CHAPTER 5

DISTANCE AND ACCUMULATED CHANGE

5.1 DISTANCE

a. Constant velocity

Let's take another look at Mary's trip to the lake on her bicycle (see Section 4.1) in Fig. 1.

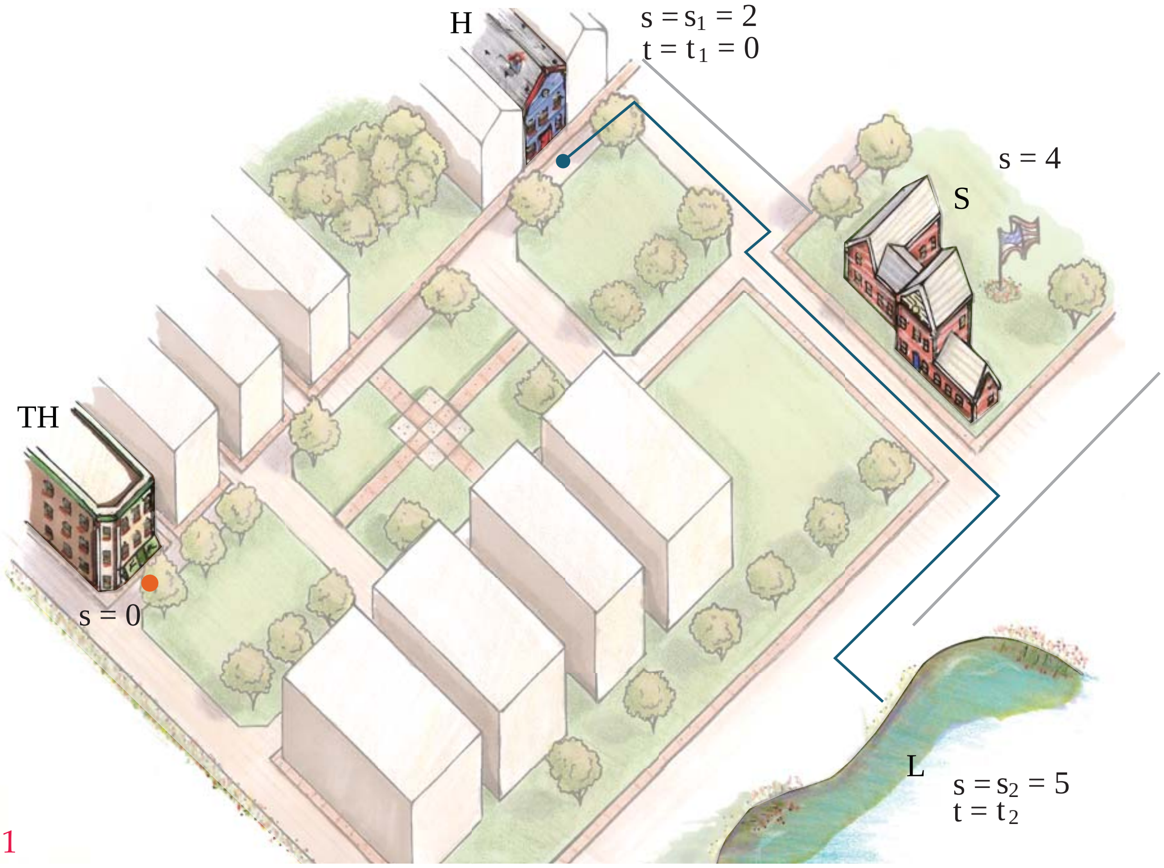


Fig.1

This time she starts from her house at time t = 0 and position s = 2 miles (remember Town Hall is located at s = 0) and travels past her school (S) to the lake (L) at s = 5 (5 miles from Town Hall (TH)) with a constant speed arriving there at t = 0.25 hours (fifteen minutes). We compute the speed of her bike from the familiar formula: rate x time = distance or

$$\text{Speed} = \frac{5-2}{0.25-0} = 12 \text{ miles/hr.}$$

where rate can be interpreted as speed.

Alternatively, if we know that the speed of her bike is 12 mi/hr we can find the distance traveled in fifteen minutes by the formula rate x time = distance, or distance = 12 x 0.25 = 3 miles. You learned all of this in Jr. H.S. However it does not tell the whole story. A quantity called velocity tells us not only her speed but also in which direction she is going. If she rides her bike in the direction of the lake then her velocity is positive and the magnitude of the velocity is her speed. If she travels away from the lake at the same speed, her velocity is negative. If her speed is constant, then the familiar rate x time formula becomes, velocity x time = displacement, or v x t = $s_2 - s_1$.

