CHAPTER 8

SHEAR FORCE AND BENDING MOMENTS FOR BEAMS WITH CONTINUOUS FORCES

8.1 INTRODUCTION

The last section was devoted to the computation of shear forces and bending moments for weightless beams with concentrated forces. In this chapter we extend these ideas to beams with continuous forces in which the beams are no longer considered to be weightless. In the next section we will show how the evaluation of beams with continuous loads arise naturally from the case of beams with concentrated loads.

8.2 FROM WEIGHTLESS BEAMS WITH CONCENTRATED FORCES TO BEAMS WITH CONTINUOUS FORCES

Let's revisit the beam from the Section 7.2 assumed to be weightless with a single concentrated load of 100 lb. load acting in the middle of the beam shown again in Fig. 1. As we saw, the shear force was,

$$\overline{V} = \begin{cases} 50, & 0 \le x < 5 \\ -50, & 5 < x \le 10 \end{cases}$$

shown again in Fig. 2.

Fig. 1,2

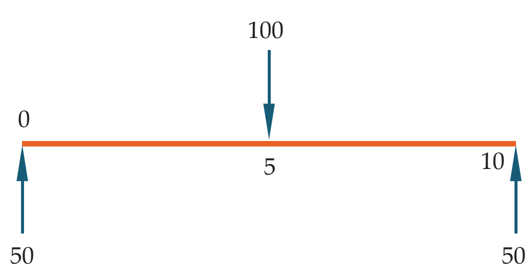


Fig.1

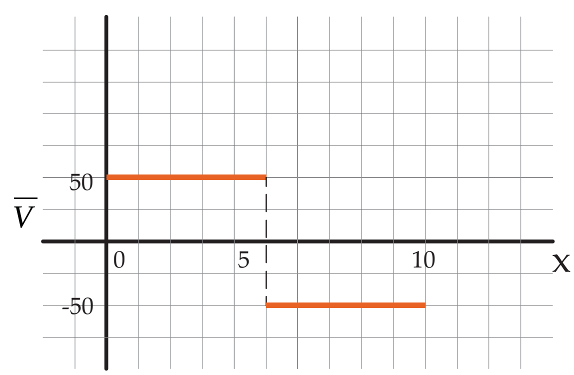


Fig.2

In Remark 2 of the last lesson we observed that the decrease in the shear force across the concentrated load equals the force, i.e., $\Delta\overline{V} = -\overline{F} = -100.$.

Now consider the 100 lb. concentrated force to be distributed across the five equally spaced points, x = 1,3,5,7,9 on the beam with 20 lbs. acting at each point shown in Fig. 3a. The decrease in shear force at each point is now given by, $\Delta\overline{V} = -20.$ and shown in Fig. 3b. Next distribute the 100 lb. load between the ten equally spaced points, x = ½, 3/2. 5/2, 7/2, 9/2,11/2,13/2,15/2,17/2,19/2 with 10 lb.acting at each point, as shown in Fig. 4a, with the decrease in shear force at each point given by $\overline{V} = -10$ and shown in Fig. 4b.

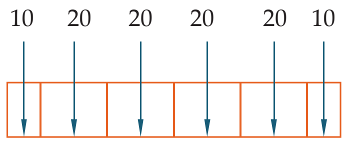


Fig.3a

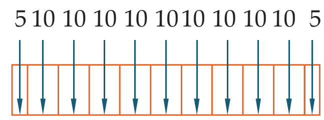


Fig.4a

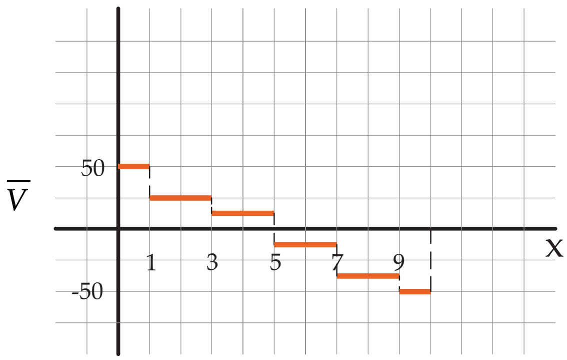


Fig.3b

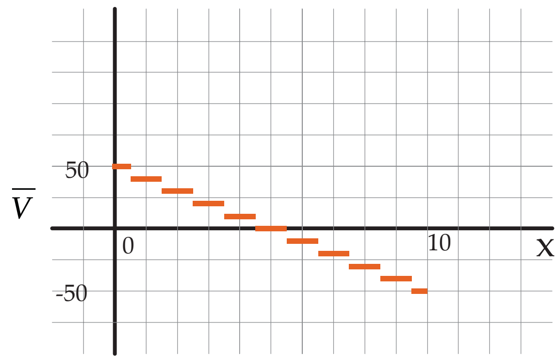


Fig.4b

If we continue to distribute the 100 lb. load evenly across the beam so that the force density is 100/10 = 10 lbs./ft., this is the same thing as assuming that the beam is no longer weightless but has a constant density of 10lbs./ft. The shear force diagram is now shown in Fig. 5. Notice that as the forces acting on the beam become more distributed, the zig-zags from the shear force curve lessen and finally disappear so that the shear force becomes a straight line from 50 lb. at x = 0 to -50 lb. at x = 10, or

