CHAPTER 9

THE DERIVATIVE

9.1 THE ORIGINS OF CALCULUS

All great discoveries have their origins with an important problem that needs to be solved. The problem generally has a long history with other great scientists contributing to its solution. The history of calculus perfectly illustrates this dynamic. Copernicus (1473-1543) upset religious doctrine by suggesting that the planets revolved about the Sun rather than the Earth as center. Galileo (1564-1642) found supporting evidence for this hypothesis by building the first telescope and observing the moons of Saturn revolving around Saturn like a miniature solar system. Galileo also studied the action of gravity on balls rolling down inclined planes. Kepler (1571-1630) inherited the observational data of the great astronomer, Tycho Brahe (1546-1601), and in his "first law" hypothesized that the planets traveled in ellipses around the sun. Kepler also found by empirical methods his "second law" that as the planets revolved around the sun they swept out equal areas in equal times, and his most important "third law" that the square of the periods of the planets around the sun equaled the cube of their mean distances from the sun as measured in units by which the distance of the earth to the sun is one unit. This is where Isaac Newton (1642-1727) entered the picture.

Newton was one of the co-discoverers of calculus along with Wilhelm Leibnitz.(1646-1716) When a plague raged through England, Newton returned from Cambridge to the seaside village of his birth to avoid the ravages of the plague. There he spent his 22nd through 24th years with a great deal of time to think about things, and he had the most fruitful time of his life in terms of scientific discovery. He wished to find a theory by which he could predict the exact motion of the planets as they revolved around the sun, based on the work of the great scientists that came before him. To do this he first hypothesized his three laws of motion, most notably his famous law, $F = ma$, to make the bold conjecture that it was an inverse square gravitational force field between sun and planets that propelled the movement of the planets about the sun. He did this by working backward in his theory from Kepler's third law to discover the universal law of gravitation, a single law by which objects developed forces of attraction between each other due to a gravitational field. As a result, from Newton's Laws one is able to derive Kepler's three laws by pure thought not merely observation.. To solve his equations of motion, Newton had to invent the subject of calculus. It is also interesting that since other scientists were not knowledgeable about the new calculus, Newton wrote his great book, the Principia, in the form of a book of geometry downplaying his calculus discoveries.

As important as Newton's theories were to science, they were even more important to philosophy. This was the first time that science was found to have precise predictive power rather than merely observational and model building abilities. After Newton, practitioners of every field, be it chemistry, geology, psychology, social science, etc. sought the simple "underlying" laws of their fields of study.

In our study of calculus and structures, we have seen how the forces of gravity act on the beam and how areas and slopes are related to the computation of shear force and bending moment. We have also found that our inability to compute the slope of tangent lines to curves and areas under curves prevented us from evaluating beams beyond a certain level of complexity. In this chapter we begin to rectify this situation by finding a method to determine the slope of the tangent line to the graph of any function. To accomplish this we will introduce a quantity known as the derivative which will be the slope of the line tangent to the graph of a function. The study of derivatives is known as the *differential calculus*. Chapters 15 and 16 will then be devoted to a shortcut method to compute integrals and using them to find the signed area under curves by what is known of as the *integral calculus*. We will then return to using the calculus to evaluate the bending moment of a beam. In the process we will show how the effect of gravitational forces on our structures and the computation of areas lead naturally to the derivation of internal forces and moments just as they did for Newton in his discovery of the motions of the planets.

9. 2 INTRODUCTION

We have seen in Chapter 1 that the line is the simplest mathematical element. As long as the graph of a function is smooth (no cusps or sharp edges) at least locally in the vicinity of a point the graph of the function is very much like a line. Mathematicians have made good use of this property of functions to explore their natures by relating the functions locally to lines. The slope of the tangent line to the graph of a function turns out to be the key to understanding complex functions. In this chapter we will study how to determine the slope of the tangent line to a curve.

9. 3 THE DERIVATIVE: A NEW APPROACH

If you look at the graph of a smooth function, y = f(x), such as the one in Figure 1, (no discontinuities or cusps ("sharp edges"), you will notice that in the vicinity of any point the curve looks like a straight line.