The displacement is her position at the end of the trip, s_2, as compared to her position at the beginning, s_1. With this formula velocity is positive when traveling from left to right and negative when traveling from right to left. Therefore if Mary returns home from the lake at the same speed that she traveled to the lake, the trip from the lake to Mary's house has a velocity of -12 mi/hr and (-12) x 0.25 = -3. She ended her trip at $s_2 = 2$ and began at $s_1 = 5$ so the displacement is $s_2 - s_1 = 2 - 5$ or -3 miles.

Let's now draw a graph of velocity vs time for the trip to the lake and then back home (see Fig. 2). Notice that the displacement on the trip to the lake equals the signed area under the curve or 12 x 0.25 = 3 while the displacement for the trip back home from t = 0.25 to t = 0.5 equals again the signed area under the curve or -3. Furthermore the total signed from t = 0 to t = 0.5 hr. equals 3 + (– 3) = 0. But Mary began her trip at $s_1 = 2$ (home) and ended it at $s_2 = 2$ (home) for a total displacement of $s_2 - s_1 = 0$, and you will notice that this is value of the displacement is also equal to the signed area under the curve in Fig. 2..

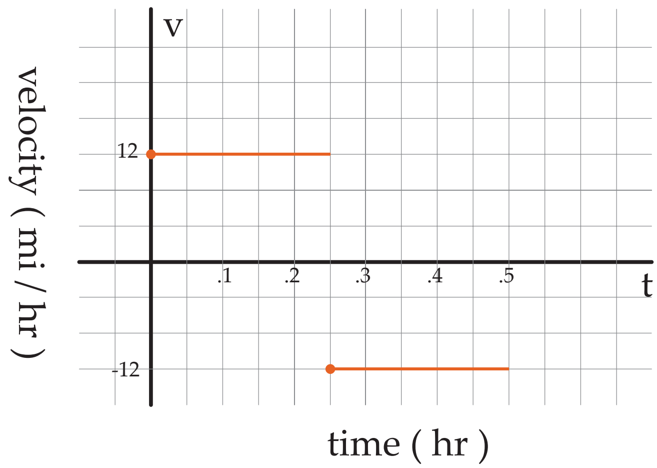


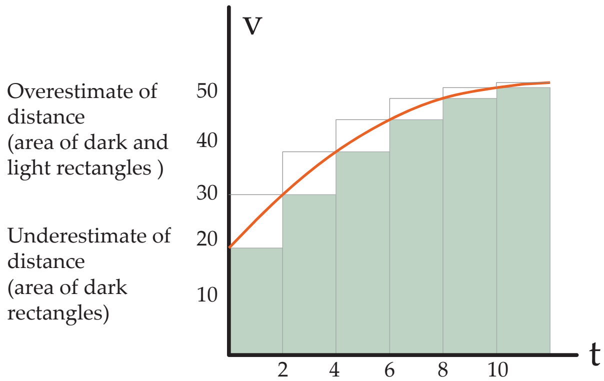
Fig. 2

b. Variable velocity

We have just learned something of great importance to the understanding of calculus, that the signed area under the curve of velocity vs time equals the total displacement. Although the velocity was constant in the example above, this holds true even if the velocity varies in time as the next example illustrates.

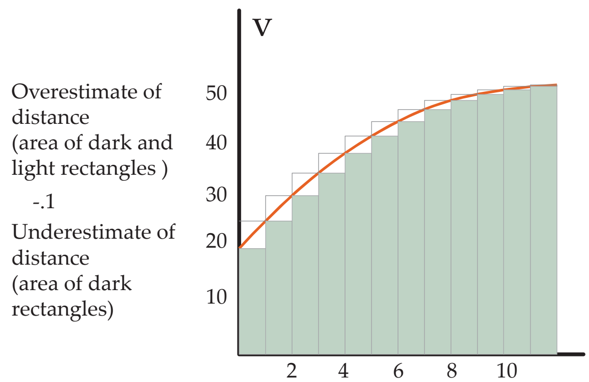
Table 1. Velocity of car every two seconds

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50



Shaded area estimates distance traveled.
Velocity measured every 2 seconds

Fig. 3a



Shaded area estimates distance traveled.
Velocity measured every second

Fig. 3b

This data is plotted in Fig. 3a. Unlike the previous trip to the lake, this time the velocity varies over time so we cannot use the simple formula, rate x time = distance, which only holds for constant velocities. However, if we approximate the graph of velocity vs time by a series of constant values that change their values every two seconds as in Fig. 3a, then we can apply that simple formula to each segment and add the results. If we take the values beneath the curve we get an underestimate of the displacement,

$$\text{Total Displacement} = s_2 - s_1 \geq 20(2) + 30(2) + 38(2) + 44(2) + 48(2) = 360 \text{ feet}$$

Notice that this is the $A_L^{(5)}$ approximation to $\int_0^{10} v(t)dt$.

