CHAPTER 21

CENTER OF GRAVITY AND CENTROIDS

21.1 INTRODUCTION

In Section 8.6 we showed how to determine the balance point, or centroid, for a general two dimensional region. If certain of these regions are considered to be the form of a density distribution over a horizontal beam, the x-coordinate of the balance point is the center of gravity of the distribution. In this chapter we will show how to use integrals to compute the center of gravity of a general density distribution over a horizontal beam. We will then show how to compute the coordinates of the centroid for certain complex planar regions.

21.2 CENTER OF GRAVITY FOR DISCRETELY PLACED WEIGHTS ON A BEAM

We have been using a beam balance to represent the balance of moments. The balance in Fig. 1 is at equilibrium with the center of the beam as the balance point or center of gravity. The sum of the moments about the center of the balance is zero, i.e.,

$$(2)(7) + (3)(4) + (4)(1) - (1)(2) - (4)(5) - (1)(8) = 0$$

Let us now consider a general beam, shown in Fig. 2, with four weights: w_1, w_2, w_3, w_4, acting at four positions: x_1, x_2, x_3, x_4. Where is the location of the center of gravity x_c?

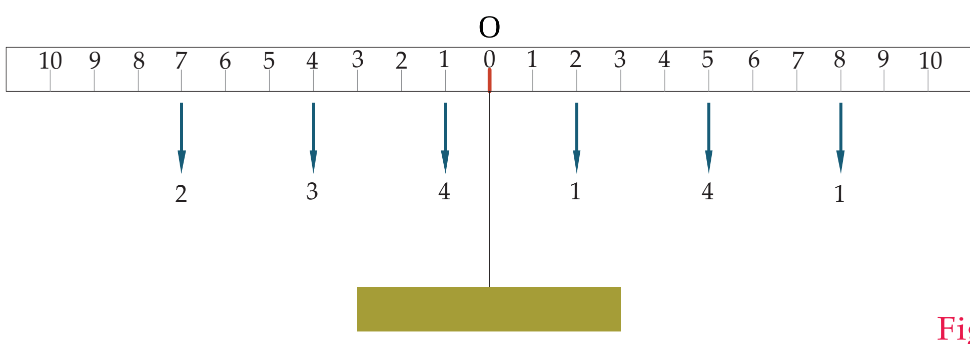


Fig. 1

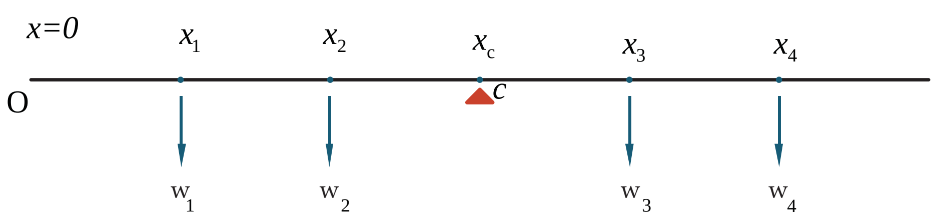


Fig. 2

Assume that the center of gravity is located at point C. By the definition of center of gravity, the sum of the moments about the center of gravity located at point C is zero. Therefore,

$$w_1(x_c - x_1) + w_2(x_c - x_2) - w_3(x_3 - x_c) - w_4(x_4 - x_c) = 0 \qquad (1)$$

Rearranging Eq.1,

$$(w_1 + w_2 + w_3 + w_4)x_c = w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 \qquad (2)$$

If we let M be the sum of the weights,

$$M = (w_1 + w_2 + w_3 + w_4)$$

then we can solve Eq. 2 for x_c,

$$x_c = \frac{w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4}{M} \qquad (3)$$

Let us check to see if this equation predicts the location of the center of gravity of the beam in Fig. 1. The twenty equidistant positions on the beam can be renumbered: $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, ..., x_{19} = 19, x_{20} = 20$. Weights are located as follows:

x = 3, w = 2; x = 6, w = 3; x = 9, w = 4; x = 12, w = 1; x = 15, w = 4; x = 18, w = 1

where, M = 15 and the center of gravity is located at x = 10. Let's see if this agrees with Eq. 3,

$$x_c = \frac{6+18+36+12+60+18}{15} = 10$$

Check.

We can also rewrite Eq. 2 as,

$$Mx_c - w_1x_1 - w_2x_2 - w_3x_3 - w_4x_4 = 0 \qquad (4)$$

Eq. 4 states that if moments are taken about the left end of the beam, O, all the weight of the beam can be thought of as acting at the center of gravity so that a counterclockwise moment at the center of gravity just balances the clockwise moments about O.

