Final Exam

During class (1 - 3:55 pm) on 6/27, Mon
Room: 412 FMH (classroom)

Bring scientific calculators
No smart phone calculators are allowed.
Exam covers everything learned in this course.

Review session: Thursday during class

Rotational Motion

-- Torque
-- Kinetic energy, potential energy, mechanical energy for rotation
-- Conservation of mechanical energy for rotation
-- Angular momentum
-- Conservation of angular momentum
How to find torque

From figure,

\[ \tau = rF_T = rF \sin \theta = r_p F \]

Line of action

If we know components

\[ \vec{r} = (r_x, r_y, 0) \quad \vec{F} = (F_x, F_y, 0) \]

\[ \vec{\tau} = (0, 0, r_x F_y - r_y F_x) \]

Reminder: Rotational variables are vectors, having direction

The angular displacement, speed, and acceleration

\( (\theta, \omega, \alpha) \)

are vectors with direction.

The directions are given by the right-hand rule:

Fingers of right hand curl along the angular direction (See Fig.)

Then, the direction of thumb is the direction of the angular quantity.
iClicker quiz

All forces have the same magnitude. Which force has the greatest magnitude of torque with respect to the pivot?

(a) A  
(b) B  
(c) C  
(d) All forces have the same torque  
(e) Not enough information

5.1. A particle located at the position vector \( \vec{r} = (1m, 2m) \) has a force \( \vec{F} = (2N, 3N) \) acting on it. The torque in N.m about the origin is?
Gravitational force on extended objects

**Torque** on extended object by gravitational force

→ Assume that the total gravitational force effectively acts at the center of mass.

**Gravitational potential energy** of extended object

→ $MgH$, where $H$ is the height of the center of mass and $M$ is the total mass.

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**iClicker Q**

Axis of rotation

Horizontal uniform rod of length $L$ & mass $M$

Find the torque by gravitational force.

- A. $LMg$
- B. $(L/2)Mg$
- C. $2LMg$
- D. $(3/2)LMg$
- E. None of the above

Find the angular acceleration. $\tau_{end,rod} = \frac{1}{3}ML^2$
Example: Torque and Angular Acceleration of a Wheel

- Cord wrapped around disk, hanging weight
- Cord does not slip or stretch
- Let \( m = 1.2 \text{ kg} \), \( M = 2.5 \text{ kg} \), \( r = 0.2 \text{ m} \)
- Find acceleration of mass \( m \), find angular acceleration \( \alpha \) for disk, tension, and torque on the disk

Formula: \( I = \frac{1}{2} Mr^2 \)

Example: A uniform circular disk of radius \( r \) and mass \( M \) is pulled by constant horizontal force \( F \) applied to the center of mass, and is rolling without slipping. \( M = 2 \text{ kg} \), \( r = 0.5 \text{ m} \), \( I_{(cm, disk)} = \frac{1}{2}MR^2 \), \( F = 5 \text{ N} \).

a) Find the angular acceleration.
b) Find the static friction force.

iClicker: If there’s no friction from the surface on the disk, and if the disk is initially not rotating, would it rotate about center of mass by applied force?

A) Yes
B) No
Kinetic energy for rotation

If the rigid body rotates with respect to an axis fixed in space

\[ K = \frac{1}{2} I \omega^2 \]

### Kinetic energy for rotation

If the rigid body rotates with respect to a moving axis through the center of mass, such as rolling

\[ K = K_{rot} + K_{cm} \]

\[ K_{rot} = \frac{1}{2} I_{cm} \omega^2 \]

\[ K_{cm} = \frac{1}{2} M v_{cm}^2 \]

For rolling without slipping

\[ \Delta s = R \Delta \theta \]

\[ v_{cm} = R \omega \]
For any rotational or translational motion,

\[ U_{\text{gravity}} = Mgh_{cm} \]

\[ E_{\text{mech}} = K + U_g \]

\[ \Delta E_{\text{mech}} = W_{\text{non-conservative}} \]

If \( W_{\text{non-conservative}} = 0 \) , then \( E_{\text{mech}} \) is conserved.

**Example of energy conservation**

Axis of rotation

Horizontal uniform rod of length L & mass M is released from rest.

Find its angular speed at the lowest point, assuming no friction between axis of rotation and the rod.
Example: Use energy conservation to find the speed of the bowling ball as it rolls w/o slipping to the bottom of the ramp

Given: \( h = 2 \) m

- Formula: For a solid sphere
  \[ I_{cm} = \frac{2}{5} MR^2 \]

Hint:
- Rotation accelerates if there is friction between the sphere and the ramp
- Friction force produces the net torque and angular acceleration.
- There is no mechanical energy change because the contact point is always at rest relative to the surface, so no work is done against friction

iClicker Q:
A solid sphere and a spherical shell of the same radius \( r \) and same mass \( M \) roll to the bottom of a ramp without slipping from the same height \( h \).

True or false? : “The two have the same speed at the bottom.”