$$\overline{V} = 50 - 10x$$

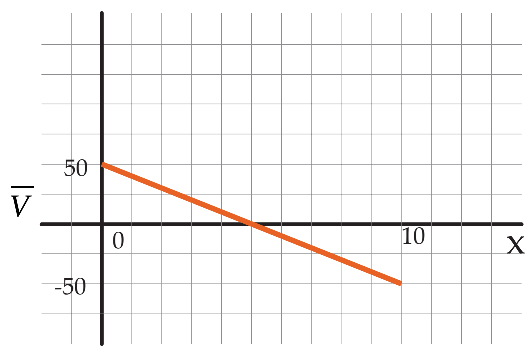


Fig.5

The force density can be thought to be the constant density ρ of the beam, 10 lb./ft. Fig. 6 illustrates the curve of ρ vs x where the density is constant. The forces of gravity (weight) are continuous along the length of the beam and are denoted by small arrows. If we divide the beam into five equal parts, each part will weigh 20 lbs. which can be thought to be acting at the middle of the interval as in Fig. 3a so that ,

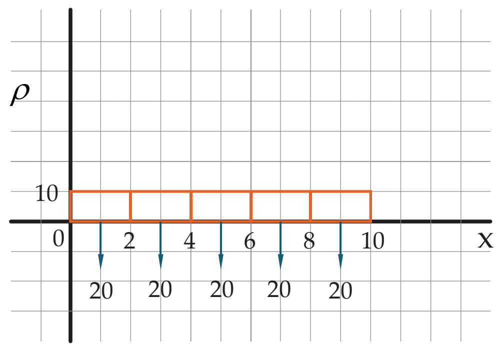


Fig.6

$$\text{Total force} = A_M^{(5)} = (10)(2) + (10)(2) + (10)(2) + (10)(2) + (10)(2) = 100$$

If we divide the beam into ten equal parts, each part will weigh 10 lbs. which can be thought to be acting in the middle of each interval as in Fig. 4 so that .

$$\text{Total force} = A_M^{(10)}$$

Notice that the total force acting on the beam of 100 lb. is just the area under the density curve between x = 0 and x = 10, i.e.,

$$\overline{F}_{total} = \int_0^{10} \rho(x)dx \quad \text{where} \quad \rho(x) = 10 \text{ lb ./ft.}$$

The total force or weight of the beam on the interval [0,x] is $\overline{F} = 10x$.

8.3 A CONTINUOUS BEAM PROBLEM WITH INIFORM LOAD

Example 1:

We wish to evaluate the shear force and bending moment of the 10 ft. beam in Fig. 7 acted upon by a continuous force density of $\rho = 10$ lb./ft. Since the load is evenly distributed, each reaction force is 50 lb.

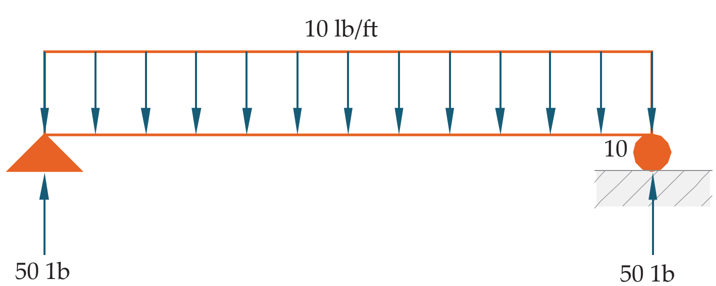


Fig.7

a) Computation of Shear Force for Continuous Force by Statics

Next we compute the shear force by the balance of forces. To find the shear force at a typical value of x take a section of the beam over the interval [0,x] as shown in Fig. 8. Notice that we are assuming that $\overline{V}$ acts downwards.

