CHAPTER 15

FINDING THE AREA UNDER A CURVE

15.1 FINDING THE AREA UNDER A CURVE

In large part the mathematical theory of architectural structures is based on two observations:

i) The derivative or slope of the bending moment curve is the shear force, i.e.,

$$\frac{d\overline{M}(x)}{dx} = \overline{V}(x) \tag{1}$$

ii) The signed area beneath the shear force curve between x_1 and x_2 is the difference between the moments, i.e.,

$$\overline{M}(x_2) - \overline{M}(x_1) = \int_{x_1}^{x_2} \overline{V}(x)dx \tag{2}$$

Example 1

When the beam in Chapter 8 is analyzed it gives rise to the bending moment,

$$\overline{M} = 50x - 5x^2, \qquad 0 \le x \le 10 \tag{3}$$

Once we know the bending moment we are able to find the shear force, i.e.,

$$\frac{d\overline{M}(x)}{dx} = \overline{V}(x)$$
$$\overline{V} = 50 - 10x, \quad 0 \le x \le 10$$

We can now use Eq. 2 to find the signed area under the $\overline{V}$ vs x curve between any two points without having to add rectangles. For example to find the signed area between x = 3 and x =8, using Eq. 3, $\overline{M}$ (3) = 105 and $\overline{M}$ (8) = 80, and from Eq. 2,

[Signed area between x = 3 and x = 8] = $\overline{M}(8) - \overline{M}(3) =$ 80 – 105 = -25

Check this area in Fig. 8a of Chapter 8.

In this example we had the luxury of being able to compute the area directly using areas of triangles or trapezoids. In the next example, this is no longer possible.

Example 2

For the beam subjected to a linearly increasing continuous force of $\rho = 6x$ in Section 8.4, .

$$\overline{M} = 144x - x^3, \qquad 0 \le x \le 12 \tag{4}$$

From $\overline{M}$ we are able to find $\overline{V}$,

$$\frac{d\overline{M}(x)}{dx} = \overline{V}(x)$$

$$\overline{V} = 144 - 3x^2 , \quad 0 \le x \le 12 \qquad (5)$$

Since the $\overline{V}$ vs x curve is a parabola we do not know a formula to compute the area. However, Eq. 2 comes to the rescue again. To find the signed area between x = 3 and x = 8, from Eq. 4, $\overline{M}(3)$ = 405 and $\overline{M}(8)$ = 640,

[Signed area between x = 3 and x = 8] = $\overline{M}(8) - \overline{M}(3) = 405 - 640 = -235$

15.2 FINDING THE AREAS UNDER THE GRAPH OF A GENERAL FUNCTION

If you know a function F(x), then you can find the signed area under the graph of the derivative of this function.

Equations 1 and 2 can be generalized to graphs of any function y = f(x).

$$\boxed{F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x)dx} \qquad (6)$$

where,

$$\boxed{\frac{dF(x)}{dx} = f(x)} \qquad (7)$$

In this way, the area under y = f(x) can be found without having to add rectangles. This is the great discovery of Isaac Newton, and a proof will be given in Chapter 17.

We will give several examples of the application of Eq. 6 and 7.

Example 3: Consider F(x) = x^2 where we have determined in Chapter 9 that

$$\frac{dx^2}{dx} = 2x$$

If we let F(x) = x^2 then from Eq. 7, y = f(x) = 2x. We draw this pair of F(x) and f(x) curves in Fig. 1.