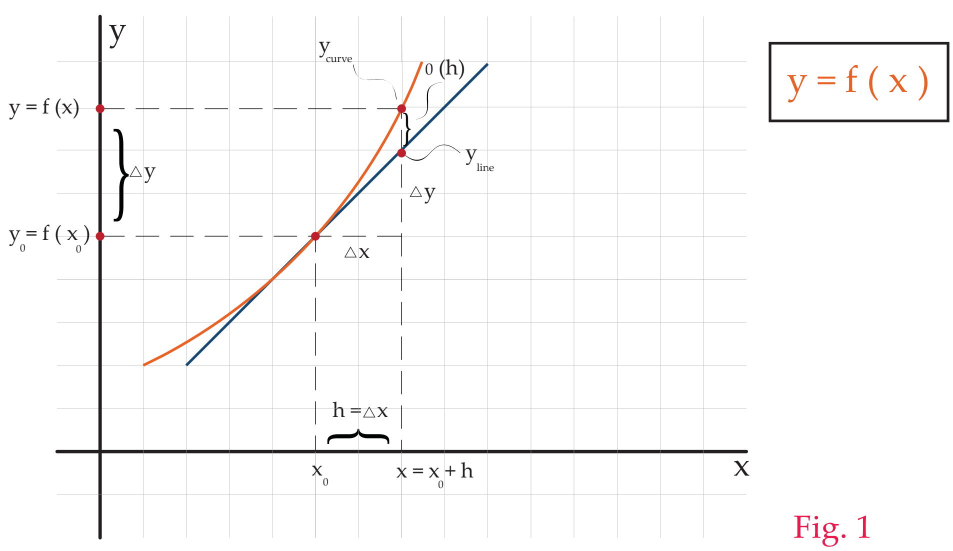


Fig. 1

That straight line is called the tangent line to the curve at x_0. Using the point slope method, its equation is,

$$y_{line} - y_0 = m(x - x_0) \text{ where } y_0 = f(x_0) \qquad (1)$$

Since we will only be considering points x near x_0, I will let $x - x_0 = h$ where h is a small number and rewrite Eq. 1 as,

$$y_{line} = f(x_0) + mh$$

And since the value of y_{curve} is very close to y_{line} for small h, it follows that,

$$y_{curve} = y_{line} + o(h)$$

or,

$$y_{curve} = f(x) = f(x_0 + h) = f(x_0) + mh + o(h) \qquad (2)$$

The function o(h), called the "little o function" , is the small difference between y_{curve} and y_{line} near

x_0. In this equation $f(x_0)$ is usually a big number, mh is a small number (because h is small),

and o(h) is a,"very small" number. In fact, the o(h) is so small that,

$$\boxed{\lim_{h \to 0i} \frac{o(h)}{h} = 0} \qquad (3)$$

In other words, in a race to 0, o(h) always beats h and functions with this property will be said to be "much smaller" than h. Clearly h^2 and h^3 are o(h) functions. Other o(h) functions are less obvious and will be discussed in Section 9.5.

The o(h) functions have the following algebraic properties:

i. $o_1(h) + o_2(h) = o(h);$

ii. $o_1(h)o_2(h) = o(h);$

iii. $ko_1(h) = o\ (h)$ where k is a constant.

The proof is obvious (why?),

In Eq.2 the slope of the tangent line to the curve, m, is called the derivative of f(x) at the

point x_o and denoted by $f'(x_o)$ or sometimes $\frac{df(x_0)}{dx}$ or simply as $\frac{dy}{dx}|_{x=x_0}$. In Chapter 4

we saw that the tangent line to a curve y = f(x) at some point x_0 represents the rate of change of y with respect to x at x_0. So derivatives are now also identified not only with slopes but also with "rate of change."

Our job will be to determine m for a variety of functions f(x). The work will be enormously simplified by setting o(h) to zero and every time a term comes up that is "much smaller" than h, i.e., passes the test of Eq. 3 (for example, h^2, h^3, etc.) I also eliminate that term from our equation. Therefore, I will let o(h) = 0 and rewrite Eq. 2 as,

$$\boxed{f(x_o + h) = f(x_o) + mh} \qquad (4)$$

I refer to Eq. 4 as the *Derivative Machine* since all derivatives can be derived from this equation by solving for m. In fact we will compute derivatives using the Derivative Machine by a six step process:

1. Compute $f(x_0)$
2. Compute $f(x_0+h)$
3. Insert these in the Derivative Machine
4. Simplify this expression algebraically
5. Set to zero all terms that are o(h)
6. Solve algebraically for m

To justify the elimination of o(h) in the Derivative Machine return to Eq. 2 without setting o(h) to 0. Rewriting Eq. 2,

$$m = \frac{f(x_0+h)-f(x_0)}{h} - \frac{o(h)}{h}$$

.We have seen in Section 4.2 that the limit of the first term leads to the derivative, $f'(x_0)$ while the limit of the second term vanishes by definition so that m approaches f '(x_0) as h approaches 0, or

$$m = \lim_{lh\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

9.4 SOME APPLICATIONS OF THE DERIVATIVE MACHINE.

Example 1: Compute the derivative of $f(x)=x^2$.

Let us apply the six step process.

1. $f(x_0)=x_0^2$
2. $f(x_0+h)=(x_0+h)^2$
3. $(x_0+h)^2 = x_0^2 + mh$ (inserting in the Derivative Machine)
4. $x_0^2+2hx_0+h^2=x_0^2+mh$ or $2hx_0 + \boldsymbol{h}^2 = mh$
5. $2hx_0 = mh$ (eliminating h^2 which is an o(h) function)
6. $m = 2x_0$ (dividing both sides by h)

Therefore f '(x) = 2x or $\frac{dx^2}{dx} = 2x$ where I have replaced x_0 by x since x_0 was an arbitrary value of x.