21.3 CENTER OF GRAVITY FOR CONTINUOUSLY DISTRIBUTED WEIGHTS ON A BEAM

Consider a continuous load with density $\rho(x)$ on a beam of length L as shown in Fig. 3. Divide L into four equal parts where $\Delta x = \frac{L}{4}$ and take the left endpoint of each interval to evaluate $\rho(x)$. If we assume that density is constant within each interval we get weights w_i where $w_i = \rho(x_i)\Delta x$ acting at point x_i so that the total mass M is given approximately by the Riemann sum $A_L^{(4)}$,

$$M = \rho(x_1)\Delta x + \rho(x_2)\Delta x + \rho(x_3)\Delta x + \rho(x_4)\Delta x$$

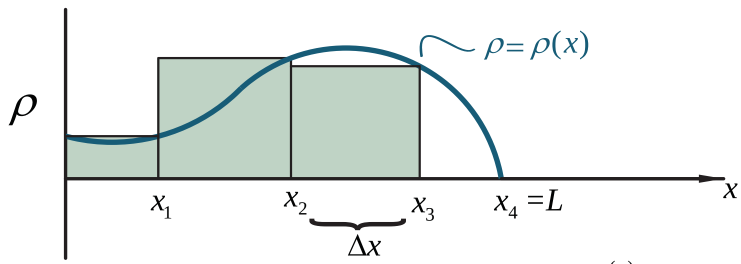


Fig. 3

As we divide L into more and more intervals, the Riemann sum $A_L^{(n)}$ approaches the integral, the exact value of M, i.e.,

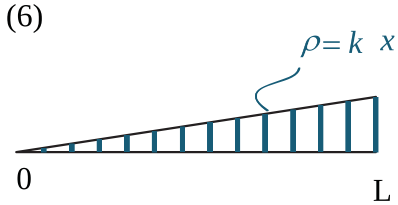
$$M = \int_0^L \rho(x)dx \tag{5}$$

The sum, $w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4$ also represents the Riemann sum of the integral,

$$\int_0^L x\rho(x)dx$$

Therefore, we can use Eq. 3 to compute x_c:

$$x_c = \frac{\int_0^L x\rho(x)dx}{\int_0^L \rho(x)dx} \tag{6}$$



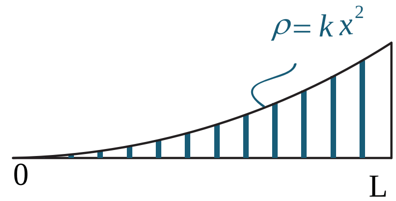
Example 1: Compute x_c if $\rho(x) = kx$ for a beam of length L.

$$x_c = \frac{\int_0^L x(kx)dx}{\int_0^L kxdx} = \frac{kx^3/3|_0^L}{kx^2/2|_0^L} = \frac{2}{3}L$$

This confirms what we learned in Section 8.6 (see Fig. 4a).

Example 2: Compute x_c if $\rho(x) = x^2$ for a beam of length L = 1 (see Fig. 4b).

$$x_c = \frac{\int_0^1 x(kx^2)dx}{\int_0^1 kx^2 dx} = \frac{kx^4/4|_0^1}{kx^3/3|_0^1} = \frac{3}{4}$$



Problem 1: Compute x_c for the following density distributions over the interval [0,1].

a) ρ = kx^3, b) ρ = x(2-x)

21.4 A SHORTCUT METHOD FOR FINDING THE CENTER OF GRAVITY FOR $\rho(x)$ MADE UP OF COMPOUND REGIONS

Consider a $\rho(x)$ distribution made up of a rectangle labeled Region 1 and a triangle labeled Region 2 as shown in Fig. 4.

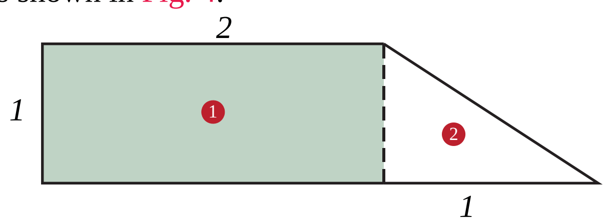


Fig. 4

It can be shown that the center of gravity is the weighted average of the center of gravities of each region taken by itself, i.e.,

$$x_c = \frac{x_{1c}A_1 + x_{2c}A_2}{A_1 + A_2} \tag{7}$$

where x_{c1}, A_1 are the center of gravity and Area of Region 1 while x_{c2}, A_2 are the center of gravity and area of Region 2. For the regions in Fig. 4,

$$x_{c1} = 1, A_1 = 2 \quad \text{and} \quad x_{c2} = \frac{7}{3}, A_2 = \frac{1}{2}$$