We can also get an overestimate by taking the values above the curve,

$$\text{Total Displacement} = s_2 - s_1 \leq 30(2) + 38(2) + 44(2) + 48(2) + 50(2) = 420 \text{ feet}$$

This is the $A_R^{(5)}$ approximation to $\int_0^{10} v(t)dt$.

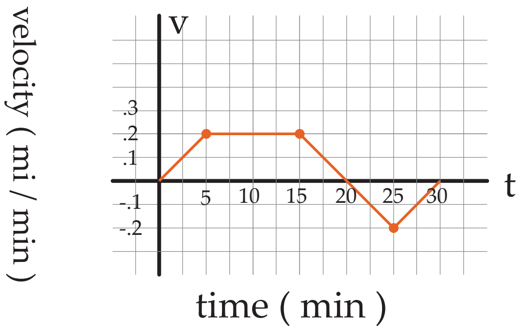
In Fig. 3b, the record of velocity vs time for the car is recorded every second and now the signed area under the curve is a better approximate to the actual total displacement.

Although in this example the velocity was always positive (traveling from left to right) it would also work if the velocity changed sign to negative (traveling from right to left) where now the integral would be interpreted as *signed area*.

So we have a nice application of signed area under a curve: *If the function is thought of as the velocity of a car then the signed area under the v vs t curve is the total displacement of the car over the duration of the trip.*

Problem 1: The graph in Figure 4 shows the velocity record of Mary's trip to the lake on her bike. She starts out from her house which is located at s = 2 mi (remember Town Hall is located at s = 0) at t = 0 minutes. If her school (S) is two miles from her house in the direction of the lake and the trip to the lake takes 20 minutes,

a) When will she pass her school?
b) How far is the lake from her house?
c) What is the value of $\int_0^{30} v(t)dt$? What is the meaning of this value in terms of Mary's trip?



Mary's trip to the lake Fig. 4

5.2 ACCUMULATED CHANGE

The velocity of a car or bike is its rate of change of distance with respect to time. What we have discovered for the signed area under the velocity vs time curve is true for the signed area under any rate of change vs time curve, or, for that matter, any (rate of change of y) vs x curve where y = f(x). The signed area under the rate of change, R(x), of y vs x, will always equal the total change in y over the interval of x under consideration. To see why this is true consider the following example.

Example 1: Consider the function $y = f(x) = x^2 + 1$ over the interval, $0 \le x \le 6$ with the average rate of change, $R_{avg} = \dfrac{\Delta y}{\Delta x}$, computed between successive values of f(x) at intervals of $\Delta x = 2$.

Table 2a

x	0	2	4	6
$f(x) = x^2 + 1$	1	5	17	37
R_{avg}		$\frac{4}{2}$	$\frac{12}{2}$	$\frac{20}{2}$

Next compute: $\int_0^6 R(x)dx \approx A_M^{(3)} = (2)\,\frac{4}{2} + (2)\,\frac{12}{2} + (2)\,\frac{20}{2} = 36 = 37 - 1$

Now compute the $R_{avg} = \dfrac{\Delta y}{\Delta x}$, between successive values of f(x) at intervals of $\Delta x = 1$.

Table 2b

x	0	1	2	3	4	5	6
$f(x) = x^2 + 1$	1	2	5	10	17	26	37
R_{avg}		$\frac{1}{1}$	$\frac{3}{1}$	$\frac{5}{1}$	$\frac{7}{1}$	$\frac{9}{1}$	$\frac{11}{1}$

Next compute: $\int_0^6 R(x)dx \approx A_M^{(6)} = (1)\frac{1}{1}+(1)\frac{3}{1}+(1)\frac{5}{1}+(1)\frac{7}{1}+(1)\frac{9}{1}+(1)\frac{11}{1} = 36 = 37 - 1$

Finally compute the $R_{avg} = \frac{\Delta y}{\Delta x}$, between successive values of f(x) at intervals of $\Delta x = 1/2$.