Example 2: Compute the derivative for

$f(x) = \frac{1}{x}$ Applying the six step process:

1. $f(x_0) = \frac{1}{x_0}$

2. $f(x_0 + h) = \frac{1}{x_0 + h}$

3. $\frac{1}{x_0 + h} = \frac{1}{x_0} + mh$

4. $x_0 = x_0 + h + mhx_0(x_0 + h)$ (multiplying both sides of the equation by $x_0 (x_0 + h)$

5. $0 = h + mhx_o^2$ (simplifying and eliminating mh^2)

6. $m = -\frac{1}{x_0^2}$

Therefore, $f'(x) = \frac{d\frac{1}{x}}{dx} = -\frac{1}{x^2}$

Example 3: Compute the derivative of f(x) = c (the constant function)

1. $f(x_0) = c$
2. . $f(x_0 + h) = c$
3. $c = c + mh$
4. 0 = mh
5. No o(h) functions to eliminate
6. m = 0

Therefore the derivative of the constant function is 0, i.e., $\frac{dc}{dx} = 0.$.

Problem 1: Apply the Derivative Machine and the six step process to find the derivative of :

a) $f(x) = \sqrt{x}$ (show that $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$); b) f(x)= 3x + 4; c) $f(x) = x^3$ (show that $\frac{dx^3}{dx} = 3x^2$)

Problem 2: Apply the Derivative Machine to find the derivatives of the following functions:

a) $f(x) = 2x^2$
b) $f(x) = 5x^2 + 2x$
c) f(x) = 7 for all values of x (i.e., the constant function)
d) f(x) = x

9.5 SOME UNUSUAL o(h) FUNCTIONS

To find the derivative of sin x, cos x, e^x, and ln x by the derivative machine requires us to recognize o(h) functions that are not as obvious as h^2or h^3. For example show that sin h – h, cos h – 1 and e^h — h – I are o(h) functions. To do this we must show that $\lim_{h \to 0i} \frac{o(h)}{h} = 0.$ .1 suggest that you do this by setting up a table in which, as h gets smaller, and smaller

$\frac{\sin h - h}{h}$ and $\frac{e^h - h - 1}{h}$ shrink towards 0. Use your calculator to complete Table 1 to show this.

Table 1

h	$\frac{\sin h - h}{h}$	$\frac{\cos h - 1}{h}$	$\frac{e^h - h - 1}{h}$
0.1			
0.01			
0.001			
0.0001			

Problem 3: Which of the following functions are o(h) functions, i.e., which ones pass the test of Eq. 3?

a) $3h^2 – 5h^3$
b) sin h
c) sin h^2
d) 2h + 3
e) h sin h

9.6 APPLICATION OF THE UNUSUAL 0(h) FUNCTIONS TO FINDING THE DERIVATIVE OF cos x, e^x, and In x.

Example 4: Making use of the fact that sin h – h, and cos h - 1 are o(h) functions, show that $\frac{d\cos x}{dx} = -\sin x$.

1. $f(x_0) = \cos x_0$
2. $f(x_0 + h) = \cos(x_0 + h)$
3. $\cos(x_0 + h) = \cos x_0 + mh$ (inserting 1 and 2 into the Derivative Machine)
4. $\cos x_0 \cos h - \sin x_0 \sin h = \cos x_0 + mh$ (using a trig identity) or,
   $\cos x_0 (\cos h - 1 + 1) - \sin x_0 (\sin h - h + h) = \cos x_0 + mh$ or,
   $\cos x_0 (\cos h - 1) + \cos x_0 - \sin x_0 (\sin h - h) - h \sin x_0 = \cos x_0 + mh$
5. $\cos x_0 - \boldsymbol{h} \sin x_0 = \cos x_0 + mh$ or $- \boldsymbol{h} \sin x_0 = mh$ (setting o(h) functions to 0 and cancelling $\cos x_0$.)
6. $\boldsymbol{m} = -\sin x_0$

Therefore, $\frac{d\cos x}{dx} = -\sin x$

Example 5: Show that $\frac{de^x}{dx} = e^x$

1. $f(x_0) = e^{x_0}$
2. $f(x_0 + h) = e^{x_{00} + h}$
3. $e^{x_{00} + h} = e^{x_0} + mh$
4. $e^{x_0} e^h = e^{x_0} + mh$
5. $e^x (e^h - h - 1 + h + 1) = e^{x_0} + mh$
6. $e^x (e^h - h - 1) + he^{x_0} + e^{x_0} = e^{x_0} + mh$
7. $he^{x_0} = mh$ (setting o(h) function to 0 and doing some algebra)
8. $m = e^{x_0}$

Therefore, $\frac{de^x}{dx} = e^x$

Example 6: Show that $\frac{d \ln x}{dx} = \frac{1}{x}$

1. $f(x_0) = \ln x_0$
2. $f(x_0 + h) = \ln(x_0 + h)$
3. $\ln(x_0 + h) = \ln x_0 + mh$
4. $e^{\ln(x_0+h)} = e^{(\ln x_0 + mh)}$
   $e^{\ln(x_0+h)} = e^{\ln x_0} e^{mh}$
   $(x_0 + h) = x_0(e^{mh} - mh - 1 + mh + 1)$
   $(x_0 + h) = x_0(e^{mh} - mh - 1) + x_0(mh + 1)$
5. $h = x_0 mh$ (setting o(h) function to 0 and doing some algebra)
6. $m = \frac{1}{x_0}$

