Real Analysis: Solved Problems From Royden & Fitzpatrick 4th Edition

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Problem 50

A function $f : E \to \mathbb{R}$ is Lipschitz if $\exists\ c \geq 0$ for which $|f(x) - f(y)| \leq c|x - y|, \forall\ x, y \in E$, but whenever $|x - y| < \delta$ we have that $|f(x) - f(y)| \leq c|x - y| < c\delta = \epsilon, \forall x, y \in E$. Therefore, f is uniformly continuous on E. In order to show that there are uniformly continuous functions that are not Lipschitz we just have to find or create such a function. Consider $f(x) = \sqrt{x}, 0 \leq x \leq 1$, which is uniformly continious, since every continuous function on a closed and bounded interval is uniformly continuous, but we can also show that is actually uniformly continuous, as $\forall x, y \in [0, 1]$,

$$\begin{aligned} |f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| &= \sqrt{|\sqrt{x} - \sqrt{y}||\sqrt{x} - \sqrt{y}|} \\ &\leq \sqrt{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|} = \sqrt{|x - y|} \end{aligned}$$

Therefore, whenever $|x - y| < \delta$ we have that $|f(x) - f(y)| \leq \sqrt{|x - y|} < \sqrt{\delta} = \epsilon$. Since, δ does not depend on x, y, f is uniformly continuous. Suppose, that f is Lipschitz then $\exists\ c \geq 0 : \forall x, y \in E$

$$\begin{aligned} |f(x) - f(y)| < c|x - y| \iff c \geq \frac{|f(x) - f(y)|}{|x - y|} &= \frac{|\sqrt{x} - \sqrt{y}|}{|x - y|} = \frac{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|}{|x - y||\sqrt{x} + \sqrt{y}|} \\ &= \frac{1}{|\sqrt{x} + \sqrt{y}|} \end{aligned}$$

But the quantity $\frac{1}{|\sqrt{x}+\sqrt{y}|}$ is unbounded on [0,1] and thus we can make it as large as we want so that there is no c that can exceed it. Hence, f is not Lipschitz.

Problem 53

$\rightarrow$ From Heine-Borel theorem we know that if a set E is closed and bounded every open cover of E has a finite subcover.

$\leftarrow$ Let $\mathcal{H}$ be any collection of open sets such that $E \subset \cup_{H \in \mathcal{H}} H$. We assume that $\exists\ n \in \mathbb{N} : E \subset \cup_{i=1}^{n} H_i,\ H_i \in \mathcal{H}, i = 1, 2, ..., n.$. Define, $\mathcal{H} = \{x \in E : x - \epsilon < x < x + \epsilon\}$ which is an open cover for E. By assumption, there is a finite subcover, $H_i = \{x \in E : x_i - \epsilon < x < x_i + \epsilon\} : E \subset \cup_{i=1}^{n} \{x \in E : x_i - \epsilon < x < x_i + \epsilon\}$. Then, $E \subset (\min_{i=1,..,n} x_i - \epsilon, \max_{i=1,..,n} x_i + \epsilon)$. Therefore, $\forall x \in E \min_{i=1,..,n} x_i - \epsilon < x < \max_{i=1,..,n} x_i + \epsilon$. Hence, E is bounded.

Suppose that E is not closed. Therefore, it doesn't contain all its limit points. We assume, without loss of generality that it doesn't contain one limit point say x_0. Let, $H = (-\infty, x_0 - \frac{1}{n}) \cup (x_0 + \frac{1}{n}, \infty)$ be an open cover for E. Then, there is a finite subcover of H such that $E \subset \cup_{n=1}^{m} (-\infty, x_0 - \frac{1}{n}) \cup (x_0 + \frac{1}{n}, \infty) \implies E \subset (-\infty, x_0 - \frac{1}{m}) \cup (x_0 + \frac{1}{m}, \infty)$. By the density of the reals in between $(x_0 - \frac{1}{m}, x_0)\ \exists\ \alpha \in E$ which is not covered by the finite subcover. Therefore, there is no finite subcover of E, a contradiction, because we assumed that E is closed.

Hence, E is closed and bounded.

