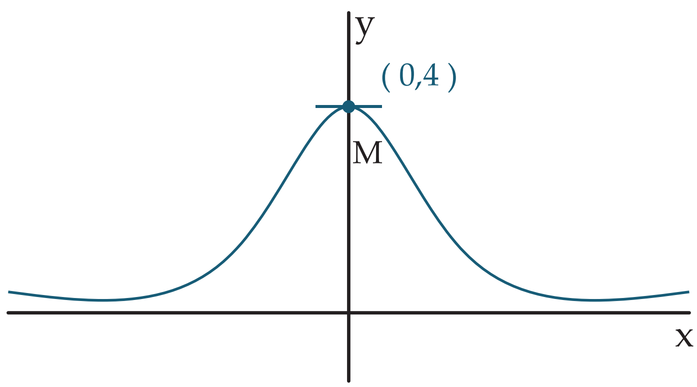

Fig. 2

Check: The function, $f(x) = \frac{4}{x^2+1}$ is even. Notice that the graph of t his function is symmetric with respect to the y-axis.

Remark: We will show in Example 6 of Section 16.2 that the area under this curve over the interval [0,1] equals π.

Example 2

Sketch the graph of ,

$$y = \frac{x^2}{x^2 - 1} \qquad (3)$$

a) Find a easy point: x= 0, y = 0 and place this point on Fig. 3
b) Find the behavior of t he function for large and small values of x. For large and small values of x only the leading terms in the numerator and denominator matter so that Eq. 3 can be rewritten as Eq. 4,

$$f(x) = \frac{x^2}{x^2 - 1} \approx \frac{x^2}{x^2} = 1 \qquad (4)$$

Therefore, $\lim_{x \to \infty} f(x) = 1$. The line y = 1 is called a horizontal asymptote. To find whether the curve approaches y = 1 above or below it, place a large value of x into Eq. 4, i.e., when x = $\pm$5, y = 25/24 so the curve approaches the horizontal asymptote from above it. Places marks above y = 1 for large values of x as shown in Fig. 2

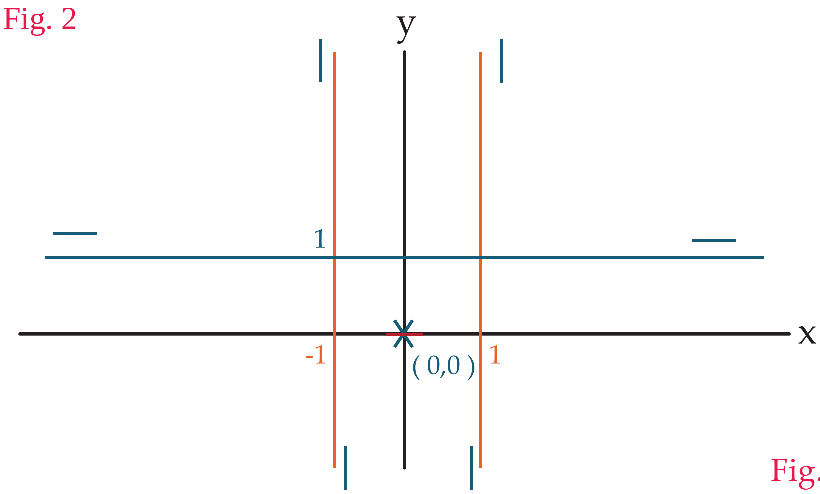

Fig. 3

b) This graph brings a new element into the graphing procedure. Rewriting Eq. 3,

$$y = \frac{x^2}{(x-1)(x+1)}$$

you will notice that the denominator equals 0 when x = $\pm$1. Therefore the graph has what is known as vertical asymptotes at $x = \pm 1$. These vertical asymptotes are vertical lines as shown in Fig. 2. To see how the graph of the function behaves as x approaches these vertical lines first let x approach 1 from above 1, i.e., let x = 3/2 in Eq. 3. We see that y = 9/5. Since this result is positive y approaches $+\infty$, i.e. , as x → 1^+, y → $+\infty$. Also as x→ 1^- let x = 1/2 in Eq. 3. We see that y = -1/3 a negative number so that y → $-\infty$. Likewise, we find that as $x \to -1^+, y \to -\infty$. This is illustrated in Fig. 3 by marks placed along the asymptotes.