Therefore, $$x_c = \frac{(1)(2) + (7/3)(1/2)}{5/2} = \frac{19}{15}$$

Example 3: The unit square divided into Region 1 and Region 2 is shown in Fig. 5. We showed in Example 2 that $x_{c2} = \frac{3}{4}\; A_2 = \frac{1}{3}$. Since A = 1, it follows that $A_1 = \frac{2}{3}$. Then since A = 1 and $x_c = 1/2$, replacing this in Eq. 7,

$$1/2 = \frac{(x_{c1})(2/3) + (3/4)(1/3)}{1}$$

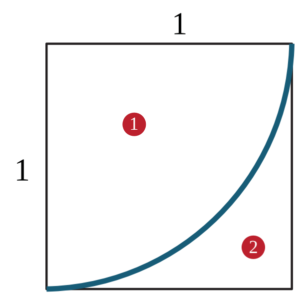


Fig. 5

Solving for x_{c1} we find that $x_{c1} = 3/8$.

Summary:

The values of x_{c1} and A for the rectangle, right triangle and right parabola, shown in Fig. 6, are given in Table 1.

Table 1

	x_c	A
Rectangle	b/2	bL
Right triangle	$\frac{2}{3}L$	$\frac{1}{2}bL$
Right parabola	$\frac{5}{8}L$	$\frac{2}{3}bL$

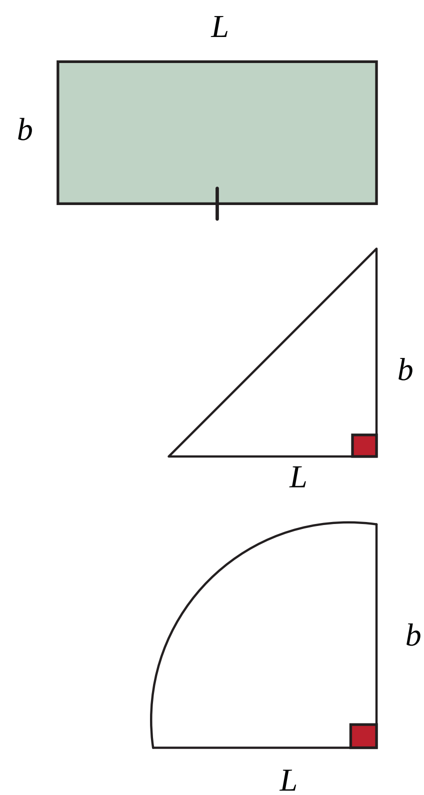


Fig. 6

Problem 2 : For the density distributions corresponding to the complex regions in Figures 7a,b, and c compute x_c.

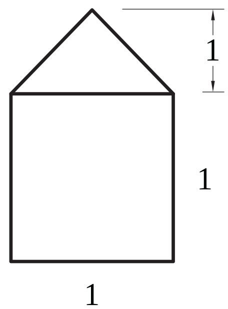


Fig. 7a

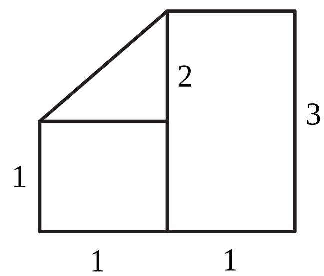


Fig. 7b

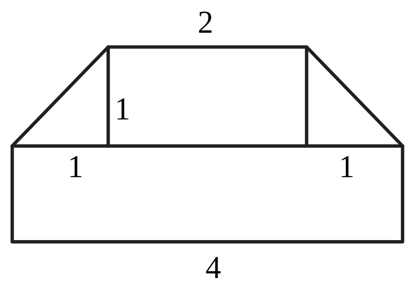


Fig. 7c

21.5 CENTROIDS

If the density is constant (we can let) the balance point is called the *centroid* of the region. We can show that the x-coordinate of the centroid of the complex regions shown in Figures 4,5,and 7 are identical with the centers of gravity of the corresponding density distributions. The y-coordinate of the region is gotten by turning the region through 90 degrees and recomputing the value of the center of gravity. For example,
The balance point of the rectangle of length L and height b has,

$$x_c = L/2 \text{ and } y_c = b/2$$

where (x_c, y_c) are now interpreted as the coordinates of the centroid.

Example 4 For the compound region in Fig. 4, $x_c = 19/15$ as before while,

$$y_c = \frac{(1/2)(2) + (1/3)(1/2)}{5/2} = \frac{7}{15} \; .$$

Therefore the coordinates of the centroid are (19/15, 7/15).

Problem 3: For the complex regions shown in Fig. 7a,b,c, compute the coordinates of the centroid.