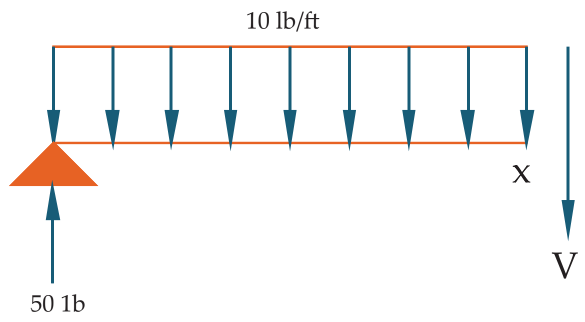


Fig.8

From this figure,

$$-50 + 10x + V = 0 \quad \text{or} \quad V = 50 - 10x$$
$$\overline{V} = 50 - 10x, \quad 0 \le x \le 10. \qquad (1)$$

Notice that over the interval [0,5), V is positive so that our assumption is correct and $\overline{V}$ acts downwards. Between x = 5 and x = 10, V is negative so we must reverse the direction of $\overline{V}$, i.e., it acts upwards. The graph of $\overline{V}$ vs x is shown in Fig. 9a.

Remark 1: Notice $\overline{V}$ is a linear function of x and that the slope of the $\overline{V}$ vs x curve is -10 and this is the negative of the force density. This will always be the case.

b) Computation of Bending Moment

i) Method 1 – the Statics Method .

We again consider the portion of the beam over the interval [0,x] as shown in Fig. 10.. Notice that we are assuming that the bending moment at x is counterclockwise. Also the weight of the beam over interval [0,x] is 10x. By using an important result from statics, that all the weight of the beam can be thought to act at the center of gravity or the balance point of the beam located at x/2, and using the result that $\overline{V} = 50 - 10x$, the balance of moments is given by,

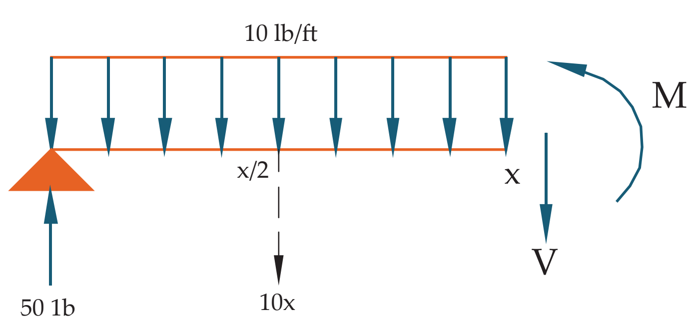


Fig.10

$$-(10x)\frac{x}{2}-(50-10x)x+M=0 \quad \text{or} \quad M=-5x^2+50x$$

$$\overline{M}=-5x^2+50x\,, \qquad (2)$$

a downward pointed parabola. Since M is positive, our assumption that the bending moment is counterclockwise is correct. $\overline{M}$ is graphed in Fig. 9b .

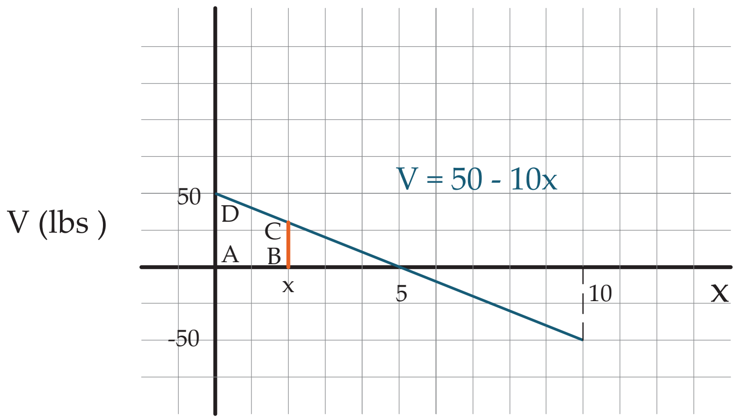


Fig.9a

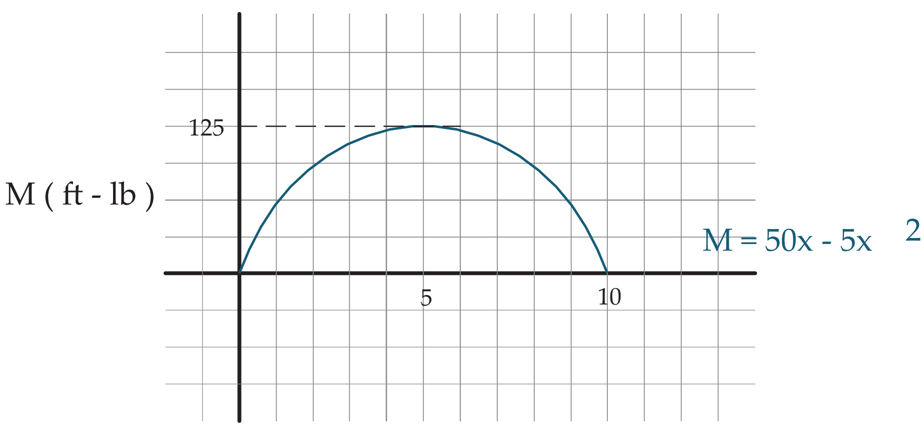


Fig.9b

ii) Method 2 – The Area Method

We now use Method 2, the area method, to compute the moment using the fact that $\overline{M}_2 - \overline{M}_1$ equals the area under the $\overline{V}$ vs x curve from x_1 to x_2., i.e., $\overline{M}(x_2)-\overline{M}(x_1) = \int_{x_1}^{x_2}\overline{V}dx$. Since $\overline{M}$ = 0 when x = 0 (true for the ends of all free standing beams), we can take $\overline{M}(x_2)$ to be the moment at a typical value of x, and the difference of moments over the interval [0,x] is the area of the trapezoid ABCD or,

