CHAPTER 17

JUSTIFICATION OF THE AREA AND SLOPE METHODS FOR EVALUATING BEAMS

17.1 THE SLOPE METHOD

The following theorem provides the justification for Method 3, the slope method.

Theorem 1: If $\overline{V}(x)$ is continuous at x and a continuous force of density $\rho(x)$ acts on the beam then $\frac{d\overline{M}(x)}{dx} = \overline{V}(x)$ or $\int \overline{V}(x)dx = \overline{M}(x) + c$

Proof: Consider the portion of the beam between an arbitrary value of $x = x_0$ and $x = x_0 + h$ for h small with shear forces and bending moments shown in Fig. 1.

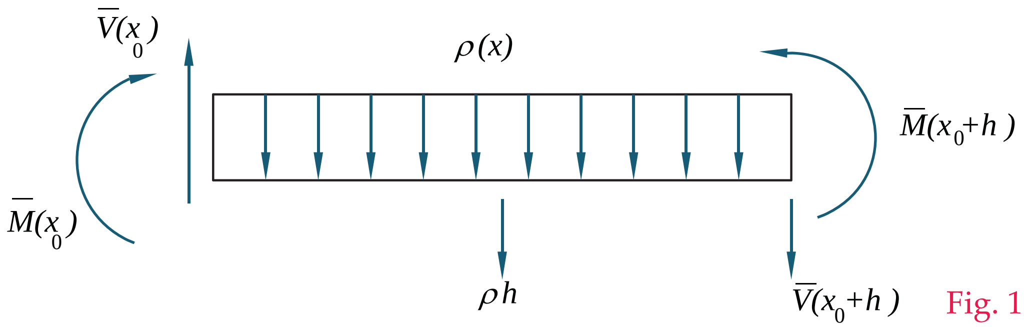

Fig. 1

The sum of the moments about x_0 is,

$$-\overline{M}(x_0) + \overline{M}(x_0 + h) - \overline{V}(x_0 + h)h + \rho(x^*)hk = 0 \qquad (1)$$

Note from the mean value theorem for Integrals in Section 3.7

The total force acting on the interval [x, x + h] equals $\int_{x_0}^{x_0+h} \rho(x)dx$. From the Mean Value Theorem for Integrals derived in Theorem 2 of Chapter 3, $\rho(x^*)h = \int_{x_0}^{x_0+h} \rho(x)dx$ where x* is a point on $[x_0,x_0+h]$ and k is the distance from the center of gravity of the force distribution to point x_0. The magnitude of k is clearly less than h.

Algebraically rearranging this equation we get,

$$\overline{M}(x_0 + h) = \overline{M}(x_0) + (\overline{V}(x_0 + h) - \overline{V}(x_0) + \overline{V}(x_0) - \rho(x^*)k)h \qquad (2)$$

But since, we have assumed $\overline{V}(x_0)$ is continuous at x_0,

$$(\overline{V}(x_0 + h) - \overline{V}(x_0))h = o(h) \quad \text{and} \quad \rho(x^*)hk = o(h) \quad \text{(why?)}$$

and Eq. 2 reduces after eliminating the o(h) term to,

$$\overline{M}(x_0 + h) = \overline{M}(x_0) + \overline{V}(x_0)h$$

It follows from the derivative machine that $\overline{V}(x_0) = \overline{M}'(x_0)$ and thus,

$$\frac{d\overline{M}(x)}{dx} = \overline{V}(x) \text{ for any value of x.}$$

17.2 THE AREA METHOD

The next theorem provides the justification for Method 2, the Area Method which we have renamed the Calculus Method.

Theorem 2: $\overline{M}(x_2) - \overline{M}(x_1)$ = [signed area under the V(x) curve from x_1 to x_2]

or $\overline{M}(x_2) - \overline{M}(x_1) = \int_{x_1}^{x_2} \overline{V}(x)dx$.

Proof: Consider the graph of $\overline{V}$ (x) from x_1 to x shown in Fig. 2. Let the signed area under the curve from x_1 to x be $A(x_1,x)$, another function of x.

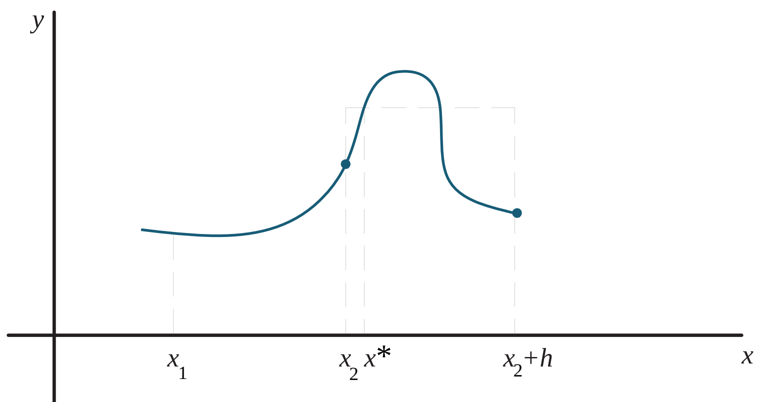

Fig. 2

Let us find the derivative of $A(x_1,x)$ at x_2 in the usual way,

$$A(x_1, x) = A(x_1, x_2) + mh + o(h) \qquad (3)$$

Let x = x_2 + h and rewrite Eq. 3 to get,

$$A(x_1, x_2 + h) - A(x_1, x_2) = mh + o(h)$$

or,

$$\frac{A(x_1, x_2 + h) - A(x_1, x_2)}{h} = m + \frac{o(h)}{h} \qquad (4)$$

But $A(x_1, x_2 + h) - A(x_1, x_2)$ is the area under the $\overline{V}$ (x) curve from x_2 to x_2 + h as shown in the detail of Fig. 2. Note that there is a value x* where $x_2 \le x^* \le x_2 + h$ such that the area between , x_2 and x_2 + h equals the area of the rectangle, $\overline{V}(x^*)h$, i.e.,

