CHAPTER 19

DEFLECTION OF BEAMS

19.1 THREE EXPERIMENTS

Let's do three experiments.

a) Take a sheet of paper and stretch it between two desks (see Fig. 1). The paper can hold only a few ounces of weight before it deflects and falls to the floor. By altering the geometry of the paper how large a weight can you maintain before collapse? Use books for weights as measured by a scale.

b) A ball is attached to a string of varying lengths (see Fig. 2) . After giving the ball an impulse the ball moves in a circle, how does the speed of revolution depend on the length of the string given identical impulses? Then take a light object such as a paper clip and give it an impulse by flicking your finger and watch the clip accelerate. Next do the same for a heavy object like a moderate sized stone; it barely moves when you flick it. How do you explain the linear motion of the paper clip or stone and the circular motion of the ball that results from external forces acting on them.

c) I have been using a plastic ruler lying flat to simulate a beam (see Fig. 3). Place a weight on the ruler and see it deflect. Now place the ruler on its edge and notice how imperceptible is the deflection. How do you explain this striking difference?

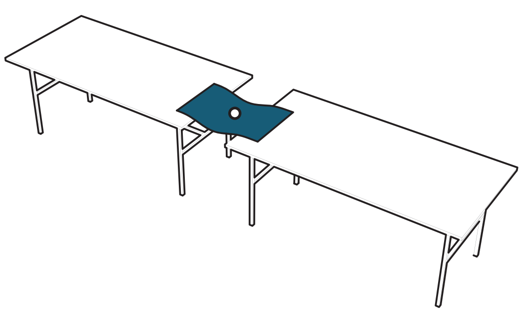

Fig. 1

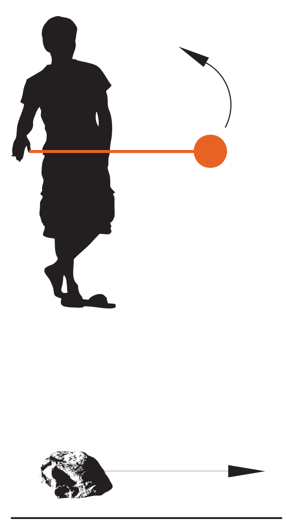

Fig. 2

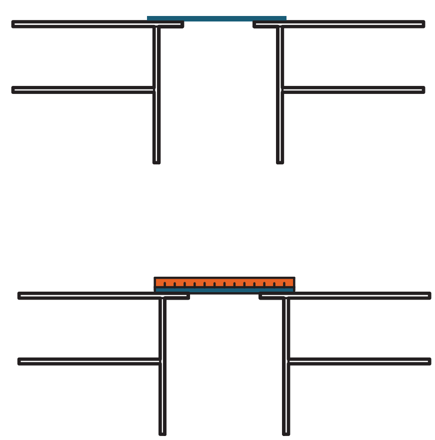

Fig. 3

19.2 A FORMULA FOR THE DEFLECTION OF BEAMS

When a load is placed on a beam the beam responds by deflecting. Depending on the material of the beam, the deflection will be large or small as determined by a parameter known as *Young's Modulus*, E. Certainly a steel beam deflects less that a wooden beam for a given set of external forces. Table 1 lists values of E for various materials. It is not surprising that the degree of deflection depends on the bending moment, $\overline{M}$, which, we have seen is also the result of the deflection of the beam. Finally the degree of deflection will depend on the geometry of the beam as measured by a quantity called the *moment of inertia, I.* Moment of inertia will be discussed in the next section.