$$M = \frac{1}{2}x(50 + 50 - 10x) \quad \text{or} \quad \overline{M} = 50x - 5x^2,$$

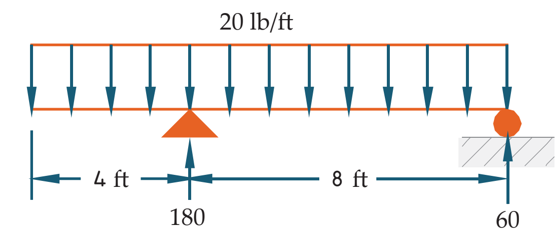
identical to Eq. 2. Note that the area of a trapezoid is ½ multiplied by the product of the altitude and the sum of the bases.

Remark 2: In order to use Method 3, the slope method, we would have to know how to find the slope of the tangent line to a parabola. To do this we will have to wait until Chapter 9 where calculus will help us with that task.

Problems:

Find $\overline{V}$ and $\overline{M}$ for these beams with their loadings:

1.



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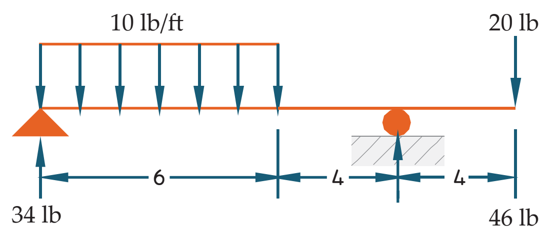

8.4 BEAMS WITH CONTINUOUS BUT NON-UNIFORM DENSITIES

Next consider the same 12 ft. beam in which the density of material within it or upon it varies linearly with the length. For example let's say the density varies as ρ=6x so that the density at x= 0 is ρ=0 and at x=12 is ρ=72 lb/ft. This is shown in Fig. 11.

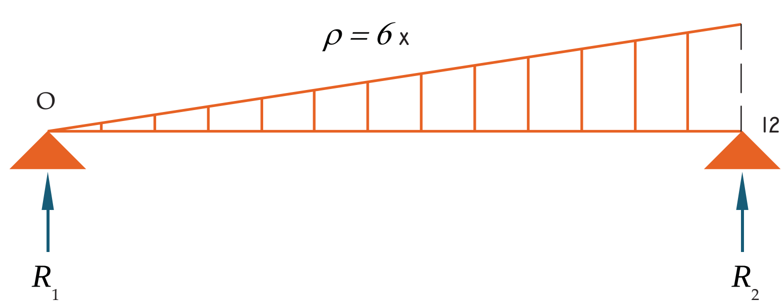

Fig.11

Again the total force $\overline{F}$ due to the weight of the beam equals the area under the density curve or $\overline{F}$ = 1/2(12)(72) =432 lb. since the area is a triangle. The total force (weight of the beam) over the interval [0,x] is ½(6x)(x).

In general the density curve will be an arbitrary function $\rho(x)$ of the density, and the total force $\overline{F}$ due to the weight on the portion of the beam between x_1 and x_2 is the area under the graph of this function between x_1 and x_2 ,

$$\overline{F} = \int_{x_1}^{x_2} \rho(x)dx$$

Example 2: External force with linear density:

Find the shear force and bending moment of a beam of length 12 ft. with density given by ρ =6x shown in Fig.11.

We must first find the reaction forces. The total force is the area under the density curve or $\overline{F}$ = ½ x 12 x 72 = 432 lb. Again, the total gravity force can be taken to act at the center of gravity which is located at x = 2/3 x 12 = 8 ft. Center of gravity will be discussed in Section 8.6. The reaction forces satisfy the equations,

$$R_1 + R_2 = 432$$
$$12R_2 - (432)(8) = 0$$

Therefore , $R_1 = 144, R_2 = 288$

a) Shear force

To find the shear force on the interval [0,x] shown in Fig.12, assume that $\overline{V}$ acts downwards and compute the balance of forces.

