CHAPTER 18

EVALUATION OF A FREE STANDING BEAM BY USE OF INTEGRALS

18.1 INTRODUCTION

Now that we are able to use the differential calculus to find the slope of a tangent line to the graph of a function, and we can use the integral calculus to compute the signed area under the graph of a function without having to add rectangles or trapezoids, we will be able to solve the beam problems by calculus.

18.2 ADDITIONAL PROPERTIES OF THE INTERGAL

a) If you reverse the limits in an integral its sign changes, i.e.,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{1}$$

b) If you consider the graph of the continuous function f(x) on the interval [a,b] and place another point c in the interval so that $a \le c \le b$ as in Fig. 1, then it is clear that,

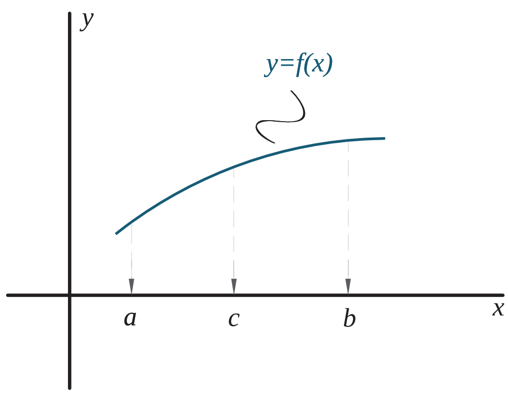

Fig. 1

SA[a,b] = SA[a,c] + SA[c,b]

where SA means signed area, or

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{2}$$

If F(x) is an anti-derivative of f(x) then Eq. 2b can be rewritten as,

$$\int_a^b f(x)dx = (F(c) - F(a)) + (F(b) - F(c)) \tag{3}$$

or,

$$\int_a^b f(x)dx = F(b) - F(a) \tag{4}$$

Now let us consider a discontinuous function shown in Fig. 2,

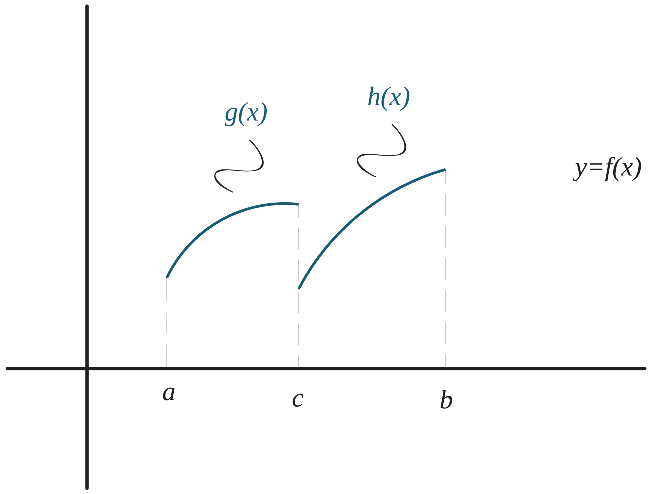

Fig. 2

Where,

$$f(x) = g(x) \quad \text{for } a \le x \le c$$
$$\quad = h(x) \quad \text{for } c < x \le b$$

with antiderivative,

$$F(x) = G(x) \quad \text{for } a \le x \le c$$
$$\qquad H(x) \quad \text{for } c < x \le b$$

For this function, Eq. 2a continues to hold but Eq. 2b is now written,

$$\int_a^b f(x)dx = \int_a^{c^-} g(x)dx + \int_{c^+}^b h(x)dx$$

and Eq. 3 becomes,

$$\int_a^b f(x)dx = (G(c^-) - G(a)) + (H(b) - H(c^+)) \qquad (5)$$

You will notice that I have used the notation c^+ and c^- in these formulas. These are needed because the value of f(x) and F(x) jump at c. The notation $f(c^-)$ means that we take the value of f(x) just to the left of x = c and $f(c^+)$ is the value just to the right of x = c. To be exact,

$$f(c^-) = \lim_{x \to c^-} f(x) \quad \text{and} \quad f(c^+) = \lim_{x \to c^+} f(x)$$