Materials	Modulus of Elasticity (E) lbf/in^2 (psi)	Notes
Steel	29,000,000	
Wood	1,300,000 to 1,900,000	Woods will varry depending on the type of wood, because some woods are harder then others, can also depend on the size also, type of beam...ect
Concrete	3,122,000 (for 3,000 psi concrete)	NOTE: The vaule of the modulus of elasticity of concrete (Ec) is not constant, but varies with the 28-days compressive strenght (f'c) and the unit weight of the concrete. for the normal weight (about 145 lbs/cf), the value of Ec is approximatley $\sqrt{57,000}$ (f'c). when the f'c is 3,000 psi, the computed value is 3,122000 psi

Table 1

A measure of the deflection of the beam is its curvature. We have already seen that the second derivative of a function determines if the curve is concave up or down. In fact, it can be shown that for small deflections the curvature of the beam is the magnitude of the 2nd derivative where curvature is defined locally in the vicinity of a point as the inverse of the radius of the circle that best fits the curve at that point. By this definition, a straight line can be fit by an infinitely large circle so that the line has zero curvature, while the curvature of the circle is just the inverse of its radius. It can be shown that the curvature is directly proportional to the bending moment, $\overline{M}$, and inversely proportion to the product of E and I, i.e.,

$$\boxed{\frac{d^2y}{dx^2} = \frac{\overline{M}}{EI}} \qquad (1)$$

From this formula, we are able to compute the deflection y. We will demonstrate this in the Section 19.4.

19.3 MOMENT OF INERTIA

In each of the experiments in Section 17.1 the trick to reducing the deflection of the paper or ruler is to rearrange its geometry so that the mass of the paper or ruler is more remote from the neutral axis.

The mass m of the object being propelled in a straight line governs the acceleration a that is the result of an external force $\overline{F}$. The mass is said to be a measure of the inertia and the product of mass an acceleration is said to be an inertial force, i.e., F = ma or $a = \frac{F}{m}$.

Therefore the acceleration is inversely proportional to its mass For the ball attached to the string, the longer the string the smaller will be the angular acceleration of rotation for a give impulse. Newton's law says,

$$\boxed{F = ma} \qquad (2)$$

But acceleration a is the instantaneous rate of change of velocity, i.e,

$$a = dv/dt \qquad (3)$$

where velocity is the rate of change of position, i.e.,

$$v = ds/dt \qquad (4)$$

where s is the arc length of the ball in its circular trajectory. But the arc length of a segment of a circle is, $s = r\theta$ where r is the radius of the string and θ is the angle of the ball in its trajectory so that ,

$$v_c = r\frac{d\theta}{dt}$$

where v_c is the circular velocity, r is a constant, and $\frac{d\theta}{dt} = \omega$ is the speed of the ball in its circular path or circular velocity,

$$v_c = r\omega \qquad (5)$$

Replacing Eq. 5 into Eq 3 and 2,

$$mr\frac{d\omega}{dt} = F$$

where $\frac{d\omega}{dt}$ is called the angular acceleration.

A measure of the tendency of a particle (the ball) to rotate about an axis at the end of the string at O is given by the moment about O. If we assume that the force is acting perpendicular to the string, the moment on the ball about the end of the string is Fr also referred to as the torque T on the ball so that replacing this in Eq. 2,

$$T = mr^2\frac{d\omega}{dt} \quad \text{or} \quad T = I\frac{d\omega}{dt}$$

where $I = mr^2$ is the moment of inertia or the tendency to resist the angular acceleration of the particle (ball).

Notice that the moment of inertia depends on the square of the radius. For a given torque, the angular acceleration will be diminished as the string length r increases. Likewise, the more remote are particles of the paper or the ruler in experiments a) and c) from the neutral axis, the less will be its tendency to rotate or bend. That is why by folding the paper, or reorienting the ruler some of the paper's or ruler's mass is relocated away from the neutral axis decreasing its tendency to bend . This is also why I-beams as shown in Fig. 4 are used for construction, since much of the mass of the beam is located at the ends increasing its moment of inertia. Moment of inertia also explains why a dancers draw their arms towards their bodies when they wish to spin faster as shown in Fig 5.

