CHAPTER 22

THE CONJUGATE BEAM METHOD

22.1 INTRODUCTION

To find the deflection of a beam you must solve the equation,

$$\boxed{\frac{d^2y}{dx^2} = \frac{\overline{M}}{EI}.}$$

Two integrations will solve this problem as was shown in Chapter 19. But there is another method that avoids having to carry out this involved computation when the beam that we are considering is a *free standing beam*. First solve for the bending moment of the beam and then consider a new beam problem called the *conjugate beam problem* in which the "old" bending moment divided by EI is taken to be the "new" load $\rho(x)$ on the beam. The "bending moment" of this new beam problem turns out to be the deflection.

22.2 THE CONJUGATE BEAM METHOD

Let's see how the conjugate beam method works for the familiar free standing beam problem in which a constant density of 10 lb/ft acts on a 10 foot beam as shown in Fig. 1

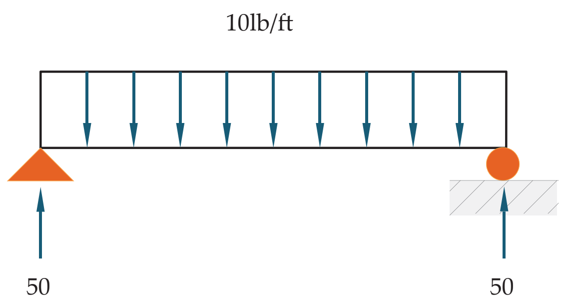


Fig. 1

By an analysis that we have carried out several times in this book,

$$\overline{V} = 50 - 10x, \quad 0 \le x \le 10 \qquad \text{where } \overline{V} = 0 \text{ when x = 5, and} \qquad (1)$$
$$\overline{M} = 50x - 5x^2, \quad 0 \le x \le 10 \qquad \text{where } \overline{M}_{\max} = 125 \text{ at x = 5.}$$

The bending moment diagram is shown in Fig. 2.

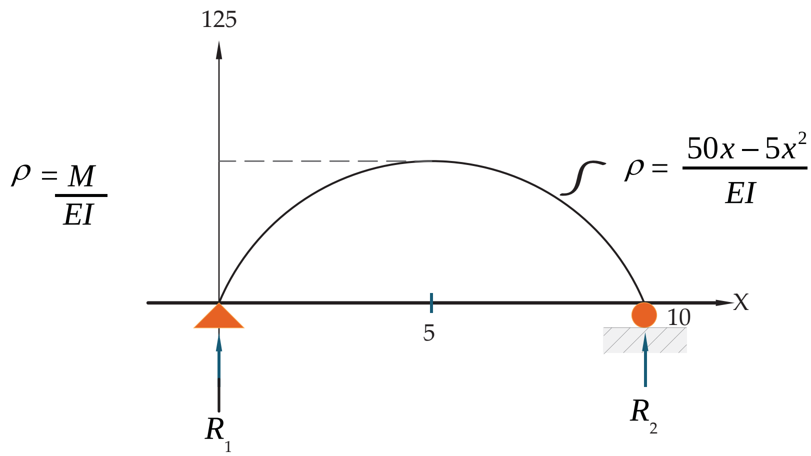


Fig. 2

However we now consider the bending moment, $\overline{M}$, divided by EI to be the load $\rho = \dfrac{50x - 5x^2}{EI}$ of a new beam problem. We can then compute the bending moment of this new beam in which case *its* bending moment will represent the deflection of the original beam problem shown in Fig. 1. However, we must first determine the total weight W acting on the beam in order to compute the resistance forces R_1 and R_2.

$$W = \int_0^{10} \rho(x)dx = \frac{1}{EI}\int_0^{10}(50x - 5x^2)dx$$
$$= (25x^2 - \frac{5}{3}x^3)\Big|_0^{10} = \frac{2500}{3EI}$$

Therefore, $R_1 = R_2 = \dfrac{1}{2}W = \dfrac{2500}{6EI}$ (2)

We are generally only interested in the maximum bending moment which we can compute by statics. Consider this conjugate beam redrawn in Fig. 3.