But remember that a constant can be added to the anti-derivative of a function to give another anti-derivative of the same function. This means that we can always find a constant to add to G and H so that F(x) will be continuous. When F(x) is continuous it follows that,

$$G(c^-) = H(c^+) \text{ , } \quad F(a) = G(a) \text{ , and } \quad F(b) = H(b)$$

so we have proven the following theorem:

Theorem: If F(x) is a continuous antiderivative of the continuous or discontinuous function f(x), then,

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (6)$$

In the next section we will show how this theorem is the key to using integrals to solve for the bending moment of a beam. After all, when there is a concentrated load, the shear force is a

discontinuous function. Since $\frac{d\overline{M}(x)}{dx} = \overline{V}(x)$, $\overline{M}$ (x) is the anti-derivative of $\overline{V}$ (x). We shall assume that $\overline{M}$ (x) is a continuous function so that it follows from Eq. 6 that,

$$\int_{x_1}^{x_2} \overline{V}(x)dx = \overline{M}(x_2) - \overline{M}(x_1) \qquad (7)$$

18.3 SOLUTION OF THE BEAM PROBLEM USING INTEGRALS

We have learned how to find the slope of the tangent line to the graph of any function by using derivatives, and how to find the signed area under the graph of any function using integrals. We are now able to apply these ideas to finding the bending moments acting on a beam once we have determined the shear forces. The shear forces are simply gotten by considering a balance of forces acting on a portion of the beam. The bending moments can then be computed by either Method 2, the AreaMethod, or by Method 3, the Slope Method.

Example 1:

We apply Method 2 and 3 to the beam problem with a linear force density $\rho = 6x$ described in Section 8.4. This problem was discussed again in Chapter 16. The beam is shown again in Fig. 3

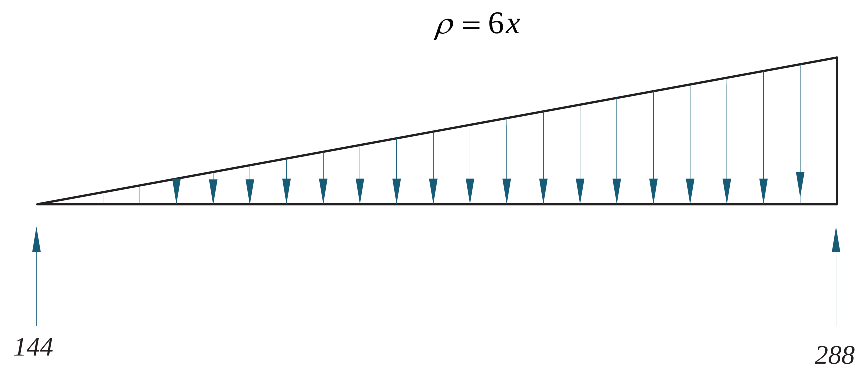

Fig. 3

The shear force for this beam was determined to be,

$$\overline{V} = 144 - 3x^2, \quad 0 \le x \le 12$$

a) Let us find the bending moment $\overline{M}$ by Method 3, the Slope Method where the slope of the bending moment is the shear stress, i.e,

$$\frac{d\overline{M}(x)}{dx} = 144 - 3x^2$$

Taking antiderivatives,

$$\overline{M} = \int (144 - 3x^2)dx$$
$$= 144x - x^3 + c$$

Since the beam is free standing, M = 0 when x = 0. Solving for c,

$$0 = 0 - 0 + c \quad \text{or } c = 0.$$

Therefore, $\overline{M}(x) = 144x - x^3, \quad 0 \le x \le 12$

b) Now let's do this problem by Method 2, the Area Method where the signed area under the graph of $\overline{V}$ vs x equals the bending moment for a free standing beam. Consider the portion of the beam on the interval [0,x]. The signed area under this portion of the graph of $\overline{V}$ vs x is,

$$\overline{M}(x) - \overline{M}(0) = \int_0^x \overline{V} dx = \int_0^x (144 - 3x^2) dx$$

$$= \int_0^x (144 - 3x^2) dx$$

$$= (144x - x^3) |_0^x$$

$$= 144x - x^3$$

But for a free standing beam, $\overline{M}(0) = 0$ resulting in,

$$\overline{M}(x) = 144x - x^3$$

Calculus can be applied to finding the bending moments of any beam however complex is its force distribution as we illustrate for the following two examples. In what follows we choose to use Method 2, the Area Method which I now refer to as the *Calculus Method.*

Example 2.

