Physics 106 Week 3
Rotational Dynamics II
SJ 7th Ed.: Chap 10.7

• Newton’s Second Law for Rotation
• Problem solving methods

Examples.
• Heavy pulley Atwood’s machine
• Double pulley Atwood’s machine

Torque on extended object by gravitational force

Æ Assume that the total gravitational force effectively acts at the center of mass.

Gravitational potential energy of extended object
Æ M g H, where H is the height of the center of mass and M is the total mass.
Axis of rotation

Horizontal uniform rod of length L & mass M

Find the torque by gravitational force.

A. LMg
B. (L/2)Mg
C. 2LMg
D. (3/2)LMg
E. None of the above

\[ I_{cm,rod} = \frac{1}{12}ML^2 \]
Moments of Inertia of Various Rigid Objects

### TABLE 10.2

<table>
<thead>
<tr>
<th>Object Description</th>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid cylinder</td>
<td>$I = \frac{1}{2}Ml^2$</td>
</tr>
<tr>
<td>Hollow cylinder</td>
<td>$I = \frac{1}{2}Ml^2 + \frac{1}{2}Mr^2$</td>
</tr>
<tr>
<td>Rectangular plate</td>
<td>$I = \frac{1}{12}Mr^2$</td>
</tr>
<tr>
<td>Thin spherical shell</td>
<td>$I = \frac{2}{3}Mr^2$</td>
</tr>
</tbody>
</table>

**iClicker Q**

**Axis of rotation**

Horizontal uniform rod of length $L$ & mass $M$

Find the angular acceleration.

A. $(2/3)g/L$
B. $2(g/L^2)$
C. $g/L$
D. $(1/2)(g/L)$
E. $(3/2)(g/L)$
Example of energy conservation

Axis of rotation

Horizontal uniform rod of length L & mass M is released from rest.

Find its angular speed at the lowest point, assuming no friction between axis of rotation and the rod.

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Axis of rotation

Horizontal uniform massless rod of length L

Uniform circular disk of diameter L & mass M

Find the torque by gravitational force.

A. LMg
B. (L/2)Mg
C. 2LMg
D. (3/2)LMg
E. None of the above
Example

A thin uniform rod (length = 1.2 m, mass = 2.0 kg) is pivoted about a horizontal, frictionless pin through one end of the rod. (The moment of inertia of the rod about this axis is ML$^2$/3.) The rod is released when it makes an angle of 37° with the horizontal. What is the angular acceleration of the rod at the instant it is released?

Example

Axis of rotation

Horizontal uniform rod of length L & mass M

Uniform circular disk of diameter L & mass M

Find the net torque.
Find the total moment of inertia
Find the angular acceleration.
Example: Torque and Angular Acceleration of a Wheel

- Cord wrapped around disk, hanging weight
- Cord does not slip or stretch
- Let \( m = 1.2 \text{ kg}, M = 2.5 \text{ kg}, r = 0.2 \text{ m} \)
- Find acceleration of mass \( m \), find angular acceleration \( \alpha \) for disk, tension, and torque on the disk

Formula: \( I = \frac{1}{2} Mr^2 \)

Example: Torque and Angular Acceleration of a Wheel

Analysis approach:

- Break into two sub-systems
  - Wheel is accelerated angularly by tension \( T \)
  - Block is accelerated linearly by weight mg, with tension opposing

- Free body diagrams shown

- The wheel is rotating and so we apply \( \Sigma \tau = I \alpha \)
  - The tension supplies the torque via tangential force

- The mass is moving in a straight line, so apply Newton's Second Law \( \Sigma F_y = ma = mg - T \)

- How to connect the two problems above?
  - Need a constraint linking linear acceleration to \( \alpha \)
    \( a = \alpha r \)
Example: Heavy pulley Atwood’s machine
application of Newton’s 2nd Law

Given the numerical values:
• \( m_1 = 5.0 \text{ kg} \),
• \( m_2 = 5.5 \text{ kg} \),
• \( r = 0.2 \text{ m} \),
• \( M = 6.0 \text{ kg} \),
• \( I = 0.20 \text{ kg-m}^2 \)

a) Find the accelerations of the hanging weights. Are they up or down?
b) Find the angular acceleration of the pulley. Is it CW or CCW?
c) Find the tensions in each cord.
d) How long does it take the 5.5 kg weight to fall 0.6 m from rest?