Therefore, $\frac{d \ln x}{dx} = \frac{1}{x}$

9.7 TANGENTS AND DERIVATIVES

Consider the line, y = mx + b as it crashes into the x-axis at angle θ in Fig. 2. The slope of the line is ,

$$m = \frac{rise}{run} = \tan\theta .$$

Now consider the tangent line to the graph of a function y = f(x) at the point x = x_0. in Fig. 2. If we consider small values of θ then,

$$\tan\theta \approx \theta$$

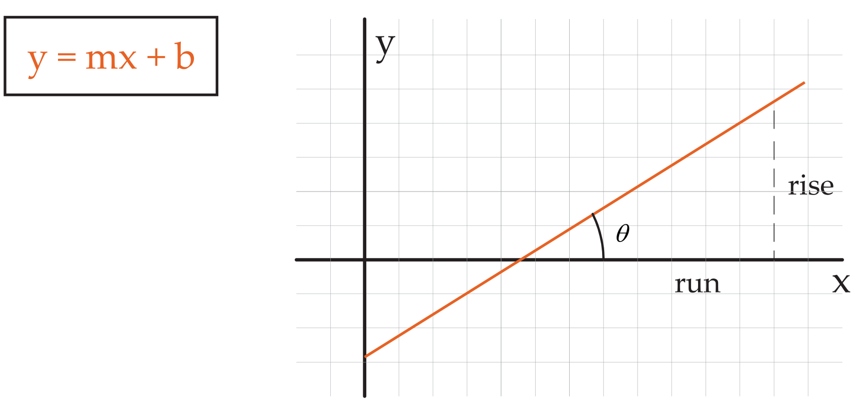


Fig. 2

You can check this approximation by taking your calculator and inputting a small angle (in radians) into the tangent function. Notice that $\tan\theta$ and θ have nearly the same values.

Since slopes are identified with derivatives,

$$\frac{dy}{dx} \approx \theta \qquad (5)$$

When beams have small deflections, θ will be small and Eq. 5 will hold true. This will play a role in Chapter 19 on the deflection of beams.

9.8 DERIVATIVE AS A RATE OF CHANGE

Consider the Derivative machine:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o_1(h) \qquad (6)$$

where y = f(x) , $f'(x_0) = \frac{dy}{dx}|_{x_0}$, $f(x_0 + h) - f(x_0) = \Delta y$, and $h = \Delta x$

Rewriting Eq.6 in terms of these new symbols ,

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} - o(h) \text{ or } \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \text{ for small h.}$$

But we know from Lesson 4 that $\frac{\Delta y}{\Delta x}$ is the average rate of change of y with respect to x over the interval $[x_0, x_0 + h]$ while $\frac{dy}{dx}|_{x_0}$ is the instantaneous rate of change of y with respect to x at x = x_0.

The *derivative can now be interpreted as either the slope of a tangent line to a curve at any point or as the instantaneous rate of change of a function at a point in its domain.*

9.9 HIGHER DERIVATIVES

In Chapter 4 we looked at the table of values for function, y = f(x), at integer values of x and computed Δy. We repeat the table for y = x^2. but also compute the change of the change of y or $\Delta(\Delta y) = \Delta^2 y$ and the third change of y, $\Delta^3 y$.

Table 2

x	-2	-1	0	1	2	3	4
y	4	1	0	1	4	9	16
Δy	-3	-1	1	3	5	7	
$\Delta^2 y$	2	2	2	2	2		
$\Delta^3 y$	0	0	0	0			

Since $\Delta x = 1$, Δy is an approximation to the rate of change or derivative at each value of x. Likewise we find that $\Delta^2 y$ is an approximation to the rate of change of the rate of change. This is also called the derivative of the derivative or the 2nd derivative and denoted by $\frac{d^2 y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx}$.

Likewise, $\Delta^3 y$ is an approximation to the 3rd derivative, denoted by $\frac{d^3 y}{dx^3}$..

Since $y = x^2$ the derivative is $\frac{dy}{dx} = 2x$, then $\frac{d^2 y}{dx^2} = 2$, and $\frac{d^3 y}{dx^3} = 0$. By looking at this table you will see that these are indeed good approximations.

Now consider the table of $y = x^3$.

Table 3

x	-3	-2	-1	0	1	2	3	4	5
y	-27	-8	-1	0	1	8	27	64	125

Δy	19	7	1	1	7	19	37	61
$\Delta^2 y$	-12	-6	0	6	12	18	24	
$\Delta^3 y$	6	6	6	6	6	6		
$\Delta^4 y$	0	0	0	0	0			

Differentiating $y = x^3$ we get,

$$\frac{dy}{dx} = 3x^2,\ \frac{d^2 y}{dx^2} = 6x,\ \frac{d^3 y}{dx^3} = 6,\ \frac{d^4 y}{dx^4} = 0$$

9.10 WHAT DERIVATIVES TELL US ABOUT A FUNCTION

As we observed in Chapter 4, positive values of Δy indicate where a function is increasing while negative values indicate where the function is decreasing. On the other hand, where Δy increases, the curve is concave up, and concave down where Δy decreases. In other words, a curve is concave up when $\Delta^2 y$ is positive and concave down when $\Delta^2 y$ is negative.

Table 2 shows that $y = x^2$ decreases to x = 0 after which it increases, while being concave up for all x. Table 3 shows that $y = x^3$ increases for all values of x and is concave down for negative values of x, but concave up for positive values. .