c) Find the critical points:

$$\frac{dy}{dx} = \frac{-2x}{(x^2-1)^2} = 0 \qquad (5)$$

x = 0, y = 0

Place the critical point on the x-axis in Fig. 3. The critical point divides the x-axis as follows:

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d) Find where the curve increases and decreases: From Eq. 4,

for x < 0, $\frac{dy}{dx} > 0$, and for x > 0, $\frac{dy}{dx} < 0$. Mark the x-axis appropriately with +'s and –'s in Fig. 3. Therefore, the graph increases from the mark at x = $-\infty$, y = 0 until it connects with the x = -1^- asymptote, then it rises from the x = -1^+ asymptote to the critical point. Then it falls to the x = 1^- asymptote and falls again from the x = 1^+ asymptote until it connects with the mark as x = $+\infty$, y = 0. Putting this all together we get Fig. 4.

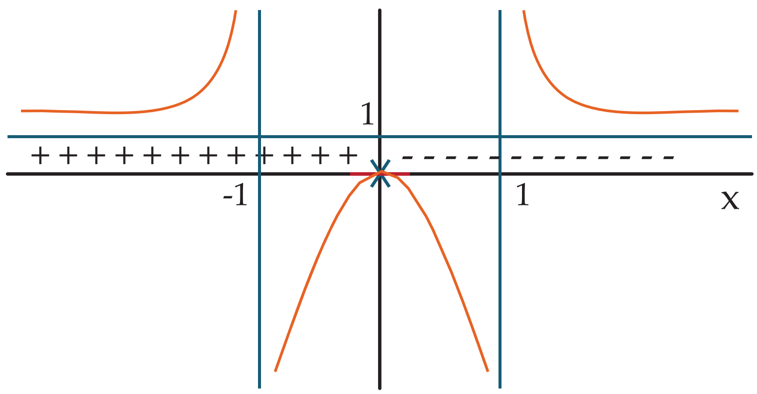

Fig. 4

Check: Notice that the graph of the even function, $f(x) = \frac{x^2}{x^2 - 1}$, is symmetric with respect to the y-axis.

Example 3

Sketch the graph of ,

$$y = xe^{-x} \quad \text{for} \quad x \geq 0 \qquad (5)$$

An easy point is y = 0 when x = 0. Place this point in Fig. 5.

Find the behavior of the function for large values of x: Referring back to Example 12 of Chapter 9, using L'Hopital's Rule, $\lim_{x \to \infty} xe^{-x} = 0$. Place a mark above the x-axis for large x in Fig. 5.

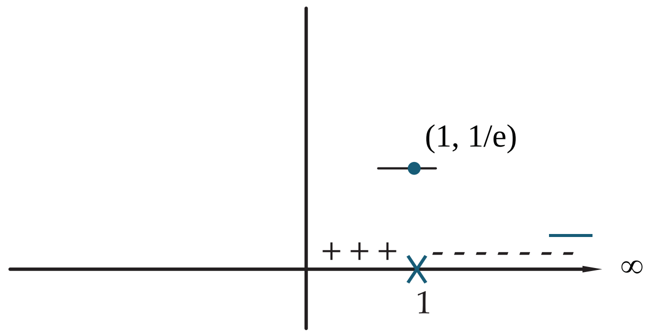

Fig. 5

Find the critical   Using the product rule,

$$\frac{dy}{dx} = e^{-x} - xe^{-x}$$

$$= e^{-x}(1-x) \qquad (6)$$

$$\frac{dy}{dx} = 0 \quad \text{when } x = 1 \text{ and } y = 1/e$$

Place a short horizontal line at (1, 1/e) to mark the critical point. And place a mark on the x-axis in Fig. 5 at x = 1.

