CHAPTER 16

FINDING AREAS II

16.1 FINDING THE AREA BY USING SHEAR FORCE

In Chapter 15, Part I we found the area under the graph of the $\overline{V}$ (x) function given the $\overline{M}$ (x) function. Now we specify the $\overline{V}$(x) directly and then find the signed area under its graph between x_1 and x_2. To accomplish this we use $\overline{V}$ to find $\overline{M}$ and then apply the ideas from Chapter 15. Since ,

$$\frac{d\overline{M}(x)}{dx} = \overline{V}(x)$$

then,

$$\overline{M}(x) = \int \overline{V}(x)dx + c$$

where,
$$\frac{d\overline{M}(x)}{dx} = \overline{V}(x) \qquad (1)$$

then,
$$\overline{M}(x) = \int \overline{V}(x)dx + c \qquad (2)$$

Applying the ideas from Chapter 15,

$$\overline{M}(x_2) - \overline{M}(x_1) = \int_{x_1}^{x_2} \overline{V}(x)dx = \text{[signed area from } x=x_1 \text{ to } x = x_2\text{]} \qquad (3a)$$

Let's apply this to finding the area between x = 3 and x = 8 for the continuous beam from Section 8.4 with linearly increasing density, $\rho = 6x$. From Chapter 8.4 (repeated in Eq. 5 of Chapter 15.1,

$$\overline{V} = 144 - 3x^2, \quad 0 \le x \le 12 \qquad (4)$$

Finding the antiderivative,

$$\overline{M}(x) = \int (144 - 3x^2)dx + c$$
$$= 144x - x^3 + c$$

Therefore,

$$\overline{M}(8) - \overline{M}(3) = \int_3^8 \overline{V}(x)dx$$
$$= (405 + c) - (640 + c)$$
$$= -235$$

Remark 1: Notice that the constant c cancelled. This will always happen when evaluating integrals.

Remark 2: This method of finding signed areas does not need a graph.

16.2 FIND THE AREA UNDER THE GRAPH OF y = f(x)

In general ,

$$F(x) = \int f(x)dx + c \quad (5)$$

where,

$$\frac{dF(x)}{dx} = f(x) \quad (6)$$

from which it follows that ,

$$\int_{x_1}^{x_2} f(x)dx = F(x)|_{x_1}^{x_2} = F(x_2) - F(x_1) \quad (7)$$

where, F(x) is any antiderivative of f(x).

Remark 1: Notice that the notation $F(x)|_{x_1}^{x_2}$ means take the value of F(x) at x = x_1 and subtract it from the value of F(x) at x = x_2, i.e., $F(x)|_{x_1}^{x_2} = F(x_2) - F(x_1)$.

Example 1:

Let f(x) = x^2 . If we wish to find the signed area between x = 0 and x = 1, i.e., $\int_0^1 x^2 dx$, then using Eq. 7 where $F(x) = \frac{x^3}{3}$ is any antiderivative of x^2,

$$\int_0^1 x^2 dx = \frac{x^3}{3}|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Remark 2: The constant of integration does not enter into the calculation since it cancels.

Example 2:

Now consider f(x) = $\frac{1}{x^2}$. Find the area between x = 1 and x = 2 (a previous homework problem approximated by rectangles in Chapter 3). An anti-derivative is,

$$F(x) = \int x^{-2}dx = \frac{x^{-1}}{(-1)} = -\frac{1}{x}$$

from which it follows from Eq. 7 that,

$$\int_1^2 x^{-2}dx = (-\frac{1}{x})|_1^2 = (-\frac{1}{2}) - (-\frac{1}{1}) = \frac{1}{2}$$

Example 3:

Find the signed area under the curve of f(x) = $x^2 - 2x$ between x = 2 and x = 3.
An antiderivative of f(x) is,

$$\int (x^2 - 2x)dx = \int x^2 dx - 2\int xdx = \frac{x^3}{3} - 2(\frac{x^2}{2}) = \frac{x^3}{3} - x^2$$

Therefore, $$\int_1^3 (x^2 - 2x)dx = (\frac{x^3}{3} - x^2)|_1^3 = (\frac{27}{3} - 9) - (\frac{1}{3} - 1) = \frac{26}{3} - 8 = \frac{2}{3}$$

Example 4:

Find the signed area of $f(x) = 4x - 4x^2 = 4x(1-x)$ between x = 0 and x = 1.

$$F(x) = \int 4x(1-x)dx = \int 4xdx - \int 4x^2 dx = 2x^2 - \frac{4}{3}x^3$$

Therefore, $$\int_0^1 4x(1-x)dx = (2x^2 - \frac{4x^3}{3})|_0^1 = (2 - \frac{4}{3}) - 0 = 2/3$$

This is the same result that we obtained in Example 2.4 of Chapter 15,

Example 5:

Find the signed area of f(x) = sin x between x= 0 and x = π.

$$F(x) = \int \sin xdx = -\cos x$$

Therefore,

$$\int_0^\pi \sin xdx = -\cos x|_0^\pi = -\cos\pi - \cos 0 = 2$$

This was a result that we found for the solution of the Buffon's Needle problem in Lab Exercises.

Example 6:

Find the area under the graph of $f(x) = \frac{4}{x^2+1}$ on the interval [0, 1].

We showed in Section 13.8 that

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$$

Therefore,

$$\int_0^{\pi} (\frac{4}{1+x^2})dx = 4\tan^{-1} x\,|_0^1 = 4(\tan^{-1} 1 - \tan^{-1} 0) = 4(\frac{\pi}{4} - 0) = \pi$$

The graph of $y = \frac{4}{1+x^2}$ over the interval [0,1] was shown in Example 1 of Section 14.2. The area under this curve over the interval [0,1] can be seen on this graph.