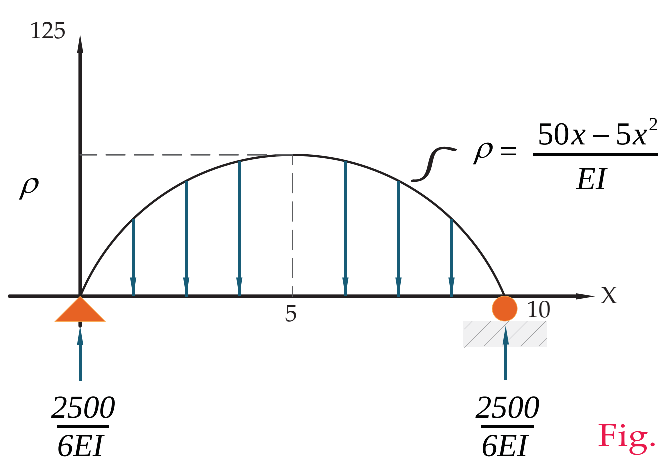


Fig. 3

To find the bending moment at x = 5 calculate the moment about point O at x = 0 (see Fig. 4).

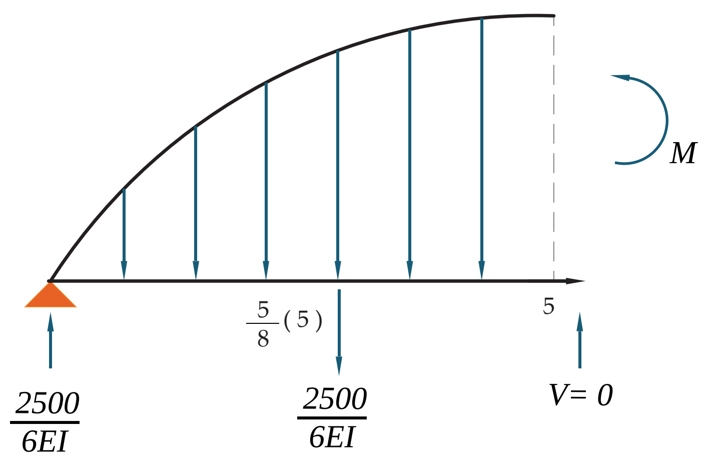


Fig. 4

Since $\overline{M}$ is a maximum at x = 5, $\overline{V}$ = 0 at that point so that the sum of the moment about O is,

$$-\frac{2500}{6EI}\left(\frac{5}{8}\right)(5)+M=0 \quad \text{or} \quad \overline{M}=\frac{1302}{EI} \qquad (3)$$

Note that we have used the fact that the centroid of a "right parabola" is $x_c=\frac{5}{8}L$ where L = 5.

This is the identical result that we obtained in Chapter 19 with a great deal of more effort.

Problem:

Apply the conjugate beam method to the following beam (Fig. 5)

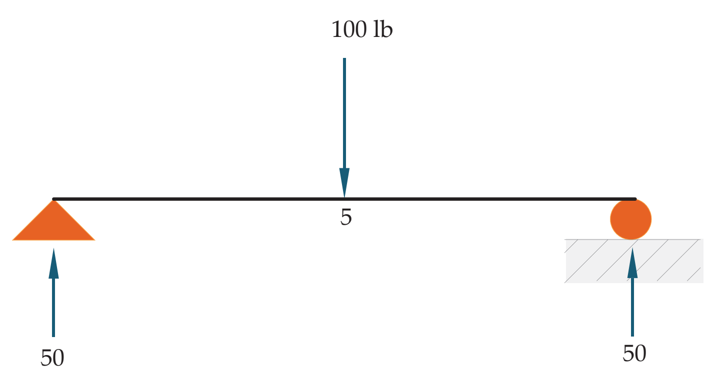


Fig. 5

where the usual analysis yields,

$$\begin{aligned} \overline{V} &= 50 \quad 0 \le x < 5 \\ &= -50, \quad 5 < x \le 10 \end{aligned} \tag{4a}$$

and,

$$\begin{aligned} \overline{M} &= 50x \qquad 0 \le x < 5 \\ &= 500 - 50x, \quad 5 < x \le 10 \end{aligned} \tag{4b}$$

The bending moment diagram is shown in Fig. 6.