Table 2c

x	0	½	1	3/2	2	5/2	3	7/2	4	9/2	5	11/2	6
$f(x)= x^2 + 1$	1	5/4	2										
R_{avg}	$\frac{1/4}{1/2}$	$\frac{3/4}{1/2}$											

I ask the student to complete this table and show again that : $\int_0^6 R(x)dx \approx A_M^{(12)} = 36 = 37 - 1$.

What we observe in these examples is that as $\Delta x \to 0$ the average rate of change of f(x) , R_{avg} = $\frac{\Delta y}{\Delta x}$, approaches the instantaneous rate of change, R(x)= $\frac{dy}{dx}$, , and $A_M^{(n)} \to \int_0^6 R(x)dx =$ f(6) – f(0) = 37 – 1 = 36 as $n \to \infty$. This is very significant and we shall see it coming up regularly in our analysis of structures. Later we will discover that it is referred to as t*he Fundamental Theorem of Calculus*. It works for any function, i.e.,

$$\int_a^b R(x)dx = f(b) - f(a)$$

where R(x) is the instantaneous rate of change of f(x) over the interval $a \le x \le b$. You can see that we have here a wonderful shortcut to finding the integral or signed area of the rate of change of f(x) provided f(x) is given. Also if you look closely at the tables above, you will see why this works. Do you see it?

Problem 2: Compute $\int_a^b R(x)dx$ for the following functions over the given intervals:

a) f(x) = $2x^2 - 3x + 1$, $1 \le x \le 3$; b) f(x) = 3sin x, $0 \le x \le \pi/2$; c) $f(x) = \frac{2x}{x^2+1}$, $0 \le x \le 1$

Example 2: If t, in hours, is the time since the start of a 20-hour period in which a bacteria population N(t) increases at a rate given by,

R = R(t) = 3 + 0.1 t^2 millions of bacteria per hour,

make an underestimate of the total change in the number of bacteria over this period using $\Delta t = 4$ then $\Delta t = 2$. The data is given in Table 2 and plotted in Fig. 4a and 4b .

Table 3. Rate of change of R vs t every two hours.

t(hours)	0	2	4	6	8	10	12	14	16	18	20
R = R(t)	3.0	3.4	4.6	6.6	9.4	13.0	17.4	22.6	28.6	35.4	43.0

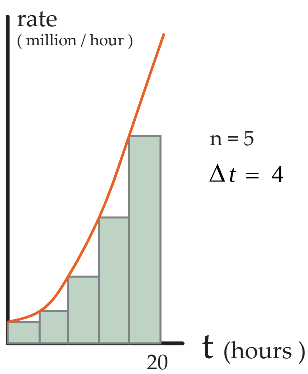


Fig. 4a

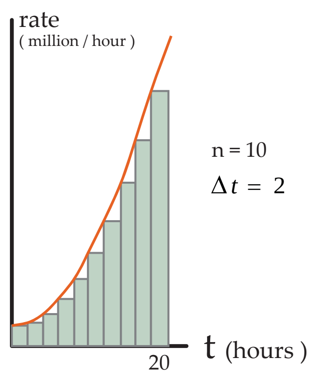


Fig. 4b

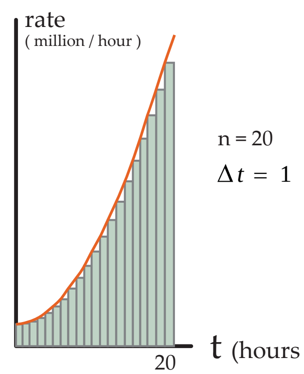


Fig. 4c

You can easily determine that an underestimate of the total change in the number of bacteria over the 20 hour period, N(20) – N(0), is approximated by $A_L^{(5)} = 252.0$ million bacteria, an approximation to $\int_0^{20} R(t)dt$ when $\Delta t = 4$. The approximation when $\Delta t = 2$ is $A_L^{(10)} = 288.0$. (You should verify these results). When $\Delta t = 1$ (not shown in the Table) as shown in Fig. 4c an even more accurate approximation to the total change in the number of bacteria is found to be, $A_L^{(20)} = 307.0$ million bacteria. Notice that as n gets larger and Δt gets smaller the area of the shaded rectangles approaches the signed area under the curve more closely. Note: In this example, since we do not know what N(t) is, we cannot use the shortcut to find the integral.