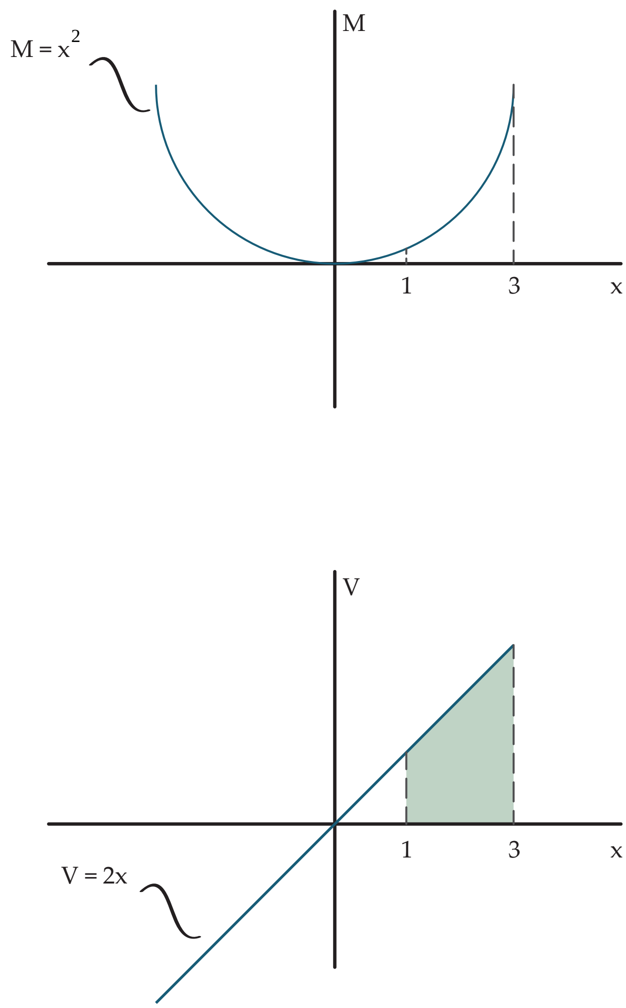


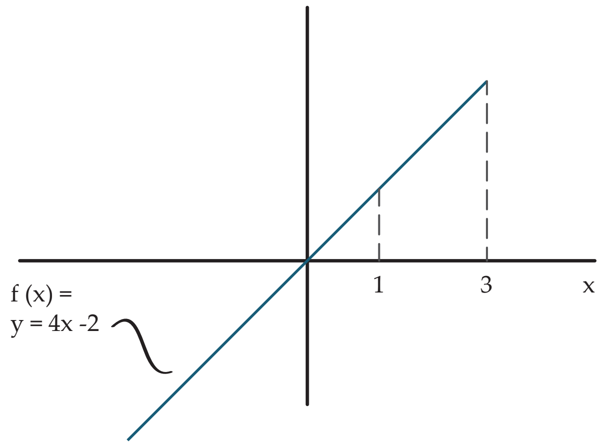
Fig. 1

We find that [signed area from x = 1 to x = 3] = $F(3) - F(1) = \int_1^3 2x dx = 3^2 - 1^2 = 8$ which agrees with the area calculated from geometry and from a previous homework problem from Chapter 3..

Example 4: For F(x) = 4x – 2,

$$\frac{d(4x-2)}{dx} = 4,$$

We can now find the area under the curve y = 4 from x = 1 to x = 3 shown in Fig. 2.



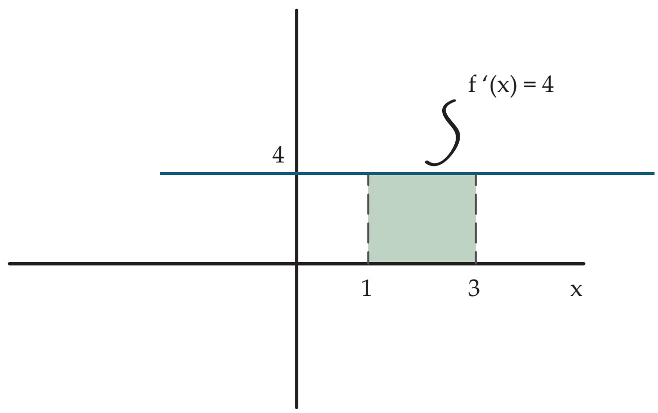


Fig. 2

$$\int_1^3 4dx = (4(3)-2)-(4(1)-2) = 8$$

Example 5: . Next consider the function $F(x) = x^2 - 2x$.

$$y = f(x) = \frac{dF(x)}{dx} = 2x - 2.$$

If we complete the square for F(x) we get y = F(x) = $(x\text{-}2)^2$ - 1 or y +1 = $(x\text{-}2)^2$ whose graph is shown in Fig. 3 along with the graph of y = f(x)..