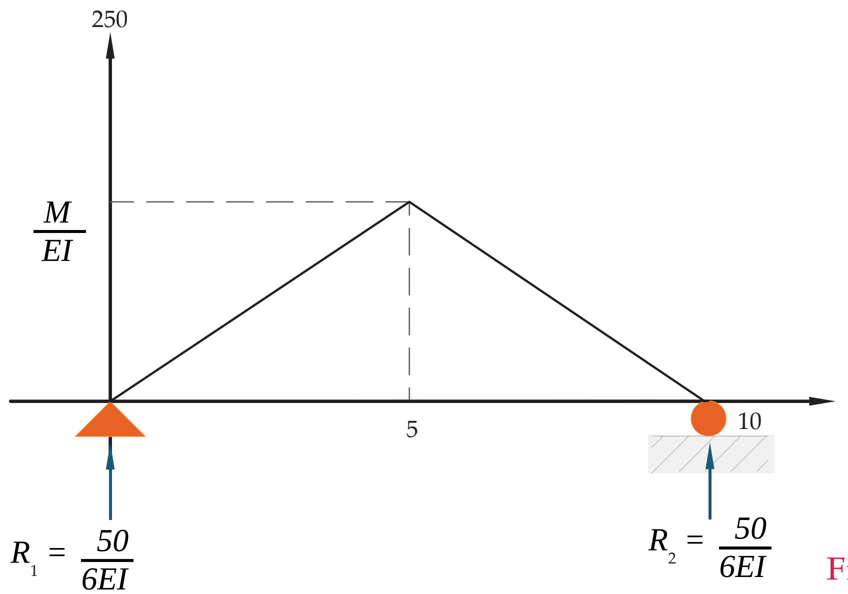


Fig. 6

22.3 PUTTING IT ALL TOGETHER

Let us now test the theory that we have developed throughout this book. Consider the beam in Fig. 7 in which a person weighing W lbs. with a carriage of'a' inches sits upon a beam of length L in., width b, and depth d. We shall assume that the beam is free standing, and we wish to determine the shear force, bending moment, maximum stress on the beam, and maximum deflection. From the maximum stress we will predict if the beam will rupture.

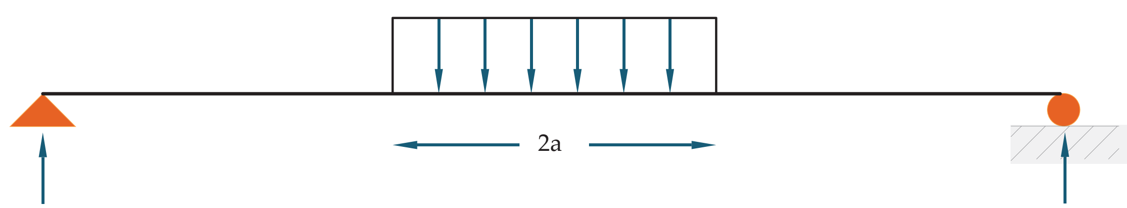


Fig. 7

We have solved many problems of this type before. However, now we are not specifying numbers but developing the solution in terms of the parameters, W, a, and L. The maximum stress , σ_{max} , is determined by Eq. 3 of Chapter 20, while the maximum deflection is determined by the conjugate beam method. I will give the results and leave it to the student to verify the results as an exercise.