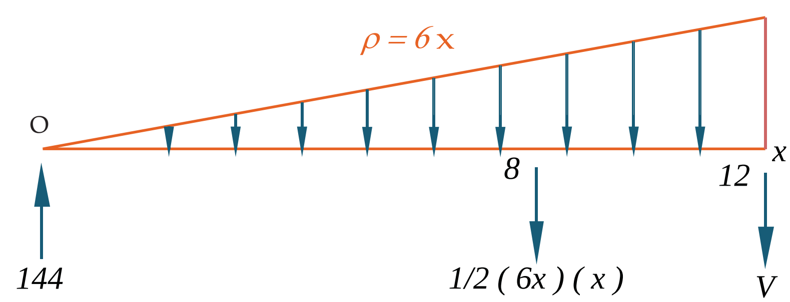

Fig.12

$$-144 + ½(6x)(x) + V = 0 \quad \text{or} \quad V = 144 - 3x^2$$

$$\text{or} \quad \overline{V} = 144 - 3x^2 \qquad (3)$$

Note that for $0 \le x < 4\sqrt{3} = \sqrt{48}$, $\overline{V}$ is positive so the force acts downwards as we have assumed. For larger values of x, $\overline{V}$ is negative so that the direction of the force reverses and acts upwards. The graph of $\overline{V}$ vs x is shown in Fig 13a.

To find the bending moment on segment [0,x] of the beam about point O (x = 0) we will use the statics method applied to Fig. 14. Assume that the weight of the beam, ½(6x)(x) acts at the center of gravity, 2/3 x.

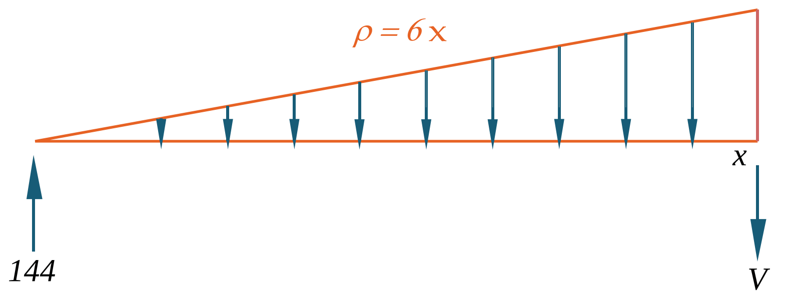

Fig.14

$$-\frac{1}{2}(6x)(x)\frac{2}{3}x - xV + M = 0 \quad \text{where } V = 144 - 3x^2.$$

After some algebra,

$$\overline{M} = 144x - x^3 \qquad (4)$$

shown in Fig. 13b.

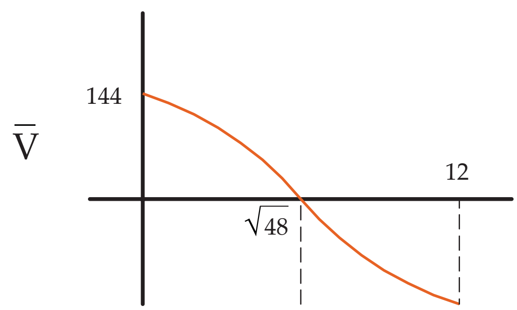

Fig.13a

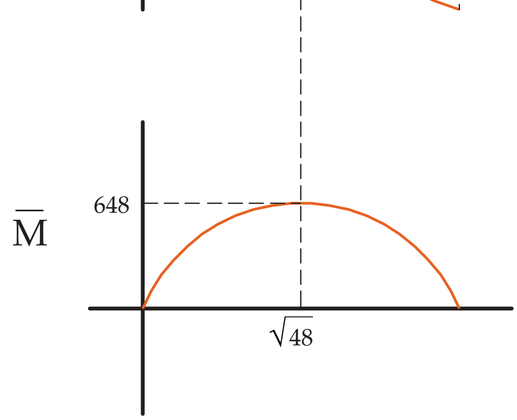

Fig.13b

Remark 3: $\overline{V}=0$ when $x=4\sqrt{3}$ so that is where the bending moment is a maximum.

Remark 4: Since $\overline{V}$ is parabolic, in order to use the area method to compute the bending moment we must be able to find the signed area under a parabola. We will learn how to do this in Chapters 15 and 16.

Remark 5: Since the shear force is not constant we are not able at this point to use the slope method. We will learn how to do this in Chapters 18.

Remark 6: The slope of the $\overline{V}$ vs x curve is the negative of the force density ρ. However, since $\overline{V}$ vs x is parabolic, we do not yet know how to find the slope. We will address this question in Chapters 9 and 10.

8.5 SOLUTION TO PROBLEM 1:

Example 3:

The beam in Problem 1 is redrawn in Fig. 15 showing the deflection.