Problem 1

The idea is to decompose B into a countable union of disjoint sets and at least one of these set has to be A(so we can have $m(A)$), in order to use the countably additivity over countable disjoint unions property. We can decompose B as $B = A \cap (B - A)$. The two sets A and $B - A$ are obviously disjoint and the union is countable therefore,

$$m(B) = m\bigg(A \cap \{B - A\}\bigg) \overset{disj.\ count.\ union}{=} m(A) + m(B - A) \overset{m(B-A)\geq 0}{\geq} m(A)$$

Problem 4

The counting measure is translation invarient because by shifting the elements on a set, by a constant, doesn't change the number of elements. Let, $\{E_n\}_{n=1}^{\infty}$ be a countable and disjoint collection of sets. Furthermore, we know that the union of a countable collection of countable sets is countable and so $\cup_{i=1}^{\infty} E_n)$ is countable.

Case 1: If all of E_n are empty and so their union and trivially $c(\cup_{i=1}^{\infty} E_n) = \sum_{n=1}^{\infty} c(E_n) = 0$.

Case 2: If at least one of the E_n has infinitely many memebers ($c(E_n) = \infty$) and so the union. Then, $c(\cup_{i=1}^{\infty} E_n) = \infty$ and $\sum_{n=1}^{\infty} c(E_n) \overset{c(.)\geq 0}{=} \infty$. Therefore, $c(\cup_{i=1}^{\infty} E_n) = \sum_{n=1}^{\infty} c(E_n)$.

Case 3: If all of the E_n are finite and non-empty then $c(\cup_{i=1}^{\infty} E_n) = \infty$ as $\cup_{i=1}^{\infty} E_n$ is a countably infinte set. Furthermore, $\sum_{i=1}^{\infty} E_n = \infty$ as a countable infinite sum. Therefore, $c(\cup_{i=1}^{\infty} E_n) = \sum_{n=1}^{\infty} c(E_n)$.

Case 4: If $\{E_n\}_{n=1}^{\infty}$ are finite and say, without loss of generality, the first m out of them are non-empty, and the rest empty, and let $n_i = number\ of\ elements\ of\ E_i$. Then, the number of elements of $\cup_{i=1}^{\infty} E_n = \cup_{n=1}^{m} E_n \cup_{n=m+1}^{\infty} E_n$ are $n_1 + ... + n_m \implies c(\cup_{n=1}^{\infty} E_n) = n_1 + ... + n_m$. Furthermore, $\sum_{n=1}^{\infty} c(E_n) = \sum_{i=1}^{m} c(E_n) + \sum_{n=m+1}^{\infty} c(E_n) \overset{c(\varnothing)=0}{=} \sum_{n=1}^{m} c(E_n) + \sum_{n=m+1}^{\infty} 0 = n_1 + ... + n_m$. Therefore, $c(\cup_{i=1}^{\infty} E_n) = \sum_{n=1}^{\infty} c(E_n)$.

Hence, the counting measure is countably additive and translation invariant.

Problem 6

We know that the set of rational numbers Q is countable and $Q \cap [0, 1] \subset Q$. Therefore, $Q \cap [0, 1]$ is countable. Furthermore, for any countable sets we know that its outer measure is 0. So, $m^*(Q \cap [0, 1]) = 0$. Also, we can decompose $[0, 1]$ as $[0, 1] = \{Q^c \cap [0, 1]\} \cup \{Q \cap [0, 1]\}$. By the countable sub-additivity of the outer measure me have that

$$m^*([0, 1]) = m^*(\{Q^c \cap [0, 1]\} \cup \{Q \cap [0, 1]\}) \leq m^*(\{Q^c \cap [0, 1]\}) + m^*(\{Q \cap [0, 1]\}) \qquad (1)$$

But the outer measure of an interval is its length and the outer measure of a countable set is zero. Therefore, (1) takes the form

$$m^*(\{Q^c \cap [0, 1]\}) \geq 1 \qquad (2)$$

Furthermore, $Q^c \cap [0,1] \subset [0,1]$ and by the monotonicity of the outer measure

$$Q^c \cap [0,1] \subset [0,1] \implies m^*(Q^c \cap [0,1]) \leq m^*([0,1]) = 1 \tag{3}$$