The x-axis divides into the segments:

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From Eq. 6, $\frac{dy}{dx} < 0$ for $x > 1$ and $\frac{dy}{dx} > 0$ for $0 < x < 1$. Increasing and decreasing intervals of the function are indicated by + and – signs.

Locate the inflection points:

$$\frac{d^2y}{dx^2} = -e^{-x} - (e^{-x} - xe^{-x}) = 0$$

$$\frac{d^2y}{dx^2} = -e^{-x}(x - 2) = 0$$

Therefore $x = 2$ and $y = 2/e^2$ is the location of an inflection point. Since $\frac{d^2y}{dx^2} > 0$ to the left of $x = 2$ and $\frac{d^2y}{dx^2} < 0$ to the right of $x = 2$, the curve is concave down up to $x = 2$ and then concave up thereafter.

From all of this evidence, we are able to sketch the graph of Eq. 5 in Fig. 6.

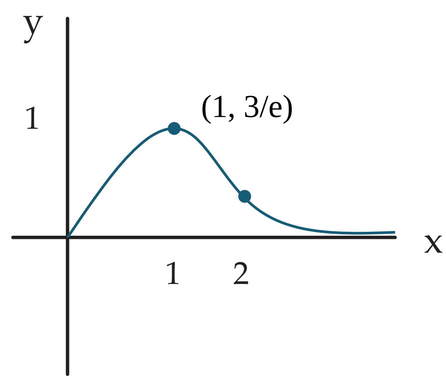


Fig. 6

Problem 1:

Sketch the graph of the following functions. Label critical points and determine if they are relative maxima or relative minima. Find all vertical and horizontal asymptotes.

a. $f(x) = x(x-1)^{2/3}$

b. $f(x) = \dfrac{x^2}{x^2+1}$

c. $f(x) = \dfrac{x^2}{x-1}$

14.3 ABSOLUTE MAXIMA AND MINIMA

If we wish to find the absolute maximum and minimum values of a function , f(x), over and closed interval [a,b] (including its endpoints) or an open interval (a,b) (not including its endpoints), then there is no need to draw a graph. We use the procedure introduced in Chapter 12. This is illustrated by the following examples.

Find the absolute maximum and minimum values of f(x) over the indicated intervals

Example 4

$f(x) = \dfrac{4}{x^2+1}$, [-1, 2]

Find critical points: According to Example 1 there is one critical point at (0, 4)

Evaluate the function at the endpoints:

y = 2 when x = -1 and y = 4/5 when x = 2

Make a table:

x	-1	0	2
f(x)	2	4	4/5

Therefore the absolute maximum is at x = 0 and the absolute minimum is at x = 2.

Example 5

$f(x) = xe^{-x}$, $[0, \infty)$

Find critical points: According to Example 3, there is one critical point at (1, 1/e)

Check endpoints:

y = 0 when x = 0 and $\lim_{x \to \infty} xe^{-x} = 0$. Using L' hospital's rule

Make a table

x	0	1	∞
f(x)	0	1/e	0

The absolute minimum is at x = 0 and ∞. The absolute maximum is at x = 1.

Example 6

$f(x) = x(x-1)^{2/3}$, $[0, 9]$

Find critical points:

$$f'(x) = (x-1)^{2/3} + 2/3x(x-1)^{-1/3}$$
$$= \frac{(x-1) + 2/3x}{(x-1)^{1/3}} = \frac{5/3x - 1}{(x-1)^{1/3}} = 0$$

f'(x) = 0 when x = 3/5

Note that the derivative is infinite when x = 1 however, the curve is continuous at that point, so the curve has a cusp at x = 1.

Check endpoints: f(0) = 0 and f(9) = 36

Make a table:

x	0	3/5	1	9
f(x)	0	$+(3/5)(2/5)^{2/3}$	0	36

The minimum value is at x = 0 and the max is at x = 9

Problems 2

Find the absolute maximum and minimum values of the following functions over the given intervals.

a) g(t) = te^{-t} , $t \geq 0$
b) $f(x) = x - \ln x$, x > 0
c) $f(t) = \dfrac{t}{1+t^2}$, for all t
d) $f(x) = e^{-x} \sin x$, $[0, 2\pi]$

14.4 WORD PROBLEMS

Example 7

Find a point on the parabola $y^2 = 2x$ that is closest to the point (1,4). What is the closest distance?