We can now make the general statement:

$$N(t_2) - N(t_1) = \int_{t_1}^{t_2} R(t)dt$$

Problems

1. A car comes to a stop six seconds after the driver applies the brakes While the brakes are on, the following velocities are recorded

Time since brakes applied (sec)	0	2	4	6
Velocity (ft/sec)	88	45	16	0

(a) Give lower and upper estimates for the distance the car traveled after the brakes were applied

(b) On a sketch of velocity against time, show the lower and upper estimates of part (a)

2. A car starts moving at time $t = 0$ and goes faster and faster Its velocity is shown in the following table Estimate how far the car travels during the 12 seconds

t (seconds)	0	3	6	9	12
Velocity (ft/sec)	0	10	25	45	75

3. The velocity of a car is $f(t) = 5t$ meters/sec Use a graph of $f(t)$ to find the exact distance traveled by the car, in meters, from $t = 0$ to $t = 10$ seconds

4. Two cars start at the same time and travel in the same direction along a straight road Figure 5 8 gives the velocity, v, of each car as a function of time, t Which car

(a) Attains the larger maximum velocity?

(b) Stops first?

(c) Travels farther?

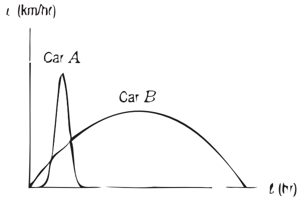


Figure 5 8

5. Two cars travel in the same direction along a straight road Figure 5 9 shows the velocity, v, of each car at time t Car B starts 2 hours after car A and car B reaches a maximum velocity of 50 km/hr

(a) For approximately how long does each car travel?

(b) Estimate car A's maximum velocity

(c) Approximately how far does each car travel?

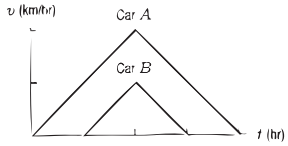


Figure 5 9

6. Figure 5 10 shows the velocity, v, of an object (in meters/sec) Estimate the total distance the object traveled between $t = 0$ and $t = 6$

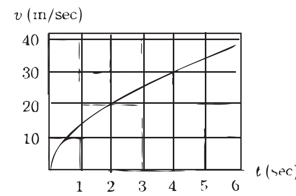


Figure 5.10

7. A car accelerates smoothly from 0 to 60 mph in 10 seconds with the velocity given in Figure 5 11 Estimate how far the car travels during the 10 second period

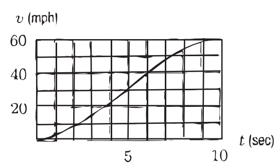


Figure 5.11

8. The following table gives world oil consumption, in billions of barrels per year [1] Estimate total oil consumption during this 20-year period

Year	1980	1985	1990	1995	2000
Oil (bn barrels/yr)	22 3	21 3	23 9	24 9	27 0

9. Filters at a water treatment plant become less effective over time The rate at which pollution passes through the filters into a nearby lake is given in the following table

(a) Estimate the total quantity of pollution entering the lake during the 30 day period

(b) Your answer to part (a) is only an estimate Give bounds (lower and upper estimates) between which the true quantity of pollution must lie (Assume the rate of pollution is continually increasing)

Day	0	6	12	18	24	30
Rate (kg/day)	7	8	10	13	18	35

10. The rate of change of the world's population, in millions of people per year, is given in the following table

(a) Use this data to estimate the total change in the world's population between 1950 and 2000

(b) The world population was 2555 million people in 1950 and 6085 million people in 2000 Calculate the true value of the total change in the population How does this compare with your estimate in part (a)?

Year	1950	1960	1970	1980	1990	2000
Rate of change	37	41	78	77	86	79

11. A village wishes to measure the quantity of water that is piped to a factory during a typical morning A gauge on the water line gives the flowrate (in cubic meters per hour) at any instant The flowrate is about 100 m^3/hr at 6 am and increases steadily to about 280 m^3/hr at 9 am Using only this information, give your best estimate of the total volume of water used by the factory between 6 am and 9 am

12. **(a)** Sketch a graph of the velocity function for the trip described in Example 1 on page 236

(b) Represent the total distance traveled on this graph

13. Graph the rate of sales against time for the video game data in Example 4 Represent graphically the overestimate and the underestimate calculated in that example

14. Roger runs a marathon His friend Jeff rides behind him on a bicycle and clocks his speed every 15 minutes Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop Jeff's data follow

Time since start (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

(a) Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the first half hour

(b) Give upper and lower estimates for the distance Roger ran in total during the entire hour and a half