Hence, combining (2) and (3) $m^*(Q^c \cap [0,1]) = 1$

Problem 11

We know that σ-algebra is closed under compliments and countable unions. Let $\mathcal{A}$ be the σ-algebra

$$(a,\infty) \in \mathcal{A} \overset{compliment}{\Longrightarrow} (-\infty, a] \in \mathcal{A} \overset{count.union}{\Longrightarrow} \bigcup_{n=1}^{\infty} \Big(-\infty, a - \frac{1}{n}\Big] = (-\infty, a) \in \mathcal{A} \overset{compliment}{\Longrightarrow} [a,\infty) \in \mathcal{A}$$

$$(-\infty, a), (b,\infty) \in \mathcal{A} \overset{count.union}{\Longrightarrow} (-\infty, a) \cup (b,\infty) \in \mathcal{A} \overset{compliment}{\Longrightarrow} [a,b] \in \mathcal{A}$$

$$(-\infty, a], (b,\infty) \in \mathcal{A} \overset{count.union}{\Longrightarrow} (-\infty, a] \cup (b,\infty) \in \mathcal{A} \overset{compliment}{\Longrightarrow} (a,b] \in \mathcal{A}$$

$$(-\infty, a), [b,\infty) \in \mathcal{A} \overset{count.union}{\Longrightarrow} (-\infty, a) \cup [b,\infty) \in \mathcal{A} \overset{compliment}{\Longrightarrow} [a,b) \in \mathcal{A}$$

$$(-\infty, a], [a,\infty) \in \mathcal{A} \overset{count.inters}{\Longrightarrow} (-\infty, a] \cap [a,\infty) = \{a\} \in \mathcal{A}$$

Hence, if a σ algebra $\mathcal{A}$ containts intevrals of the form (a,∞) then it contains all type of intervals.

Problem 18
Let $\{I_k\}_{k=1}^{\infty}$ be a countable collection of open intervals that covers E. Then, $\forall\ \epsilon > 0$

$$\sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon$$

Define $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. Then, $\mathcal{O}$ is open set containing E. By the definition of the outer measure of $\mathcal{O}$,

$$m^*(\mathcal{O}) \leq \sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon, \forall\ \epsilon > 0 \implies m^*(\mathcal{O}) \leq m^*(E) \tag{4}$$

Let $\{\mathcal{O}_n\}_{n=1}^{\infty}$ be a countable collection of such sets. Then we define $G = \cap_{n=1}^{\infty} \mathcal{O}_n$, which is a G_δ set and so is measurable. Observe, that G is also an open covering for E. By the monotonicity of the outer measure,

$$E \subset G \implies m^*(E) \leq m^*(G) \tag{5}$$

On the other hand $G \subset \mathcal{O}$, because if $x \in G$ then x is in every set $\cup_{k=1}^{\infty} I_k$ and so in $\mathcal{O}$. By monotonicity

$$m^*(G) \leq m^*(\mathcal{O}) \overset{(1)}{\leq} m^*(E) \tag{6}$$

Therefore, by (2) and (3) $m^*(E) = m^*(G)$, where G is a G_δ set that contains E. From the Inner approximation by closed and F_σ sets, E is measurable if $\exists\ F \in F_\sigma : F \subset E : m^*(E - F) = 0$. Furhermore, F has a finite outer measure since by monotonicity $F \subset E \implies m^*(F) \leq m^*(E) < \infty$. Therefore, by excision property

$$0 = m^*(F - E) = m^*(F) - m^*(E) \implies m^*(E) = m^*(F)$$

$\square$

Problem 24

We can decompose E_1 and E_2 as the union of disjoint set and because both of them are measurable we can use the countable additivity proberty of the measure

$$\begin{aligned} E_1 = \{E_1 - E_2\} \cup \{E_1 \cap E_2\} &\implies m(E_1) = m(E_1 - E_2) + m(E_1 \cap E_2) \\ &\implies m(E_1 - E_2) = m(E_1) - m(E_1 \cap E_2) \qquad (7) \\ E_2 = \{E_2 - E_1\} \cup \{E_1 \cap E_2\} &\implies m(E_2) = m(E_2 - E_1) + m(E_1 \cap E_2) \\ &\implies m(E_2 - E_1) = m(E_2) - m(E_1 \cap E_2) \qquad (8) \end{aligned}$$

Furthermore, because E_1, E_2 are measurable and so is their union and can decompose it as the union of disjoint sets where we can apply also the countable additivity property of the measure

$$\begin{aligned} E_1 \cup E_2 &= \{E_1 - E_2\} \cup \{E_1 \cap E_2\} \cup \{E_2 - E_1\} \\ m(E_1 \cup E_2) &= m(E_1 - E_2) + m(E_1 \cap E_2) + m(E_2 - E_1) \\ &\overset{(1),(2)}{\implies} m(E_1 \cup E_2) = m(E_1) - m(E_1 \cap E_2) + m(E_2) \end{aligned}$$

Note: Each of the decomposed sets belong to the σ-algebra as they can be formed by unions, intersections and compliments of the measurable sets E_1, E_2 and so they are measurable.