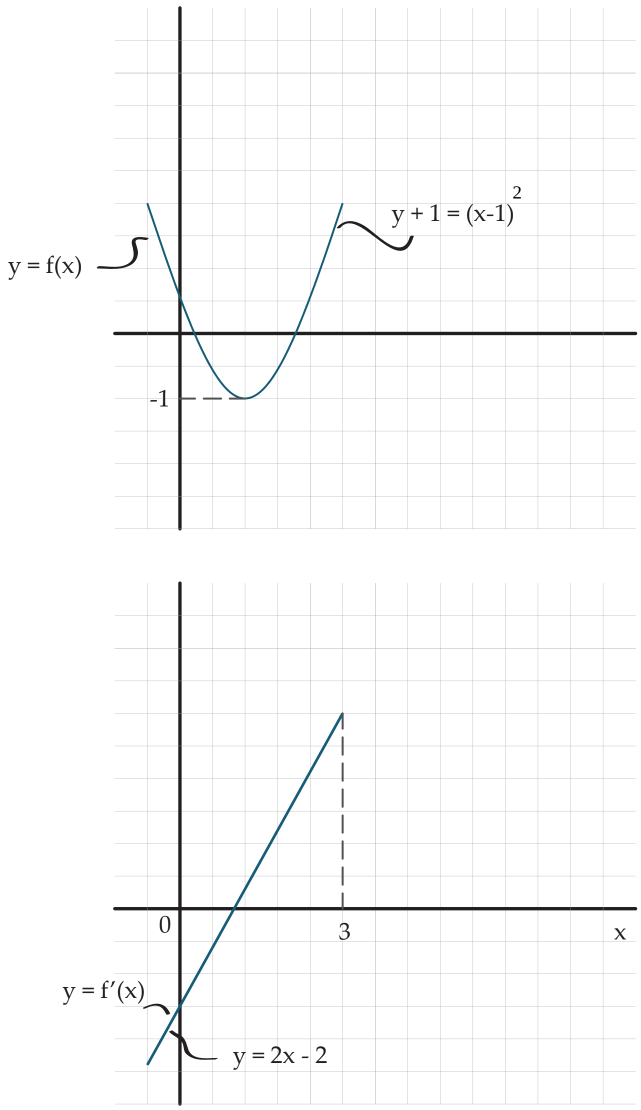


Fig. 3

[The signed area under y = 2x-2 from x=0 to x=3] = $\int_0^3 (2x-2)dx = (3^2 - 2(3)) - ((0)^2 - 2(0)) = 3$

Example 6: Consider Y = $F(x) = 2x^2 - \frac{4}{3}x^3$

Using the multiplication and addition rules,

$$y = f(x) = F'(x) = 4x - 4x^2 = 4x(1-x)$$

Graphing f(x) in Fig. 4,

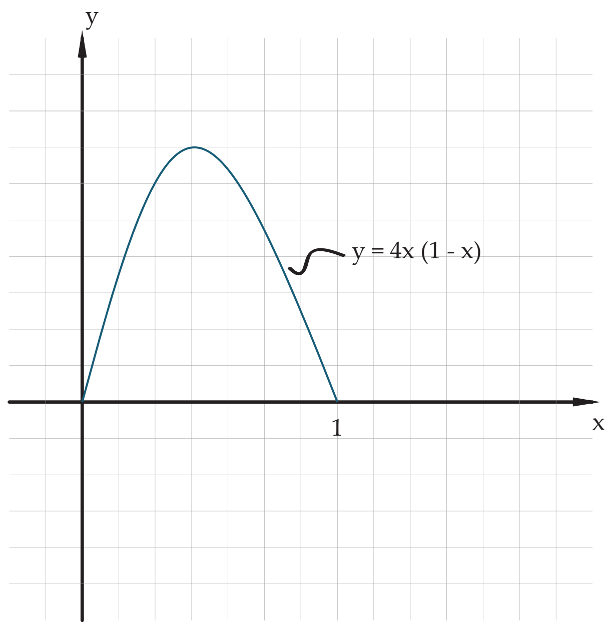


Fig. 4

We find that , $\int_{o}^{1} 4x(1-x)dx = F(1) - F(0) = (2(1)^2 - \frac{4}{3}(1)^3) - 0 = 2/3$
which agrees with a previous homework problem from Chapter 2.

Summary: We have demonstrated in general that if you know the derivative f (x) of a function F(x) then the signed area under the f(x) function between x_1 and x_2 is :

$$\text{Signed area} = F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x)dx \qquad (7)$$

Remark 1: Notice in the above graphs that where the derivative function is positive the function increases and where the derivative is negative, the function decreases.

Remark 2: Notice that where the derivative f (x) is zero the function F(x) has a horizontal tangent.