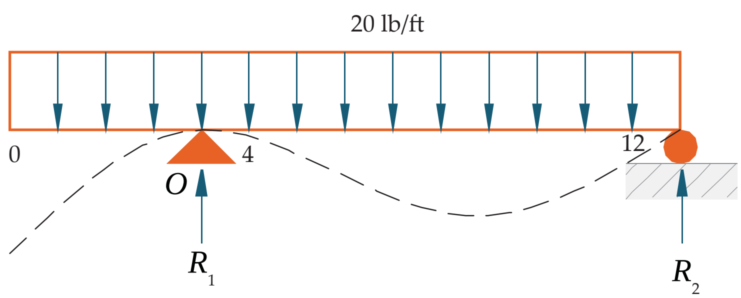

Fig.15

a) Find reaction forces

Calculate R_2 by taking moments about O (x = 4) in Fig. 15. The center of gravity of the force distribution across the beam is located at x = 6 (the balance point), and the entire force can be considered to be acting there.

$$-(20)(16)(6-4)+(R_2)(8)=0 \quad \text{or} \quad R_2=60$$

$$\text{Since} \quad R_1+R_2=240 \quad \text{it follows that } R_1=180.$$

b) Calculate the shear force, $\overline{V}$:

For x on the interval, [0,4) (see Fig. 16):

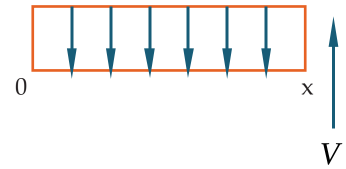

Fig.16

Assuming that the shear force acts upwards.

$$20x - V = 0 \quad \text{or} \quad V = 20x$$

Since V is positive, our assumption is correct and V acts upwards.
Therefore, $\overline{V} = -20x$.

For x in (4,12) (see Fig. 17):

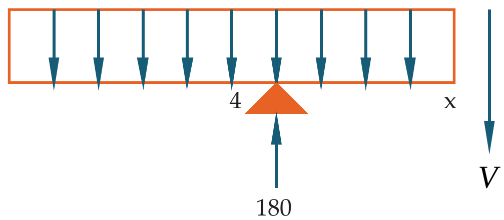

Fig.17

Assume that shear force acts downwards.

$$-180 + 20x + V = 0 \quad \text{or} \quad V = 180 - 20x$$

We see that $V = 0$ at $x = 9$ where,

$$V > 0 \text{ for x in the interval } [4,9) \text{ and } V < 0 \text{ for x in } (9,12]$$

Therefore, the shear force acts downwards in [4,9) and upwards in (9,12] ,i.e.,

$$\overline{V} = 180 - 20x$$

Therefore,

$$\overline{V} \begin{cases} -20x, & 0 \le x < 4 \\ 180 - 20x, & 4 < x < 12 \\ 0, & x = 12 \end{cases} \qquad (5)$$

The curve of $\overline{V}$ vs x is shown in Fig . 18a.

Remark 7: Notice that the shear stress is $\overline{V} = -60$ at x = 12, but because the support $\overline{R_2} = -60$ takes over, the shear force reduces to $\overline{V} = 0$ at x = 12.

c) Compute the bending moment by Method 2 (area method):

For x in [0,4):

From Fig. 18a, the area of the triangle is,

$$\overline{M} = \frac{1}{2}(-20x)(x) = -10x^2$$

For x in (4,12] :

From Fig 18a the total area is the sum of the area of the triangle over the interval [0,4] and the trapezoid over the interval [4,x].

$$\overline{M} = -160 + \frac{1}{2}(x-4)(100+180-20x)$$

After some algebra,

$$\overline{M} = -10x^2 + 180x - 720 \qquad \textbf{(6)}$$

Therefore,

$$\begin{aligned} \underline{M} &= -10x^2, & 0 \le x < 4 \qquad \textbf{(7)} \\ &= -10x^2 + 180x - 720, & 4 < x < 12 \end{aligned}$$

The graph of $\overline{M}$ vs x is shown in Fig. 18b.