$\square$

Problem 26

We can write the set $\left\{A \cap \bigcup_{k=1}^{\infty} E_k\right\}$ as $\bigcup_{k=1}^{\infty}\{A \cap E_k\}$. Therefore, by the sub-additivity property of the outer measure we have

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty}\{A \cap E_k\}\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k) \qquad (9)$$

On the other hand the finite union $\bigcup_{k=1}^{n}\{A \cap E_k\}$ is a subset of the countable union $\bigcup_{k=1}^{\infty}\{A \cap E_k\}$ and by the monotonicity property of the outer measure

$$\begin{aligned} \bigcup_{k=1}^{n}\{A \cap E_k\} &\subset \bigcup_{k=1}^{\infty}\{A \cap E_k\} \\ m^*\left(\bigcup_{k=1}^{\infty}\{A \cap E_k\}\right) &\geq m^*\left(\bigcup_{k=1}^{n}\{A \cap E_k\}\right), \ \textit{for each } n \\ &= \sum_{k=1}^{n} m^*(A \cap E_k), \ \textit{for each } n \qquad \left(\{E_k\}_{k=1}^{\infty} \textit{ countable disjoint}\right) \end{aligned}$$

The left hand side of this inequality is independent of n. Therefore,

$$m^*\left(\bigcup_{k=1}^{\infty}\{A\cap E_k\}\right) \geq \sum_{k=1}^{n} m^*(A\cap E_k) \tag{10}$$

Combining (6) and (7) we have

$$m^*\left(\bigcup_{k=1}^{\infty}\{A\cap E_k\}\right) = \sum_{k=1}^{n} m^*(A\cap E_k)$$

□

Problem 28

Without loss of generality, let $\{A_k\}_{k=1}^{\infty}$ be a collection of disjoint measurable sets, if they were not disjoint we can always construct a disjoint collection. In order to use the continuity of the measure we need somehow to contstuct either an ascending or descending set. Let, $C_k = \bigcup_{i=1}^{k} A_i$, which is obviously ascending. Furthermore, the set $\bigcup_{k=1}^{\infty} C_k$ is equal to $\bigcup_{k=1}^{\infty} A_k$. Therefore,

$$\begin{aligned} m\left(\bigcup_{k=1}^{\infty} A_k\right) &= m\left(\bigcup_{k=1}^{\infty} C_k\right) \\ &= \lim_{k\to\infty} m(C_k) \ (\textit{continuity of measure}) \\ &= \lim_{k\to\infty} m(\bigcup_{i=1}^{k} A_i) \\ &= \lim_{k\to\infty} \sum_{i=1}^{k} m(A_i) \ (\textit{finite additivity}) \\ &= \sum_{i=1}^{\infty} m(A_i) \end{aligned}$$

□

Problem 27

Consider the sequence of measurable functions $\{f_n\}$ such that $f_n(x) = \chi_{(n,\infty)}(x), \forall x \in \mathbb{R}$. Observe that $\{f_n\} \overset{p.w.}{\to} f$ where $f(x) = 0, \forall x \in E = \mathbb{R}$. Then by Egoroff's theorem $\forall \epsilon > 0$ there is a closed set F contained in $\mathbb{R}$ for which

$$\{f_n\} \overset{u}{\to} f \text{ on F and } \ m(\mathbb{R} - F) < \epsilon$$

From the uniform convergence of f_n we have that $\forall \epsilon > 0, \exists N \in \mathbb{N} : |f_n - f| < \epsilon, \forall n > N$. Choose $m : m > N$ sufficiently large and because thats true for every $\epsilon > 0$, choose $\epsilon : 0 < \epsilon < 1$. So, $\chi_{(m,\infty)} < \epsilon$ on F

$$\chi_{(m,\infty)} < \epsilon \iff x \in (-\infty, m)$$

Therefore, F must be a closed set subset of $\{x \in \mathbb{R} : x \in (-\infty, m)\}$.

$$F \subset \{x \in \mathbb{R} : x \in (-\infty, m)\} \implies \{x \in \mathbb{R} : x \in (m, \infty)\} \subset \mathbb{R} - F$$

By the monotonicity of the measure and the result of Egoroff's theorem we have

$$m\bigg(\{x \in \mathbb{R} : x \in (m, \infty)\}\bigg) \leq m(\mathbb{R} - F) < \epsilon \implies \infty < \epsilon$$

a contradiction, because we choose E to be an unbounded set, with infinite measure.