A) True
B) False. Shell is faster.
C) False. Solid sphere is faster.
D) Not enough information.

\[ I_{cm, \text{ spherical shell}} = \frac{2}{3} MR^2 \]
\[ I_{cm, \text{ solid sphere}} = \frac{2}{5} MR^2 \]
Angular momentum – concepts & definition

- Linear momentum:
  \[ p = mv \]

- Angular (Rotational) momentum:
  \[ L = \text{moment of inertia} \times \text{angular velocity} = I\omega \]

Angular momentum of a bowling ball

6.1. A bowling ball is rotating as shown about its mass center axis. Find its angular momentum about that axis, in kg.m²/s

A) 4   B) \( \frac{1}{2} \)   C) 7   D) 2   E) \( \frac{1}{4} \)

\[ \omega = 4 \text{ rad/s} \]
\[ M = 5 \text{ kg} \]
\[ r = \frac{1}{2} \text{ m} \]
\[ I = \frac{2}{5} MR^2 \]

\[ L = I\omega \]
Angular momentum of a point particle

\[ L = I \omega = m r^2 \omega = m v_T r = m v r \sin \theta = m v r_p = p_T r = p r_p \]

\[ \mathbf{v}_r = \omega \mathbf{r} \]

Note: \( L = 0 \) if \( \mathbf{v} \) is parallel to \( \mathbf{r} \) (radially in or out)

\[ L = I \omega = m v_T r = m v r \sin \theta = m v r_p = r p_T = r p \sin \theta = r_p p \]

Angular momentum of a point particle

If we know components

\[ \mathbf{r} = (r_x, r_y, 0) \quad \mathbf{p} = (p_x, p_y, 0) \]

\[ \mathbf{L} = (0, 0, r_x p_y - r_y p_x) \]
Angular momentum for car

5.2. A car of mass 1000 kg moves with a speed of 50 m/s on a circular track of radius 100 m. What is the magnitude of its angular momentum (in kg \( \cdot \) m\(^2\)/s) relative to the center of the race track (point “P”)?

A) \(5.0 \times 10^2\)  
B) \(5.0 \times 10^6\)  
C) \(2.5 \times 10^4\)  
D) \(2.5 \times 10^6\)  
E) \(5.0 \times 10^3\)

5.3. What would the angular momentum about point “P” be if the car leaves the track at “A” and ends up at point “B” with the same velocity?

A) Same as above  
B) Different from above  
C) Not Enough Information

\[ L = p \times r \sin(\theta) \]

Net angular momentum of particles

\[ L_{net} = L_1 + L_2 + L_3 + \ldots \]
Example: calculating angular momentum for particles

PP10602-23*: Two objects are moving as shown in the figure. What is their total angular momentum about point O?

\[
\tau_{\text{net}} = I \alpha = I \frac{\Delta \omega}{\Delta t} = \frac{\Delta L}{\Delta t} \quad \Delta L = \tau_{\text{net}} \Delta t
\]

Conservation of angular momentum:

Angular momentum, \( L \) is conserved if \( \tau_{\text{net}} = 0 \)

Conservation of net angular momentum for a system

Net angular momentum, \( L_{\text{net}} \) is conserved if \( \tau_{\text{net,ext}} = 0 \)
A puck on a frictionless air hockey table has a mass of 5.0 g and is attached to a cord passing through a hole in the surface as in the figure. The puck is revolving at a distance 2.0 m from the hole with an angular velocity of 3.0 rad/s. The cord is then pulled from below, shortening the radius to 1.0 m. The new angular velocity (in rad/s) is ________.

\[
\tau_{net,ext} = 0 \quad \Rightarrow \quad L = \text{constant}
\]

\[
I_i \omega_i = I_f \omega_f
\]

**Demonstration: Spinning Professor:** [Web link]

**Isolated System**

**Moment of inertia changes** [Another link]
Internal torques do not change total angular momentum...  
... it is redistributed within the isolated system  
\[ \vec{\tau}_{\text{net}} = 0 \quad \text{about z-axis} \quad \Rightarrow \quad \vec{L}_{\text{sys}} = \text{constant} \]

**Demonstration:** Weblink, more

Internal torques not 0  
they reverse \( L_{\text{wh}} \)

Spacecraft maneuvers by  
spinning the flywheel  
the craft counter-rotates

\[ \vec{L}_{\text{boy,final}} = 2\vec{L}_{\text{wheel}} \]

\[ \vec{L}_{\text{tot}} = 0 = \vec{L}_{\text{craft}} + \vec{L}_{\text{flywheel}} \]

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**Tethered Astronauts**

- **6.3.** Two astronauts each having mass \( M \) are connected by a mass-less rope of length \( d \). They are isolated in space, orbiting their center of mass at identical speeds \( v \). By pulling on the rope, one of them shortens the distance between them to \( d/2 \). What are the new angular momentum \( L' \) and speed \( v' \)?

A) \( L' = mvd/2, \; v' = v/2 \)  
B) \( L' = mvd, \; v' = 2v \)  
C) \( L' = 2mvd, \; v' = v \)  
D) \( L' = 2mv'd, \; v' = v/2 \)  
E) \( L' = mvd, \; v' = v/2 \)

\[ L = I\omega \]
\[ L = mvr \]
Example: A merry-go-round problem

A 40-kg child running at 4.0 m/s jumps tangentially onto a stationary circular merry-go-round platform whose radius is 2.0 m and whose moment of inertia is 20 kg·m².

Find the angular velocity of the platform after the child has jumped on.