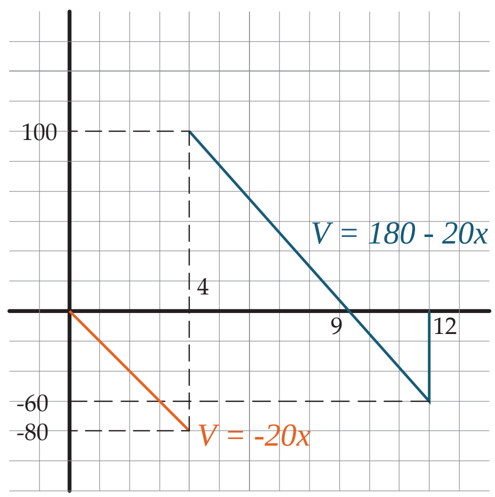

Fig.17a

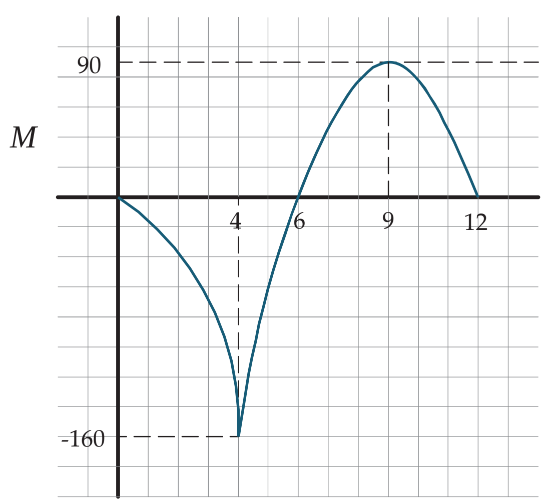

Fig.17b

Check 1: The area under the $\overline{V}$ vs x curve equals 0
Check 2: $\overline{M} = 0$ at the free ends of the beam, x = 0 and x = 12
Check 3: While the deflection goes: up - down - up, the bending moment goes in the opposite direction : down - up – down.

Notice in Fig. 18a and b that the maximum of $\overline{M}$ occurs at x = 9 where x goes from positive to negative through 0 ($\overline{V} = 0$ at x = 9) The minimum of $\overline{M}$ occurs at x = 4 where x goes from negative to positive through 0.

The maximum and minimum values of $\overline{M}$ are,

$$\overline{M}_{max} = \overline{M}(9) = 90 \quad .\text{and} \quad \overline{M}_{min} = \overline{M}(4) = -160$$

To find where $\overline{M} = 0$ set Eq. 6 equal to 0 and solve for x.

$$0 = -10x^2 + 180x - 720$$
$$x^2 - 18x + 72 = 0$$
$$(x-6)(x-12) = 0$$

Therefore x = 6 and x = 12.

Clearly $\overline{M} = 0$ at x = 12 since that is the end of the free standing beam. What about the point x = 6? We noted in Remark 8 of Chapter 7 that the deflection of the beam is concave up when $\overline{M}$ is positive and concave down where $\overline{M}$ is negative, therefore the deflection of the beam goes from concave down to concave up at x = 6 so that this must be a point of inflection of the deflection curve as can be observed in Fig. 15.

8.6 CENTER OF GRAVITY

The *center of gravity* of an object is its balance point If you could suspend the object at that point, it would not tip over in any direction. This is easy to see for a planar object. Cut an odd shape from a piece of cardboard and look for the balance point. Once you find it, that is the center of gravity of the cutout. Actually, since the density of the cardboard does not change from point to point, in this case the center of gravity is called the *centroid*. Centroids depend only on the geometry of the shape while center of gravity depends on the geometry and density of the object.

It is easy to find the centroid of a planar object. Suspend from a point A on the periphery of the object a string with a weight on one end so that the string is a kind of plumbline as shown in Fig.19. Draw on the face of the object the line marked by the plumbline. Now suspend the object from a second point B and draw a second line marked by the new plumbline. The centroid is the intersection of the two lines.