$\square$

Problem 9

For each $c \in \mathbb{R}$ consider the set

$$\begin{aligned} \{x \in E : f(x) < c\} \subset E &\implies m\bigg(\{x \in E : f(x) < c\}\bigg) \leq m(E) = 0 \\ &\implies m\bigg(\{x \in E : f(x) < c\}\bigg) = 0 \end{aligned}$$

Every set of measure 0 is measurable. Therefore, $\{x \in E : f(x) < c\}$ measurable $\implies f$ measurable. Consider, a finite collection of disjoint sets $\{E_i\}_{i=1}^n$ such that $\bigcup_{i=1}^n E_i = E$. Then,

$$0 = m(E) = m\bigg(\bigcup_{i=1}^{n} E_i\bigg) = \sum_{i=1}^{n} m(E_i) \implies \quad m(E_i) = 0 \;\; \forall i = 1, 2, ..., n$$

Since f is measurable and bounded on E the simple approximation lemma applies. So, there are simple functions ϕ, ψ on E such that $\phi \leq f \leq \psi$ on E. Let α_i, β_i be the distinct values that ϕ, ψ take in each E_i, respectively. Then,

$$\int_E \phi = \sum_{i=1}^{n} \alpha_i m(E_i) = 0 \quad \text{and} \quad \int_E \psi = \sum_{i=1}^{n} \beta_i m(E_i) = 0$$

$$\implies sup\left\{ \int_E \phi : \phi \text{ simple and } \phi \leq f \right\} = 0 \quad \text{and} \quad inf\left\{ \int_E \psi : \psi \text{ simple and } f \leq \psi \right\} = 0$$

So, the upper and lower Lebesgue integrals are equal and by definition f is Lebesgue integrable and

$$\begin{aligned} \int_E f &= sup\left\{ \int_E \phi : \phi \text{ simple and } \phi \leq f \right\} \\ &= inf\left\{ \int_E \psi : \psi \text{ simple and } f \leq \psi \right\} \\ &= 0 \end{aligned}$$

□

Problem 10

Since f is measurable and A is a measurable subset of E, $f\chi_A$ is measurable on A. Also, E has a finite measure and so A has. Then, $f\chi_A$ is a bouded (since f is bounded), measurable function on a set of finite measure and so is integrable on A. In addition, E has finite measure. Consider, a finite collection of disjoint sets $\{E_i\}_{i=1}^{n}$ such that $\bigcup_{i=1}^{n} E_i = E$. From simple approximation lemma we know that there exist simple functions ϕ, ψ such that $\phi \leq f \leq \psi$ on E and let α_i, β_i be the distinct values that ϕ, ψ take in each E_i, respectively. Then,

$$\phi\chi_A \leq f\chi_A \leq \psi\chi_A \quad \text{on E} \implies \int_E \phi\chi_A \leq \int_E f\chi_A \leq \int_E \psi\chi_A \tag{11}$$

We re-write ϕ and ψ in their canonical representation $\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, $\psi = \sum_{i=1}^{n} \beta_i \chi_{E_i}$. Then,

$$\int_E \phi\chi_A = \int_E \sum_{i=1}^{n} \alpha_i \chi_{E_i}\chi_A = \int_E \sum_{i=1}^{n} \alpha_i \chi_{E_i \cap A} = \sum_{i=1}^{n} \alpha_i m(E_i \cap A) = \int_A \phi$$

$$\int_A f = sup\left\{ \int_A \phi : \phi \text{ simple and } \phi \leq f \right\} \leq \int_E f\chi_A \quad \text{from (1)} \tag{12}$$

$$\int_E \psi\chi_A = \int_E \sum_{i=1}^{n} \beta_i \chi_{E_i}\chi_A = \int_E \sum_{i=1}^{n} \beta_i \chi_{E_i \cap A} = \sum_{i=1}^{n} \beta_i m(E_i \cap A) = \int_A \psi$$

$$\int_A f = inf\left\{\int_A \psi : \psi \text{ simple and } f \leq \psi\right\} \geq \int_E f\chi_A \quad \text{from (1)} \tag{13}$$