First we compute $\overline{V}$ and $\overline{M}$.

$$\begin{aligned}\overline{V} &= \frac{W}{2}, \quad 0 \le x \le \frac{L}{2} - a \\ &= \frac{W}{4a}(L-2x), \quad \frac{L}{2} - a \le x \le \frac{L}{2} + a \\ &= -\frac{W}{2}, \quad \frac{L}{2} + a \le x \le L\end{aligned} \tag{4a}$$

$$\begin{aligned}\overline{M} &= \frac{W}{2}x \qquad 0 \le x \le \frac{L}{2} - a \\ &= \frac{W}{4}(L-2a) - \frac{W}{16a}[(L-2x)^2 - 4a^2, \quad \frac{L}{2} - a \le x \le \frac{L}{2} + a \\ &= \frac{W}{2}(L-x), \quad \frac{L}{2} + a \le x \le L\end{aligned} \tag{4b}$$

The maximum bending moment clearly occurs at L/2 and has the value,

$$\overline{M}_{max} = \overline{M}(\frac{L}{2}) = \frac{W(L-a)}{4}.$$

From Eq. 3 of Chapter 20, the maximum stress of a beam of width b and depth d is,

$$\sigma_{max} = \frac{6\overline{M}_{max}}{d^2 b} = \frac{3W(L-a)}{2d^2 b}$$

Finally, the maximum depth is determined from the conjugate beam method. Again, the maximum deflection, y_{max} , occurs at the midpoint of the beam, L/2. Its value is the bending moment obtained from a density distribution of ,

$$\rho = \frac{\overline{M}}{EI} \quad .$$

For a wooden beam with rectangular cross section from Eq. 4 of Chapter 19, $I = \frac{d^3 b}{12}$.
and letting $E = 1{,}900{,}000$ lbs. /in^2 , The maximum deflection occurs at the midpoint of the beam by the statics method applied to the free body diagram in Fig. 8.

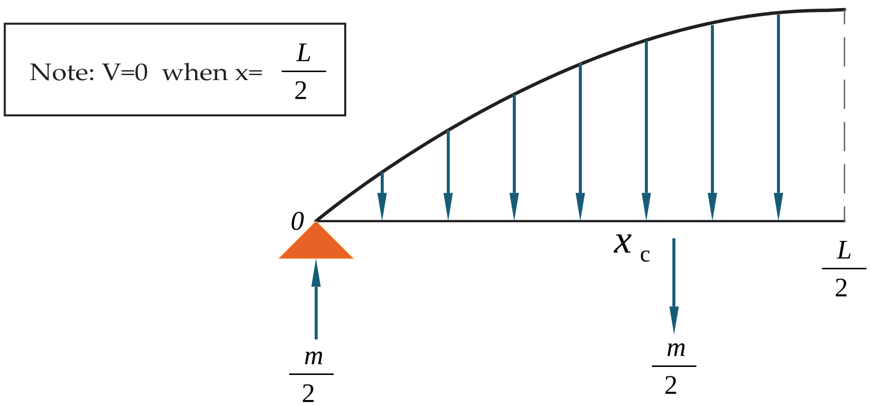


Fig. 8

The equation of moments about point O at x = 0 in Fig. 8 is ,

$$y_{max} = \frac{m}{2} x_c \, , \tag{7}$$

where m is the weight of the conjugate beam in which the density, $\rho = \frac{\overline{M}}{EI}$. The value of m is the integral of the density over the interval [0, L], i.e.,

$$m = \int_0^L \left(\frac{M}{EI}\right) dx \tag{8}$$

where M is given by Eq. 4b. After a lengthy calculation we have found that,

$$m = \frac{WL^2}{16} \tag{9}$$

Unfortunately it is very complicated to compute the center of gravity for this portion of the beam. Instead we will compute an approximate value. The portion of the beam over the interval [0,L/2] has two extreme positions: a) a = 0 in which case the moment curve is a straight line over the interval $0 \le x \le L/2$. In this case $x_c = \frac{2}{3}(\frac{L}{2})$ (see Fig. 9b); or b) when a = L/2 in which case the moment curve is a parabola over the interval $0 \le x \le L/2$ with a maximum value at x = L/2. In this case $x_c = \frac{5}{8}(\frac{L}{2})$ (see Fig. 9c)