Entry phase: Sketch graph and label point as shown if Fig. 7

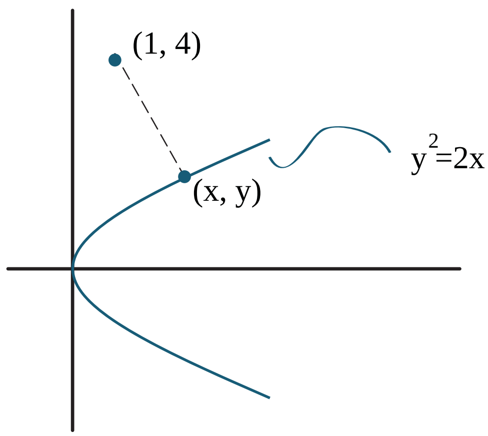


Fig. 7

Let D be the distance from (1,4) to (x,y) on the curve.

Find (x,y) such that D is a minimum. Compute D_{min}.

Attack phase:

$$D = \sqrt{(x-1)^2 + (y-4)^2} \qquad (7)$$

where $y^2 = 2x$ or $x = ½\ y^2$ (8)

Replacing x into Eq. 7,

$$D = \sqrt{(\frac{1}{2}y^2 - 1)^2 + (y-4)^2}\ ,$$ y is in the interval $(-\infty, \infty)$

D^2 and D both have minima at the same value of x so we shall compute the minimum of D^2 since the math is easier.. We will call $z = D^2$.

$$z = D^2 = (\frac{1}{2}y^2 - 1) + (Y - 4)^2$$

Find the critical point of D2:: Using the chain rule,

$$\frac{dz}{dx} = 2(\frac{1}{2}y^2 - 1)y + 2(y - 4) = y^3 - 8 = 0$$

The critical point is : y = 2

As y → ±∞ z → ∞ So there is no maximum. The minimum occurs at the critical point. Therefore , from Eq. 8 when y = 2, x = 2 and D = $\sqrt{5}$

Review phase: The answer appears reasonable by looking at the sketch in Fig. 7.

Example 8:

A wire 100 inches long is cut into two pieces and each piece is bent into a square. Where should the wire be cut so that the sum of the areas of the two squares is a maximum and a minimum?

Entry Phase:

The information of the problem is placed in the diagram in Fig. 8

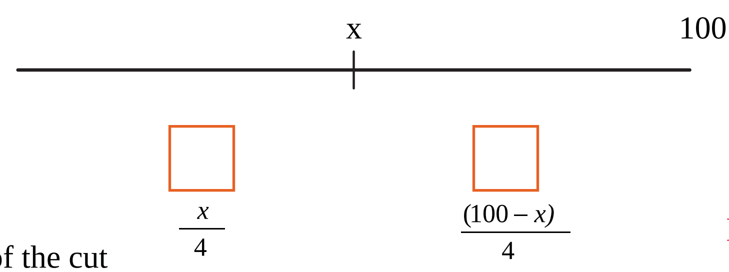


Fig.8

Let x be the position of the cut

Find x such that $A = A_1 + A_2$ is a min.

Attack Phase:

$$A = A_1 + A_2 = \frac{x^2}{16} + \frac{(100 - x)^2}{16}, \quad 0 \le x \le 100$$

Find the critical point: $\frac{dA}{dx} = \frac{2x}{16} - \frac{2(100-x)}{16} = 0$

After some algebra,

$$x = 50, \quad A = \frac{25^2}{2}.$$

Check the endpoints:

When x = 0 and x = 100, A = $(25)^2$

Make a table:

x	0	50	100
A	$(25)^2$	$\frac{(25)^2}{2}$	(25)

The Absolute Max occurs at x = 0 and x = 100. The absolute min occurs at x = 50.

Review: The answer makes sense because by symmetry both squares should have the same area.