Since we are identifying Δy with $\frac{dy}{dx}$ and $\Delta^2 y$ with $\frac{d^2y}{dx^2}$ we are able to make the following general statements about derivatives:

a) When $\frac{dy}{dx} > 0$ functions increase as x increases.

When $\frac{dy}{dx} < 0$ functions decrease as x increases.

b) When $\frac{d^2y}{dx^2} > 0$ functions are concave up.

When $\frac{d^2y}{dx^2} < 0$ functions are concave down.

What is the nature of the function when $\frac{dy}{dx} = 0$? Values of x where $\frac{dy}{dx} = 0$ are said to be *critical points* or *stationary points* of the function.. At such points the curve neither increases nor decreases but is stationary; it has a horizontal tangent line. Fig. 3a shows three possibilities for critical points: i) the curve can either have a *relative maximum* (a peak value as compared to nearby points) ii) a *relative minimum*, or iii) a horizontal plateau along a rising or falling function.

What is the nature of the function when $\frac{d^2y}{dx^2} = 0$? Fig. 3b shows three possibilities where $\frac{d^2y}{dx^2} = 0$: i) the curve changes from concave down to concave up; ii) the curve changes from concave up to concave down; iii) the curvature does not change. For cases i and ii the function is said to have an *inflection point*.

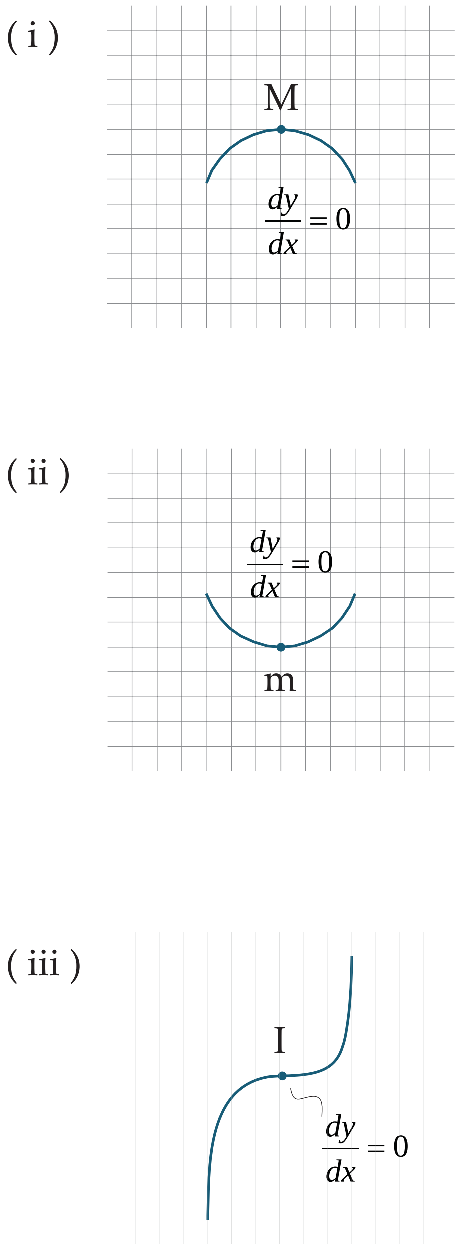


Fig. 3a

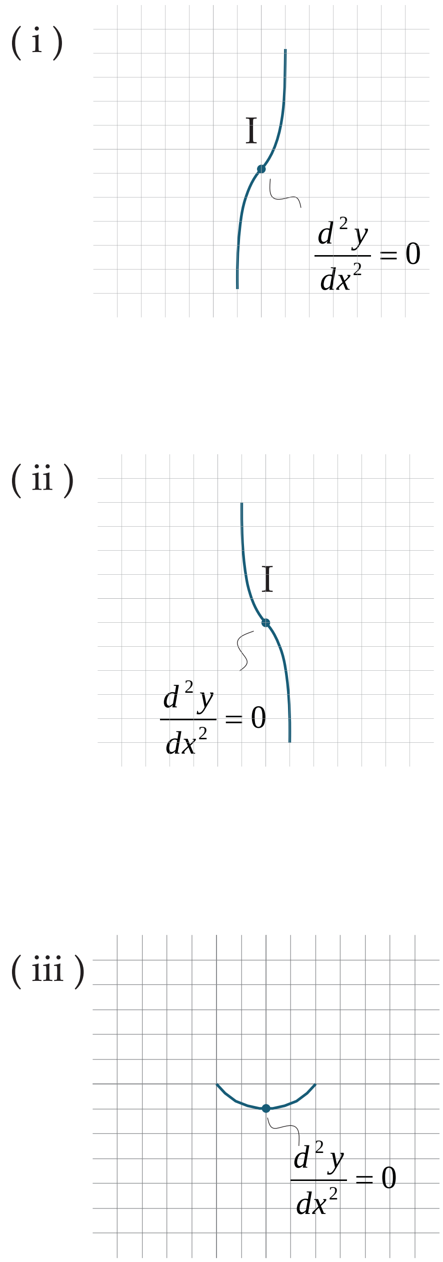


Fig. 3b

.One can get a picture of how $y, \frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ interrelate by sketching their graphs one above the other. We do this in Fig. 4 for $y = x^2$ and $y = x^3$..