Consider the beam in Fig.4. This problem was solved in Section 8.2.

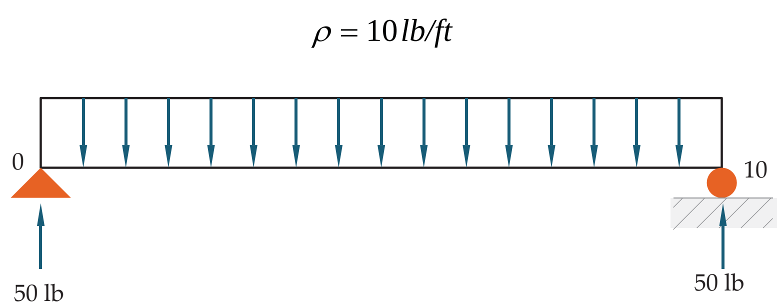

Fig. 10a

The shear force was computed as,

$$\overline{V} = 50 - 10x, \quad 0 \le x \le 10$$

To find the bending moment by the Calculus Method,:

$$\overline{M}(x) - \overline{M}(0) = \int_0^x \overline{V}(x) dx = \int_0^x (50 - 10x) dx = (50x - \frac{10x^2}{2}) |_0^x = 50x - 5x^2$$

But since $\overline{M}$ (0) = 0,

$$\overline{M}(x) = 50x - 5x^2 \ , \quad 0 \le x \le 10$$

This is same result that we obtained in Chapter 8.

Example 3: Consider the beam in Fig. 5

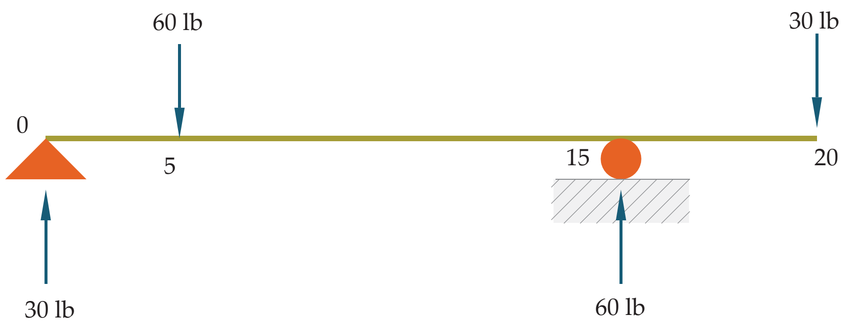

Fig. 5

The value of $\overline{V}$ (x) for the beam is easily shown to be,

$$\begin{aligned} \overline{V}(x) &= 30, \quad 0 < x < 5 \\ &= -30 \quad 5 < x < 15 \\ &= 30 \quad 15 < x < 20 \end{aligned}$$

The beam divides naturally into three regions as shown in Fig. 2.

0_________5_________________15_________20______

To compute $\overline{M}$ (x) we shall assume that the $\overline{M}$ (x) curve is continuous and use the equation:

Choose x in the interval [0,5), then ,

$$\overline{M}(x) - \overline{M}(0) = \int_0^x \overline{V}(x)dx = \int^x 30dx = 30x \,|_0^x = 30x - 30(0) = 30x$$

or $\overline{M}$ (x) = 30x + $\overline{M}$ (0). But since $\overline{M}$ (0) = 0,

$$\overline{M}(x) = 30x \quad \text{and} \quad \overline{M}(5) = 150$$

For the interval [5,15]:

Choose x in the interval (5,15)

$$\overline{M}(x)-\overline{M}(0)=\int_0^x \overline{V}(x)dx=\int_0^5 \overline{V}(x)dx+\int_5^x \overline{V}(x)dx$$