Example 9a:

A bird is released from point A, an island which is 5 km, from the nearest point B. The bird flies along a straight shoreline to point C and then continues flying along the shoreline to its nesting area D. If it uses 1.4 times as much energy to fly over water than it does over land, what path will minimize t he expenditure of energy. Points B and D are 12 km apart

Entry Phase: Fig. 9 illustrates the given information

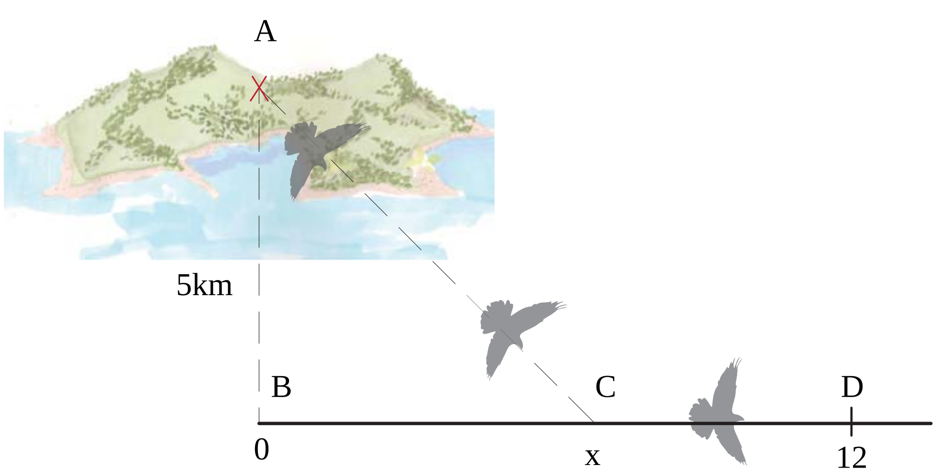


Fig.9

Find: x such that Energy E is a minimum.

Attack Phase

$$E = (1.4)\sqrt{5^2 + x^2} + (1.0)(12 - x), \qquad 0 \le x \le 12$$

Find the critical point: Using t he chain rule,

$$\frac{dE}{dx} = \frac{1.4x}{\sqrt{25 + x^2}} - 1 = 0$$

$$1.4x = \sqrt{25 + x^2}$$

After some algebra,

$$0.96\, x^2 = 25 \quad \text{or} \quad x = \frac{5}{\sqrt{0.96}} = 5.103$$

Check the endpoints:

When x = 0, E = (1.4)(5) + 12 = 19, when x = 12 , E = (1.4)(13) = 18.2

Make a table:

x	0	5.103	12
E	19	16.9	18.2

The least energy path occurs for x = 5.103 with and energy of 16.9 units

Problem 3

In Example 9a,

Let W be the energy per km over water
L be the energy per km over land

i) If the bird lands at x = 4, what is the ratio W/L?

ii) If the ratio W/L ≈ 0, i.e., W<<L, where does the bird land?

iii) If the ratio W/L is large ($\approx \infty$), i.e., L<<W, where does the bird land?

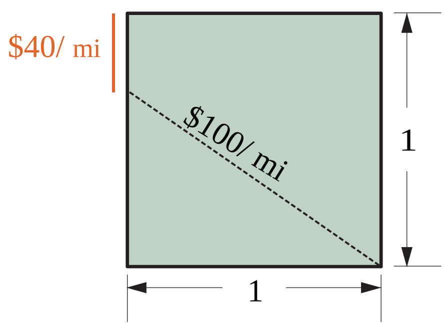
For this problem you can use the energy formula,

$$E = W\sqrt{5^2 + x^2} + L(12 - x)$$

where W and L are treated like constants.

Problem 4:

Fig. 10 shows a one-mile-square city park. A local power company needs to run a power line from the northwest corner A of the park to the southeast corner B. To preserve the beauty of the park, only underground lines may be run through the park, only overhead lines are permissible along the boundary of the park. The power company plans to construct an overhead line a distance x along the west edge of the park, then from the southern end of this line continue with a straight power lie to point B. If overhead lines cost $40 thousand per mile and underground lines cost $100 thousand per mile, how should the power company construct the line to minimize cost?