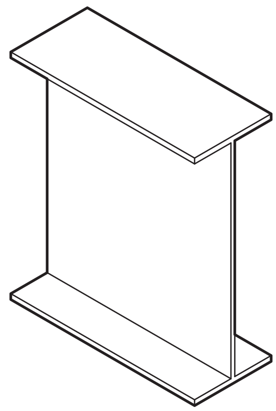

Fig. 4

Fig. 5

19.4 AN EXAMPLE OF THE DEFLECTION OF A BEAM

We reconsider once again t he beam from Chapter 8. shown below.

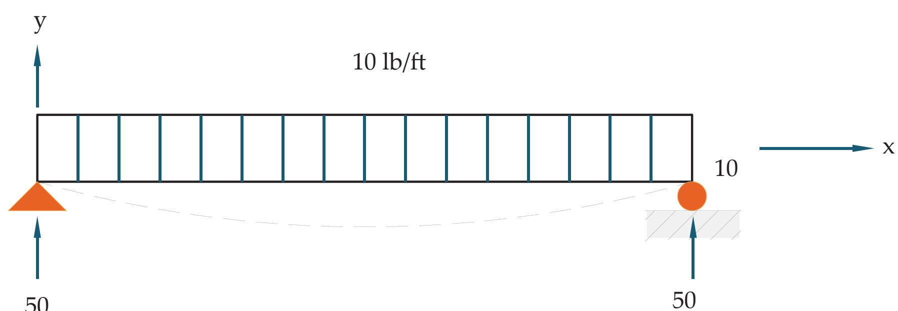

Fig. 6

The shear forces and bending moments are,

$$\overline{V} = 50 - 10x, \quad 0 \le x \le 10$$
$$\overline{M}(x) = 50x - 5x^2 \ , \quad 0 \le x \le 10$$

Let y be the deflection as measured from a coordinate system in which the beam lies along the x-axis. The deflection y will be negative for the beam in Fig. 6,. From Eq. 1,

$$\frac{d^2y}{dx^2} = \frac{1}{EI}(50x - 5x^2) \ , \quad 0 \le x \le 10$$

where y = 0 at the endpoints of the beam, i.e., y = 0 when x= 0 and x = 10.

Since $\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx}$, it follows by taking antiderivatives that ,

$$\frac{dy}{dx} = \frac{1}{EI}\int (50x - 5x^2)dx,$$

or, $$\frac{dy}{dx} = \frac{1}{EI}(25x^2 - \frac{5}{3}x^3 + c_1) \qquad (6)$$

Again, taking antiderivatives,

$$y = \frac{1}{EI}\int (25x^2 - \frac{5}{3}x^3 + c_1)dx$$

$$= \frac{1}{EI}(\frac{25}{3}x^3 - \frac{5}{12}x^4 + c_1 x + c_2) \qquad (7)$$

The boundary condition states that y = 0 when x = 0.. Therefore, from Eq. 2,

0 = 0 – 0 + 0 + c_2 or c_2 = 0.

When x = 10, y = 0. Therefore,

$$0 = \frac{25000}{3} - \frac{50000}{12} + 10c_1 \quad \text{or,}$$

$$c_1 = -\frac{2500}{6} \qquad (8)$$

Replacing Eq. 8 in Eqs. 6 and 7,

$$y = \frac{1}{EI}(-\frac{5}{12}x^4 + \frac{25}{3}x^3 - \frac{2500}{6}x_2)$$

$$\frac{dy}{dx} = \frac{1}{EI}(25x^2 - \frac{5}{3}x^3 - \frac{2500}{6})$$

The minimum value of y occurs in the middle of the beam at x = 5, or,

$$y_{\min} = \frac{1}{EI}(-1302.1)\,ft.$$

Remark: Referring to Section 9. the slope of the beam, $\frac{dy}{dx} \approx \theta$, where θ is the angle that the tangent line makes with the horizontal as shown in Fig. 5.
For example**,** $\theta = 0$ at the midpoint of the beam, and the slope of the beam at its left end , x = 0, is $\theta = -\frac{2500}{6EI}$.