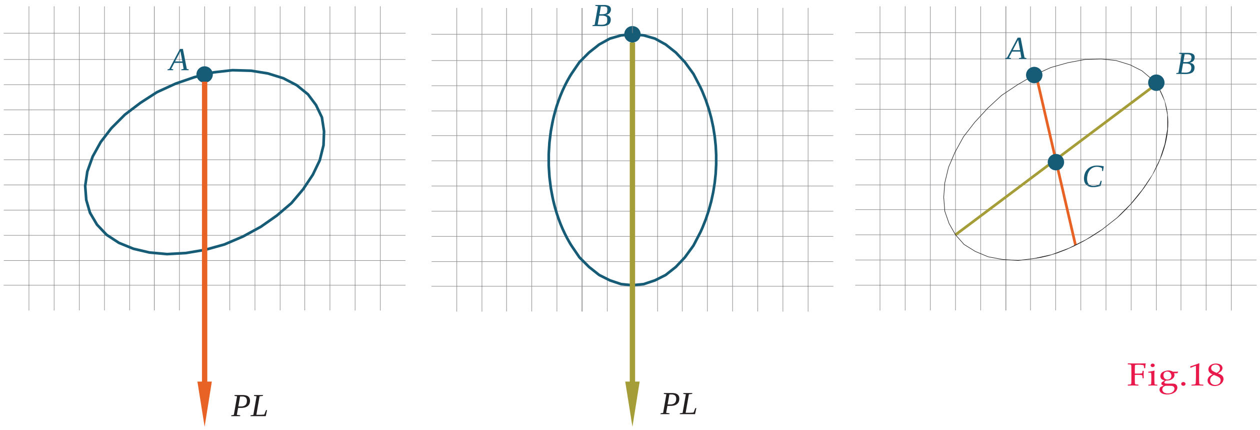

Fig.18

If the shape of the planar object is a polygon, there is another simple way to find the centroid. Let's say the polygon is a quadrilateral as shown in Fig .20. Number the points 1,2,3,4. Choose any pair of points, e.g., points 1 and 2, and mark their midpoint as point A (see Fig. 19). Choose another point, say point 3, and mark the point that is 1/3 of the distance from A to point 3. Call this point B. Finally mark point C that is ¼ of the distance from B to the remaining point 4. This will be the centroid. If the polygon has more than five vertices we would continue this process of identifying points A,B,C,D,E,… that follow the sequence : ½, 1/3, ¼, 1/5, …

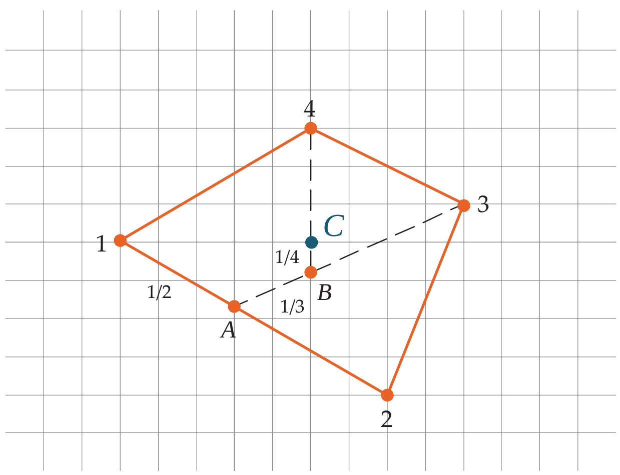

Fig.19

Remark 8: The four points in the above example do not have to lie in a plane. While any three points lie in a plane, the centroid of the tetrahedral volume created by connecting these three points to the fourth vertex to in that plane must lie ¼ of the distance from the centroid of the triangle to its fourth vertex.

If we apply this procedure to the line segment connecting two points, clearly the centroid lies at the midpoint of the line segment. Applying the procedure to a triangle, we see that the centroid is located 2/3 of the distance from any vertex to the midpoint of the opposite side, known as a median line of the triangle, a well known theorem of geometry (see Fig. 21)

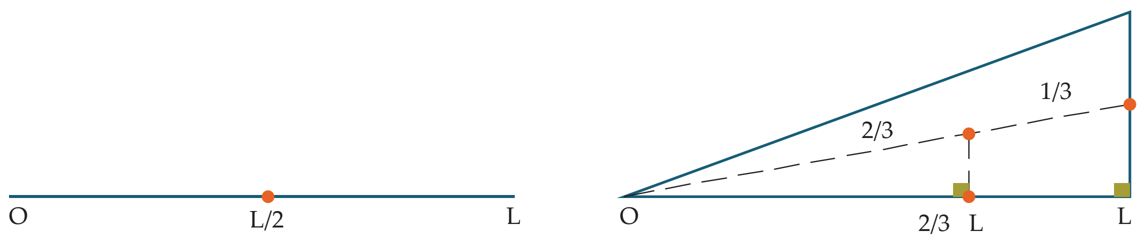

Fig.20

Remark 9: The three medians of a triangle divide the triangle into six triangles of equal area. The meeting points of the three medians is the centroid and the six equal area triangles provide the balance (see Fig. 21)

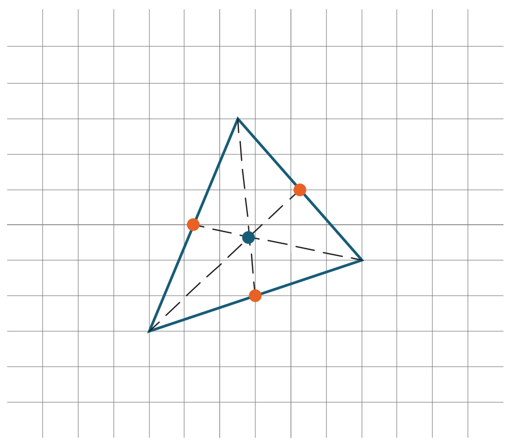

Fig.21

In certain situations, we can use this approach to finding centroids to also find the center of gravity of a linear distribution of varying density. For example, consider the triangular distribution for the example in Section 8.4, shown again in Fig. 20, where the density of the line segment [0,12] varies linearly, e.g., $\rho = 6x$. The centroid of this triangle lies 2/3 of the distance from x = 0 to the midpoint of opposite side of the distribution triangle. By similar triangles, the balance point of the distribution is located at x = (2/3)12 = 8.

We will describe how to find the centroids and centers of gravity of more complex shapes and distributions in Chapter 21.