But $\int_0^5 \overline{V}(x)dx=\overline{M}(5)-\overline{M}(0)=150-0=150$ and $\int_5^x \overline{V}(x)dx=\int_5^x -30dx=-30x\,|_5^x=-30x+150$

Therefore, $\overline{M}$ (x) = 300-30x + $\overline{M}$ (0) = 300 – 30x (since $\overline{M}$ (0)=0) and $\overline{M}$ (15) = -150

Next, choose x in the interval (15,20]

$$\overline{M}(x)-\overline{M}(0)=\int_0^x \overline{V}(x)dx=\int_0^{15} \overline{V}(x)dx+\int_{15}^x \overline{V}(x)dx$$

But $\int_0^{15} \overline{V}(x)dx=\overline{M}(15)-\overline{M}(0)=-150=-150$ and $\int_{15}^x \overline{V}(x)dx=\int_{15}^x 30dx=30x\,|_{15}^x=30x-450$

Therefore, $\overline{M}$ (x) = 30x -600

As a check on the computation, $\overline{M}$ (20)=0, as it should be for a free standing beam

Summary: $\overline{M}$ (x) = 30x , $0\le x\le 5$
= 300 – 30x, $5\le x\le 15$
= -600 + 30x, $15\le x\le 20$

18.4 GENERAL PROCEDURE FOR FINDING THE BENDING MOMENT FOR A FREE STANDING BEAM BY THE AREA METHOD.

To solve for the bending moment due to any loading of a free standing beam by the area method, follow this procedure:

1. Divide the beam into N intervals $0\le x_1\le x_2\le ...\le x_k\le x_{k+1}\le ...\le x_{N-1}\le x_N$ depending on its loading.
2. Express the shear force $\overline{V}$ (x) in each distinct segment or interval, $[x_k, x_{k+1}]$ of the beam using the balance of forces.
3. Let x be an arbitrary point in the interval $[0,x_1]$. Evaluate, $\overline{M}(x)-\overline{M}(0)=\int_0^x \overline{V}(x)dx$
4. Since $\overline{M}$ (0)=0, $\overline{M}(x)=\int_0^x \overline{V}(x)dx$ where $\int_0^x \overline{V}(x)dx$ is evaluated from the known value of $\overline{V}$ (x) in interval $[0,x_1]$.

1. The value of $\overline{M}$ (x_2) is computed from step 3 by letting x = x_2.

2. Evaluate, $\overline{M}(x)-\overline{M}(0)=\int_0^x \overline{V}(x)dx=\int_0^{x_k}\overline{V}dx+\int_{x_k}^x \overline{V}dx$ where x is an arbitrary point in the k-th interval [x_k, x_{k+1}] .

3. $\int_0^{x_k}\overline{V}(x)dx=\overline{M}(x_k)-\overline{M}(0)=\overline{M}(x_k)$ where $\int_{x_k}^x \overline{V}(x)dx$ is evaluated from the known value of $\overline{V}$ (x) in interval [x_k,x_{k+1}] and $\overline{M}(x_k)$ is computed from the previous step.

4. If the beam is free standing then $\overline{M}$ (0) = 0 and the result of step 7 is replaced in step 6 to give $\overline{M}$ (x) = $\overline{M}$ (x_k) + $\int_{x_k}^x V(x)dx$ for x on the interval [x_k,x_{k+1}]. .

5. The value of $\overline{M}$ (x_{k+1}) is computed from step 5 by letting x = x_{k+1}.

6. Continue this procedure until x_N is reached.

7. Check to see that $\overline{M}$ (x_N) = 0

Remark 1: This method does not need a diagram

Remark 2: : We assume that $\overline{M}$ (x) is continuous so that the value of $\overline{M}$ at the end of an interval equals the value of $\overline{M}$ at the beginning of the next interval.

Problem:

Use this calculus method to find $\overline{M}$ solve the beam problems 1 and 2 of Chapter 7 and problems 1 and 2 of Chapter 8.