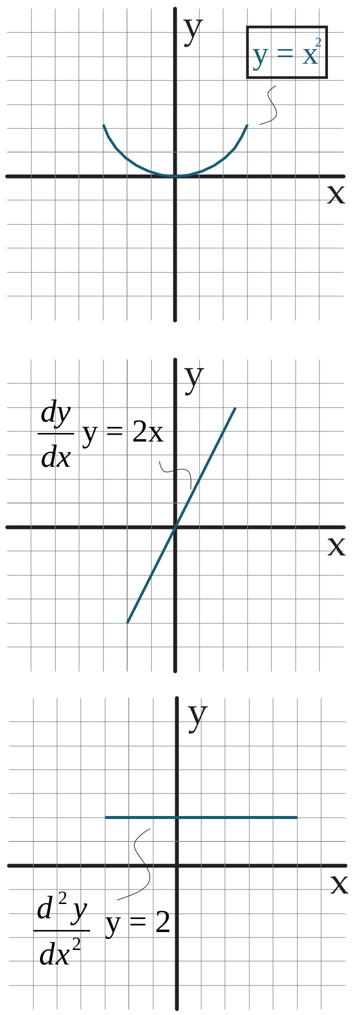


Fig.4a

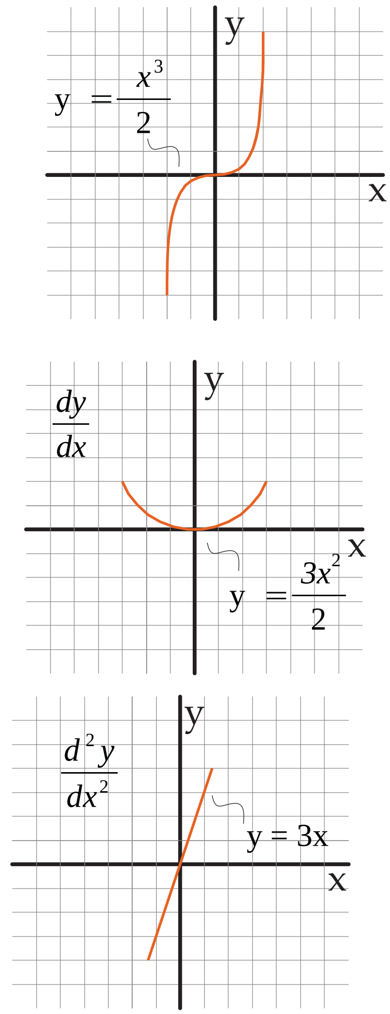


Fig.4b

Notice that:

a) $y = x^2$ increases where $\frac{dy}{dx}$ is positive and decreases where $\frac{dy}{dx}$ is negative; y is concave up since $\frac{d^2y}{dx^2}$ positive; and y has a relative minimum at x= 0 where $\frac{dy}{dx} = 0$. .

$y = x^3$ increases for all x since $\frac{dy}{dx}$ is positive for all x, y is concave down when x<0 and concave up when x > 0 since $\frac{d^2y}{dx^2}$ is negative for x < 0; and positive for x >0, y has an inflection point at x = 0 since $\frac{d^2y}{dx^2} = 0$ at x = 0 and the curve goes from concave down to concave up, y has a flat plateau at x = 0 since $\frac{dy}{dx} = 0$ at x = 0.

9.11 LIMITS

The concept of a *limit* pervades the subject of calculus even though it is often out of sight. For example, limits were hidden in the o(h) functions of this chapter and they always vanished in the computation of derivatives. Still we had to show that certain functions got nearer and nearer to 0 in terms of its decimal values as h got nearer to 0. All of our limits were computed as h "approached" 0. This is not entirely rigorous since by this approach, the computation of a limit would have to extend to an infinite process. However, this way of looking at limits is enough to satisfy our interests, and it is in the spirit of the founders of calculus who had to wait more than one hundred years before calculus was placed on an entirely rigorous and logical foundation.

What about limits of functions f(x) where x approaches some number other than 0? If the values of f(x) gets nearer and nearer in terms of decimal values to the number L as x gets nearer and nearer to x= a, we say,

$$\lim_{x \to a} f(x) = L$$

Example 7

Show that , $\lim_{x \to 1} (x^3 - 3x + 2)$ = 0.

Clearly as x gets close to x = 1, f(x) gets close to 0. In fact you can actually plug x = 1 into the function to get f(1) = 0. That was easy.

Example 8

Show that $\lim_{x \to 1} (\frac{x^2 - 1}{x - 1}) = 2$

This is not so obvious since if you plug x = 1 into this function you get $\frac{0}{0}$.which tells you nothing. There are two methods that we can use to find the limit.

Method 1:

You can solve this problem, as we did with o(h) functions, with a table

Table 4

x	f(x) = $\dfrac{x^2-1}{x-1}$
2	3
1.1	2.1
1.01	2.01
1.001	2.001

This certainly suggests that $\lim_{x \to 1}(\dfrac{x^2-1}{x-1}) = 2$

However, with a bit of algebra we can do even better.