$$A(x_1, x_2 + h) - A(x_1, x_2) = \overline{V}(x^*)h$$

Replacing this in Eq. 4 and canceling h,

$$\overline{V}(x^*) = m + \frac{o(h)}{h} \qquad (5)$$

If we now let h → 0 in a limit sense, then x* → x_2 , and $\frac{o(h)}{h} \to 0$, and Eq. 5 becomes,

$$\overline{V}(x_2) = m = \frac{dA(x_1, x_2)}{dx}$$

Since x_2 is arbitrary,

$$\overline{V}(x) = \frac{dA(x_1, x)}{dx}$$

In other words, $\overline{V}$ (x_2) is the slope (derivative) of the signed area function, $A(x_1, x)$, at x = x_2. So we see from Theorem 1 that both $\frac{dA(x_1, x)}{dx}$ and $\frac{d\overline{M}(x)}{dx}$ equal $\overline{V}$ (x). But it can be shown that if two functions have the same derivative (i.e., slope) then they must be either equal or differ by only a constant. So we have found that,

$$\overline{M}(x) = A(x_1, x) + c \qquad (5)$$

Setting x = x_1 and noting from Fig. 1 that $A(x_1, x_1) = 0$, it follows from Eq. 5 that $\overline{M}$ (x_1) = 0 + c or c = $\overline{M}$ (x_1). Therefore,

$$\overline{M}(x) - \overline{M}(x_1) = A(x_1, x)$$

where A(x_1,x) is the signed area under the $\overline{V}$ (x) curve from x_1 to x. Letting x = x_2 we have the result of our theorem,

$$\overline{M}(x_2) - \overline{M}(x_1) = A(x_1, x_2) = \int_{x_1}^{x_2} \overline{V}(x)dx$$

17.3 THE RELATIONSHIP BETWEEN SHEAR FROCE AND DENSITY

Theorem 3: If a beam experiences a continuous force with density $\rho(x)$**, then** $\frac{d\overline{V}(x)}{dx} = -\rho(x)$.

Proof: Consider the portion of the beam from x to x + h as shown in Fig.1.

The balance of forces is:

$$\overline{V}(x+h)-\overline{V}(x)+\rho(x^*)h=0$$

where x* is some point in the interval (x, x+h) such that , again using the Mean Value Theorem for Integrals (Theorem 1 from Chapter 3) , $\rho(x^*)h$ is the weight of the beam (area under the $\rho(x)$ curve) on this interval. Such a point is guaranteed to exist by a more advanced theorem of calculus.

Therefore,

$$\overline{V}(x+h)=\overline{V}(x)-(\rho(x^*)-\rho(x))h-\rho(x)h=0$$

But since $(\rho(x^*)-\rho(x))h=o(h)$ why?

Therefore, $\overline{V}(x+h)=\overline{V}(x)+(-\rho(x))h=0$

From the derivative machine,

$$\frac{d\overline{V}(x)}{dx}=-\rho(x)$$

Let us now apply this to finding the bending moment of the beam under a continuous but constant load in Chapter 8 by the slope method, i.e., Method 3.

Since $\frac{d\overline{M}(x)}{dx}=\overline{V}(x)$ it follows that $\overline{M}(x)=\int\overline{V}(x)dx+c$

In Chapter 7 we found that $\overline{V}$ = 50 – 10x so that ,

$$\overline{M}(x)=\int(50-10x)dx+c=50x-10\frac{x^2}{2}+c=50x-5x^2+c$$

But $\overline{M}$ (0)=0 so that 0 = 0 + c or c = 0 and

$$\overline{M}(x) = -5x^2 + 50x$$

agreeing with our result.

We now have the startling result that we can find the area under the $\overline{V}$ (x) curve from x_2 to x_1 by merely evaluating the bending moment, $\overline{M}$ (x_2) – $\overline{M}$ (x_1) without needing to compute the area directly. In other words if $\overline{V}$ (x) is a continuous and smooth function,

$$\overline{M}(x_2) - \overline{M}(x_1) = \int_{x_1}^{x_2} \overline{V}(x)dx$$

where $\overline{M}$ (x) is any anti-derivative of $\overline{V}$ (x) . Since the constants for the slope method c will be the same for both $\overline{M}$ (x_2) and $\overline{M}$ (x_1), they will cancel out.

This will be very helpful in applying the Calculus Method to solving structures problems and for finding areas under curves without having to add up rectangles or trapezoids. In the next Chapter we use these ideas to find areas under curves without having to add rectangles.