From (2) and (3)

$$\int_E f\chi_A \leq \int_A f \leq \int_E f\chi_A \implies \int_A f = \int_E f\chi_A$$

□

Problem 12

Let $E_0 = \{x \in E : f(x) \neq g(x)\}$ and $E - E_0 = \{x \in E : f(x) = g(x)\}$, then $m(E_0) = 0$. Since, $f = g\ a.e\ onE \implies g$ measurable. So, g is a bounded, measurable, on a set of finite measure $\implies g$ integrable. For the set of measure zero we have $\int_{E_0} f = \int_{E_0} g = 0$.

$$\int_E f = \int_{E-E_0} f + \int_{E_0} f \overset{f=g \ \ on \ \ E-E_0}{=} \int_{E-E_0} g + 0 = \int_{E-E_0} g + \int_{E_0} g = \int_E g$$

□

Problem 17

Let $E : m(E) = 0$ and define $\{f_n\} = n$ be an increasing sequence of measurable functions on E, $\{f_n\} \overset{p.w.}{\to} f = \infty$ and so the Monotone Convergence Theorem applies

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_E n = \lim_{n\to\infty} nm(E) = \lim_{n\to\infty} 0 = 0 \tag{14}$$

□

Problem 23

$f(x) = a_n,\ x \in [n, n+1)$ on $E = [1, \infty)$. Then, we can write $f(x) = \sum_{n=1}^{\infty} a_n \chi_{[n,n+1)}(x)$ where $\left\{a_n \chi_{[n,n+1)}\right\}$ is a sequence of non-negative functions as a_n is a sequence of non-negative real numbers. Then, from the corollary of the monotone convergence theorem we have

$$\int_E f = \sum_{n=1}^{\infty} \int_E a_n \chi_{[n,n+1)} \overset{int\ of\ simple\ function}{=} \sum_{n=1}^{\infty} a_n m([n, n+1)) = \sum_{n=1}^{\infty} a_n$$

□

Problem 27

From previous homework problem if f_n is a sequence of measurable functions then $inf\{f_k : k \geq n\}$ is also measurable. Define $g_n := inf\{f_k : k \geq n\}$ and $g := \lim_{n\to\infty} inf\{f_k : k \geq n\}$. Then, g_n is an increasing sequence of non-negative measurable functions and

$$g_n \overset{p.w}{\to} g$$

Therefore, the Monotone Convergence Theorem applies

$$\int_E g = \lim_{n\to\infty} \int_E g_n \leq \lim_{n\to\infty} \inf \left\{ \int_E g_k : k \geq n \right\} \tag{15}$$

In addition,

$$g_n = \inf\{f_k : k \geq n\} \leq f_n \implies \int_E g_n \leq \int_E f_n \implies \inf \left\{ \int_E g_n \right\} \leq \inf \left\{ \int_E f_n \right\} \tag{16}$$

Hence, combing (1) and (2) we have

$$\int_E \lim_{n\to\infty} \inf\{f_k : k \geq n\} = \int_E g \leq \lim_{n\to\infty} \inf \left\{ \int_E g_k : k \geq n \right\} \leq \lim_{n\to\infty} \inf \left\{ \int_E f_k : k \geq n \right\}$$

□

Problem 28

Since, f is integrable so $f\chi_C$ is. Then, by definition

$$\int_E f\chi_C := \int_E (f\chi_C)^+ - \int_E (f\chi_C)^- = \int_E f^+\chi_C - \int_E f^-\chi_C \tag{17}$$

We only need to show that $\int_E f^+\chi_C = \int_C f^+$ and $\int_E f^-\chi_C = \int_C f^-$.

$$
\begin{aligned}
\int_E f^+\chi_C &= \sup\left\{\int_E h : h \text{ bounded, measurable, with finite support}: \ h \le f^+\chi_C \text{ on E}\right\} \\
&\overset{*}{=} \sup\left\{\int_C h : h \text{ bounded, measurable, with finite support}: \ h \le f^+ \text{ on C}\right\} \\
&= \int_C f^+ \qquad (18)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_E f^-\chi_C &= \sup\left\{\int_E h : h \text{ bounded, measurable, with finite support}: \ h \le f^-\chi_C \text{ on E}\right\} \\
&\overset{*}{=} \sup\left\{\int_C h : h \text{ bounded, measurable, with finite support}: \ h \le f^- \text{ on C}\right\} \\
&= \int_C f^- \qquad (19)
\end{aligned}
$$

Therefore, combining (3), (4) and (5) we have

$$\int_E f\chi_C := \int_E f^+\chi_C - \int_E f^-\chi_C = \int_C f^+ - \int_C f^- := \int_C f$$

* If $h \le f^+\chi_C,\ h \le f^-\chi_C$, on E then $h \le f^+,\ h \le f^-$ on C.