Method 2:

Let,

$$f(x) = \frac{x^2-1}{x+1} = \frac{(x-1)(x+1)}{(x-1)}$$

So long as x does not equal 1 we can cancel (x-1) from numerator and denominator. But you will notice in the expression of a limit, x only gets closer and closer to 1 never actually having to equal it so we can do the canceling. This give the result,

$$f(x) = \frac{x^2-1}{x+1} = x+1$$

Now it is clear that as x gets nearer and nearer to x = 1 , f(x) gets nearer and near to exactly L = 2, so we can say, precisely, that,

$$\lim_{x \to 1}(\frac{x^2-1}{x-1}) = 2$$

Method 2 was elegant but it used a bit of algebraic trickery that was avoided by Method 1.

Sometimes we want to see what the function approaches as x approaches an arbitrarily large or small value, i.e., $x \to \pm\infty$.

Example: 9

Show that $\lim_{x \to \infty}(\dfrac{3x^2-5x-2}{2x^2+7}) = 3/2$

Method 1: Make a table of values where x gets large.

Table 5

x	$f(x) = (\frac{3x^2 - 5x + 2}{2x^2 + 7})$
10	1.217
100	1.474
1000	1.497
10000	1.499

It is apparent that $\lim_{x \to \infty} f(x) = 1.5$

Remark 2: There are two decimal representations of the rational number 3/2 either1.500… or 1.49999…

Method 2:

a) Divide numerator and denominator by the highest power of x (in this case x^2.), i.e.,

$$f(x) = \frac{3x^2/x^2 - 5x/x^2 + 2/x^2}{2x^2/x^2 + 7/x^2} = \frac{3 - 5/x + 2/x^2}{2 + 7/x^2}$$

b) Let x get large approaching ∞., then f(x) clearly approaches 3/2 = 1.5.

Method 3:

Since as x gets very large only the highest order power of x in the numerator and denominator matters, the others are insignificant by comparison. For example, 1000^3 is 1000 times greater than 1000^2. Therefore,

$$f(x) \approx \frac{3x^2}{2x^2} = \frac{3}{2} \ .$$

Remark 1: Method 3 is certainly the easiest way of evaluating infinite limits

Example 10:

Evaluate the following limit:

$$\lim_{x \to \infty} (\frac{3x^2 - 1}{5x^3 - 2x})$$

Using Method 3,

$$f(x) = \frac{3x^2 - 1}{5x^3 - 2x} \approx \frac{3x^2}{5x^3} = \frac{3}{5x}$$

Clearly $\lim_{x\to\infty} f(x) = 0$

Example 11:

$$\lim_{x\to\infty}\left(\frac{2x^4+3x^2-1}{5x^3-2x}\right)$$

Using Method 3,

$$f(x) = \frac{2x^4+3x^2-1}{5x^3-2x} \approx \frac{2x^4}{5x^3} = \frac{2x}{5}$$

As $x \to \infty$, f(x) gets arbitrarily large. Since f(x) does not approach any particular number we say that there is no limit. However, sometimes it makes sense to say that $f(x) \to \infty$ which gives us some information about the behavior of f(x) as x gets large.

Example 11:

Show that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

where e = 2.71828… an irrati onal number.

For this problem it is best to use a Table.

Table 6

n	f(n) = $(1+\frac{1}{n})^n$
1	2
10	2.593
100	2.705
1000	2.717
10000	2.718…

9.12 L'HOSPITAL'S RULE

We saw in Example 8 that if you insert x = 1 into $f(x) = \frac{x^2-1}{x+1}$, the result is $\frac{0}{0}$, and in Example 10 if you insert x = ∞ into $f(x) = (\frac{3x^2-5x+2}{2x^2+7})$, the result is $\frac{\infty}{\infty}$. In evaluating a limit, if the result is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then you can evaluate the limit by a theorem known as L'Hospital's Rule. What you do is to differentiate the numerator and then differentiate the denominator of the function. If this ratio of derivatives approaches a limiting value, that will also be the limit of the function.

Example 12

For example, applying L'Hospital's rule to $f(x) = \frac{x^2 - 1}{x + 1}$,

$$\lim_{x \to 1} f(x) = \frac{2x}{1} = 2$$

Example 13

Applying L'Hospital's Rule to $f(x) = (\frac{3x^2 - 5x + 2}{2x^2 + 7})$

$$\lim_{x \to \infty} f(x) = \frac{6x - 5}{4x}$$

But $\lim_{x \to \infty} \frac{6x - 5}{4x} = \frac{\infty}{\infty}$, so we can apply L'Hopital's Rule again to get,

$$\lim_{x \to \infty} \frac{6x - 5}{4x} = \lim_{x \to \infty} \frac{6}{4} = \frac{3}{2}$$