If neither of these extreme cases hold, the interval $0 \le x \le (\frac{L}{2} - a)$ will be a straight line while the remaining part of the interval, $(\frac{L}{2} - a) \le x \le L/2$, will be a parabola (see Fig. 9c). We will take the approximation to the center of gravity,

$$x_c = \frac{2}{3}\left(\frac{L}{2} - a\right) + \frac{5}{8}a$$
$$= \frac{8L - a}{24} \qquad (10)$$

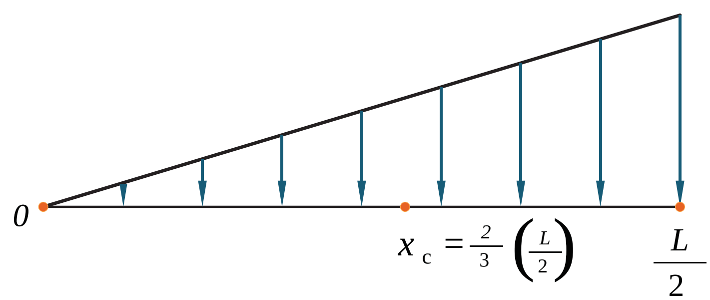


Fig. 9a

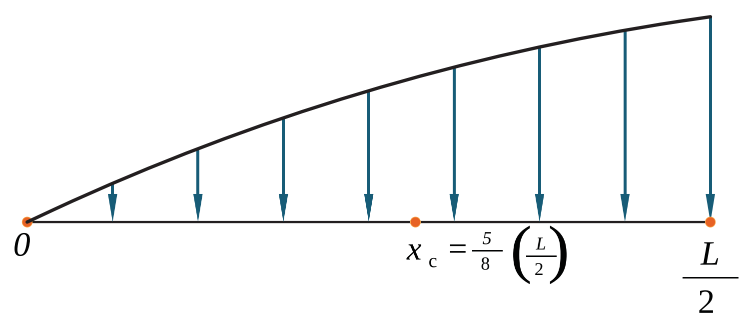
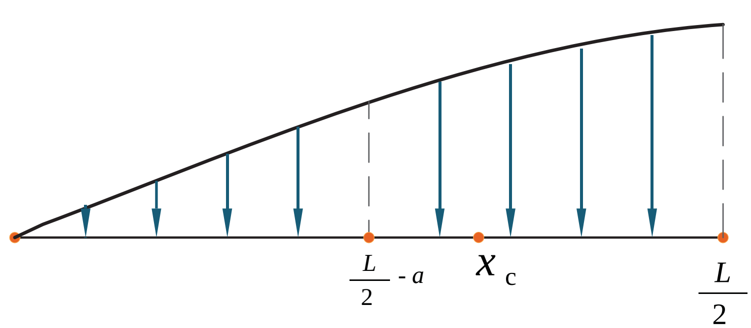


Fig. 9b



$$\boxed{x_c = \frac{8L - a}{24}}$$

Fig. 9c

You will see from Eq. 10 that for extreme cases a) $a = \frac{L}{2}$ and $x_c = \frac{5}{8}\frac{L}{2}$ and; b) a = 0 and $x_c = \frac{2}{3}(\frac{L}{2})$. This value of x_c was used in Eq. 7 to compute y_{max}. The result is,

$$y_{\max} = \frac{W(8L - a)L^2}{32Ebd^3} \tag{11}$$

22.4 CONCLUSION

This method works whenever the beam is supported by hinges and rollers, i.e., a free standing beam. However, the method can also be adapted to solve for the deflection of cantilever beams or beams with hinges. However, we will not consider these beams in this book.