Problem 5

A 100 foot length of wire is cut at point x and the left part of the wire is bent into a square while the right length of wire is bent into a circle. Where should the cut be made so that the sum of the areas of square and circle is a minimum? A maximum?

Problem 6

A Norman window has the shape of a semicircle placed on top of a rectangle. If the perimeter, P, of the window is 30 ft.., find the dimensions x and y such that the window lets in the most light, i.e., the area A = Area of rectangle + area of semicircle is a maximum. See Fig. 11.

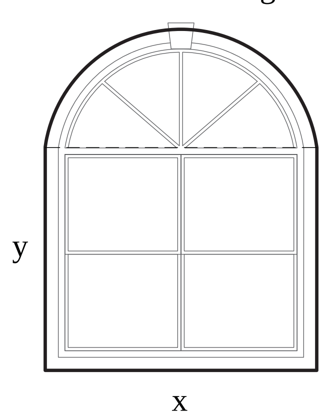


Fig.11

Problem 7

During a flu outbreak in a school of 763 children, the number of infected children , I, was expressed in terms of the number of susceptible (but still healthy) children, S, by the expression

$$I = 192 \ln\left(\frac{S}{762}\right) - S + 763$$

What is the maximum number of infected children?

Problem 8:

An apple tree produces, on average, 400 kg, of fruit each season. However, if more than 200 trees are planted per km^2, crowding reduces the yield by 1 kg. for each tree over 200.

a) Express the total yield from one square kilometer as a function of the number of trees on it. Graph this function.

b) How many trees should a farmer plant on each square kilometer to maximize yield?

Problem 9

For a parabola, $y = x^2$, find (x,y) on the parabola such that the square of the distance from (0,2) to (x,y) is a minimum. See Fig. 11.

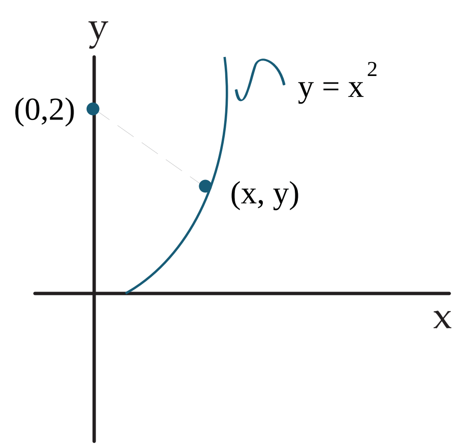


Fig. 12

Problem 10

For the function y = 1/x, find the point (x_0, y_0) on this curve such that the length of the tangent line AB to the curve at (x_0, y_0) is a minimum. See Fig. 13

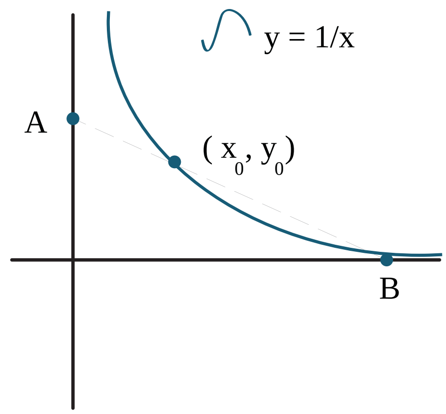


Fig. 13

Problem 11

For the rectangle with area 64 square units shown in Fig 14, line AB is drawn from vertex A to the midpoint of the opposite side. Find the values of x and y such that the length of AB is a minimum.

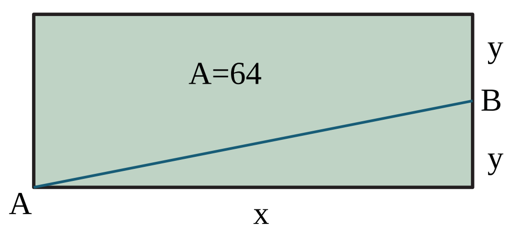


Fig. 14