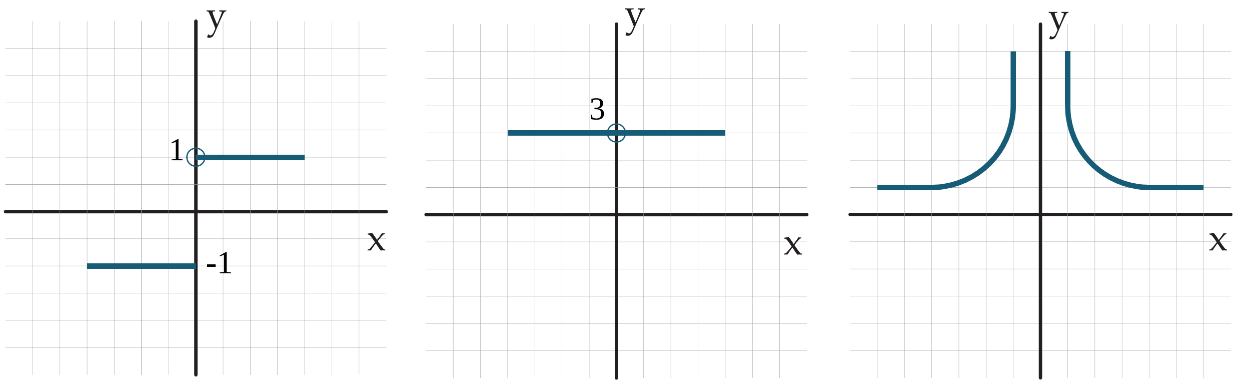
Example 14

Evaluate: $\lim_{x \to \infty} xe^{-x}$. If the function is rewritten as $f(x) = \frac{x}{e^x}$, then applying L'Hospital's Rule we have, $\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

9.13 CONTINUITY

The continuity of the graphs of functions is very much connected to the concept of limits.. Before giving a mathematical definition of a continuous function, I will first describe functions that are discontinuous.

There are three ways in which a function of one variable can be discontinuous. Examples are shown in Fig. 5 where the discontinuity of the function occurs at x = 0 in each case.



i) iii) ii)

Fig.5

Case i) f(x) = $\begin{cases} -1, & \text{for } x \le 0 \\ 1, & \text{for } x > 0 \end{cases}$

This function is well-defined at x = 0 where f(0) = -1, but it immediately jumps to f(x) = 1 when x is greater than 0. For this function, $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$ where $x \to 0^+$. means that x approaches 0 from values greater than 0 and $x \to 0^-$. means that x approaches zero from values less than 0. We have seen functions with such discontinuities in our evaluation of shear force for beams with concentrated forces. In order for a function to be continuous at x = a, it must be true that,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

Case ii) $f(x) = \dfrac{1}{x^2}$

This function explodes at x = 0. By this I mean that as you approach x = 0 from either the positive or negative side of x = 0 the function values get large beyond all limits. Sometimes we say that the value of the function at x = 0 approaches $+\infty$.

Remark 3: If the function under consideration is $f(x) = \dfrac{1}{x}$, $f(x) \to +\infty$. as $x \to 0^+$.

while $f(x) \to -\infty$. as $x \to 0^-$.

Case iii) $f(x) = \dfrac{3x}{x}$

This function has the value f(x) = 3 at all values of x except at x = 0 where it is not defined since its value is $f(0) = \dfrac{0}{0}$. So the function has a hole at x = 0, but since $\lim_{x \to 0} f(x) = 3$ the hole can easily be plugged by redefining the function to 3 when x = 0, i.e, f(0) = 3. The discontinuity in this case is said to be *removable*.

Remark 4: Clearly $\frac{0}{0}$ has no particular value. For the function in Case iii) $\frac{0}{0} = 3$. However, if our function was $f(x) = \frac{4x}{x}$, then we could say that $\frac{0}{0} = 4$. By the same reasoning $\frac{0}{0}$ can equal any number so we say that it is undefined.

We now have enough information to define a continuous function.

Definition: f(x) is continuous at x = a if and only if,

a) $\lim_{x \to a} f(x) = L$, and

b) L = f(a)

Example 15

By this definition $f(x) = x^3 - 3x + 2$ from Example 7 certainly passes the two continuity tests and is a continuous function.

Example 16

This function represents the bending moment of a beam.

$$f(x) = \begin{cases} 15x, & 0 \le x \le 6 \\ 60 + 5x, & 6 \le x \le 12 \\ 200 - 15x, & 12 \le x \le 20 \end{cases}$$

The function passes both tests for continuity so that it is a continuous function.

Example 17:

Consider $f(x) = \frac{x^2 - 1}{x - 1}$.

This not a continuous function because it fails test a) at x = 1. There is a hole in the function at x = 1 as seen in Fig. 6. However since $\lim_{x \to 1} f(x) = 2$ this point of discontinuity can be easily remedied by defining f(1) = 2.

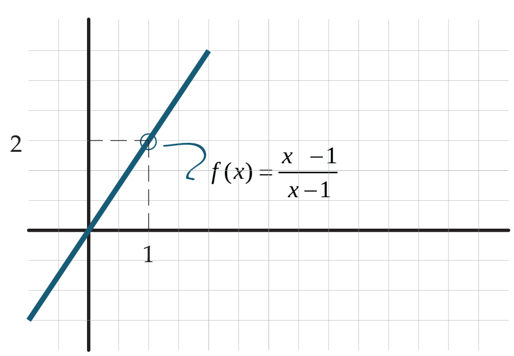


Fig.6