Strategy:
Apply Newton’s Second Law (linear and rotational form)
to all three bodies, connected by constraints. Make sure there are as many independent equations as there are unknowns

Heavy pulley Atwood’s machine: solution
Apply Second Law:

(I) \( \sum F_i = ma \) (each component)

(II) \( \sum \tau = I \alpha \) (pulley)
If \( I = 0 \) problem simplifies

For \( m_1 \) use I:
\[
m_1 a_1 = T_1 - m_1 g
\]
1. \( T_1 = m_1 (a_1 + g) \)

• positive \( \alpha \) \( \Rightarrow \) negative \( a_1 \) (down)

For \( m_2 \) use I again:
\[
m_2 a_2 = T_2 - m_2 g
\]
2. \( T_2 = m_2 (a_2 + g) \)

• positive \( \alpha \) \( \Rightarrow \) positive \( a_2 \) (up)

Constraints: cord cannot stretch or slip
\[
a_2 = -a_1
\]
\[
a_2 = +\alpha r \quad a_1 = -\alpha r
\]

\[
a_2 = \frac{m_1 - m_2 g}{m_1 + m_2}
\]
only 2 of these are independent

Note: if \( I = 0 \), then \( T_1 = T_2 \)
Result using 1 & 2 becomes:
For the Pulley use II:

\[ \tau_{\text{net}} = + T_1 r - T_2 r I = I \alpha \]

3. \[ \alpha = \frac{(T_1 - T_2) r}{I} \]

- positive \( \alpha \) \( \Rightarrow \) \( T_1 > T_2 \)

Check ability to solve:
- 5 unknowns: \( a_1, a_2, T_1, T_2, \alpha \)
- Have 5 independent equations: 1, 2, 3 and 2 constraints

**OK!**

**Atwood’s solution, continued**

**Atwood’s solution, numerical evaluation**

\[ \alpha = \frac{(5.0 - 5.5)(9.8)(0.2)}{0.20 + (5.0 - 5.5)0.2^2} \]

- \( \alpha = -1.58 \text{ rad/s}^2 \) (CW)

- \( a_1 = -\alpha r = +0.32 \text{ m/s}^2 \) (up)
- \( a_2 = -\alpha r = -0.32 \text{ m/s}^2 \) (down)

- \( T_1 = 5.0 \text{ (9.8 + 0.32)} \)
- \( T_2 = 5.5 \text{ (9.8 - 0.32)} \)

- \( T_1 = 506 \text{ N.} \) (\( a_1 \) increases tension)
- \( T_2 = 521 \text{ N.} \) (\( a_2 \) decreases tension)

How long does \( m_2 \) take to fall 0.6 m from rest?

\[ d = \frac{1}{2} a_2 t^2 \]

- \( t = \sqrt{\frac{2d}{a_2}} = 1.94 \text{ s} \)
Example: Two-level-pulley Atwood’s machine

Assume:
- \( a_A = +2 \text{ m/s}^2 \) (CCW)
- \( r_A = 2 \text{ cm}, \ r_B = 5 \text{ cm} \)

Find \( \alpha \) for the pulley:

Find \( a_B \) (mass B linear acceleration):

No slipping on double pulley means:
- \( \alpha r_A = a_A \)
- \( \alpha r_B = a_B \)
- \( \alpha \) is the same for A & B
- but \( a_A \) is different from \( a_B \)

Example: no-slipping conditions on a double pulley

No slipping on double pulley means:
- \( \alpha r_A = a_A \)
- \( \alpha r_B = a_B \)
- \( \alpha \) is the same for A & B
- but \( a_A \) is different from \( a_B \)