□

Problem 29
Consider the function $f(x) = \chi_{[n,n+1)}(x) - \chi_{[n+1,n+2)}(x)$

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \int_n^{n+1} f(x) \right| &= \sum_{n=1}^{\infty} \left| \int_n^{n+1} \chi_{[n,n+1]}(x) - \chi_{[n+1,n+2)}(x) \right| \\
&= \sum_{n=1}^{\infty} \left| \int_n^{n+1} \chi_{[n,n+1)}(x) - \int_n^{n+1} \chi_{[n+1,n+2)}(x) \right| \\
&= \sum_{n=1}^{\infty} \left| m([n,n+1]) - m([n+1,n+2)) \right| \\
&= \sum_{n=1}^{\infty} |0| \\
&= 0
\end{aligned}$$

Hence, the series converges absolutely which implies convergence, too. But the function is not integrable since

$$|f| = f^+ + f^-, \text{ where} f^+ = \chi_{[n,n+1)}(x) \text{ and} f^- = \chi_{[n+1,n+2)}(x)$$

$$\int_{[1,+\infty)} f^+ = \int_{[1,+\infty)} \chi_{[n,n+1]}(x) \overset{*}{=} \sum_{n=1}^{\infty} \int_{[n,n+1]} \chi_{[n,n+1)} = \sum_{n=1}^{\infty} 1 = \infty$$

$$*\text{simple function is integrable and } [1,\infty) = \cup_{n=1}^{\infty}[n,n+1)$$

Hence, $\int_{[1,\infty)} |f| = \infty \implies |f|$ not integrable $\implies f$ not integrable. So, both of the if and only if statements are **not** true, as we found a counter-example that disaproves each of one direction, which is enough.

□

Problem 37
We need to show that $\forall \epsilon > 0, \exists N \in \mathbb{N} : \left| \int_{E_n} f \right| < \epsilon \; \forall n \geq N$, which is equivalent of showing that $\lim_{n\to\infty} \int_{E_n} f = 0$. The countable collection of measurable set $E_n = \{x \in E : |x| \geq n\}$ is descening and $\bigcap_{n=1}^{\infty} E_n = \emptyset \implies m\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0$. In addition, f is integrable over E, so is finite a.e. and so bounded. From a previous homework problem the integral of a bounded function over a set of measure zero, is zero. So, $\int_{\bigcap_{n=1}^{\infty} E_n} f = 0$. Therefore, from the continuity of integration

$$\lim_{n\to\infty} \int_{E_n} f = \int_{\bigcap_{n=1}^{\infty} E_n} f = 0 \iff \forall \epsilon > 0 \; \exists \; N \in \mathbb{N} : \left| \int_{E_n} f \right| < \epsilon \; \forall \; n \geq N$$

$\square$

Problem 9

Consider the sequence of measurable functions $f_n(x) = \chi_{[n,n+1](x)}$. Then, $f_n \overset{p.w}{\to} f = 0$ in $E = \mathbb{R}$. Then in order f_n to converge in measure on $\mathbb{R}$ to f, we need $\forall \epsilon > 0$

$$\lim_{n\to\infty} m\Bigg(x \in \mathbb{R} : |\chi_{[n,n+1](x)}| > \epsilon \Bigg) = 0$$

Choose $\epsilon < 1$ then,

$$\lim_{n\to\infty} m\Bigg(x \in \mathbb{R} : |\chi_{[n,n+1](x)}| > \epsilon \Bigg) = \lim_{n\to\infty} m([n, n+1]) = 1$$

Therefore, f_n fails to converge in measure on $\mathbb{R}$ to f.
Another, counter example is by choosing $g_n(x) = \chi_{[n,\infty)}(x)$, then, $g_n \overset{p.w}{\to} g = 0$ in $E = \mathbb{R}$
Choose $\epsilon < 1$ then,

$$\lim_{n\to\infty} m\Bigg(x \in \mathbb{R} : |\chi_{[n,\infty)}| > \epsilon \Bigg) = \lim_{n\to\infty} m([n, \infty)) = \infty$$

$\square$