Example:
Assume:
- \( a_A = +2 \text{ m/s}^2 \) (CCW)
- \( r_A = 2 \text{ cm}, \ r_B = 5 \text{ cm} \)

Find \( \alpha \) for the pulley:

\[
\alpha = \frac{a_A}{r_A} = \frac{2}{0.02} = 100 \text{ rad/s}^2 \text{ (CCW)}
\]

Find \( a_B \) (mass B linear acceleration):

\[
a_B = \alpha r_B = \frac{a_A r_B}{r_A} = \frac{2 \times 5}{2} = 5 \text{ m/s}^2
\]

What if there IS slippage? Can not connect \( a \)'s with \( \alpha \)
Two-level-pulley Atwood’s machine
- application of Newton’s 2nd Law

For the “two-level-pulley” Atwood’s machine, two cords connect to the 10 kg pulley, whose angular acceleration is 2 rad/s² clockwise. The tension in the cord attached to mass B is 50 N

a) Draw free body diagrams
b) and apply Newton’s second law to all three bodies.
c) Calculate the rotational inertia of the pulley from the dynamics and conditions given
d) Find the unknown mass \( m_B \).

Strategy - same as before:
Apply Newton’s Second Law (linear and rotational) applying it to all three bodies, connected by constraints

Numerical values:
\( m_A = 2.0 \text{ kg} \), \( m_B = \text{ ??} \)
\( r_A = 60 \text{ cm} \), \( r_B = 40 \text{ cm} \)
\( M_P = 10 \text{ kg} \) \( I = \text{ ??} \)
\( \omega = -2 \text{ rad/s}^2 \text{ (CW)} \)
\( T_B = 50 \text{ N} \)

Problem 16.35

Given: \( 3 \) Pulley

Free Body Dynamics:

mass A:
\[
\sum F_t = T_A - T_B - M_A g = M_A a_A
\]
\[
\sum F_n = 0
\]

mass B:
\[
\sum F_t = T_B - M_B g = M_B a_B
\]
\[
\sum F_n = 0
\]
pulley:
\[
\sum F_{rot} = 0 \quad \text{Pivot force (not shown)}
\]
\[
\sum T = I_p \frac{d\omega}{dt}
\]
No slipping constraints:

\[ a_A = -a_B \quad \text{if } a \text{ is ccw } a_A \text{ is down} \]
\[ a_A = +a_B \quad \text{if } a \text{ is ccw } a_B \text{ is up} \]

5 equations
5 unknowns \( T_A, a_A, m_B, a_B, I_p \)

All is well
Problem can be solved

\( \bullet \) Find \( I_p \):

\[ I_p = \frac{T_A r_A - T_B r_B}{\alpha} \]

Use \( r_A \):

\[ I_p = m_B (m_A r_A^2 + m_B r_B^2) - m_B (m_A r_A + m_B r_B) \]

Redo:

\[ I_p = \frac{-m_B (m_A + m_B) r_A^2 + \frac{g}{\alpha} (m_A r_A - m_B r_B)}{\alpha} \]

In the above:
- don’t know \( M_B \) yet
- other quantities are known
- first term is just the rotational inertia of the hanging masses

Side remark:

\( \alpha = \frac{(M_a - M_b) g}{I_B + (M_a + M_b) r^2} \)

\( \frac{\text{SAME RESULT}}{\text{AS LAST EXAMPLE!}} \)

Find \( M_B \) using equation 2:

\[ T_B = M_B (a_B - g) \]
\[ M_B = \frac{T_B}{(a_B - g)} = \frac{50}{-2} = -25 + 98 \]
\[ M_B = 5.56 \text{ kg} \]

Now find the numerical value of \( I_p \):

\[ I_p = -\frac{(2 \times 56 + 5.56 \times 9.8^2 - 2 \times 2 \times 56 \times 9.8)}{2} \]
\[ I_p = 1.61 + 5.00 \]
\[ I_p = 3.6 \text{ kg} \cdot m^2 \]