Some Nearly Finite-Dimensional Hamiltonian Dynamics In NLS-Like Systems: Some results and one new idea

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Nonlinear Schrödinger/Gross-Pitaevskii Equation

\[ i\psi_t = -\nabla^2 \psi + V(r)\psi \pm |\psi|^2 \psi \]

Two contexts for today:

- Propagation of light in a nonlinear waveguide
  - \( \psi(x, z) \) gives the electric field envelope
  - “Evolution” occurs along axis of waveguide (\( t \to z \)) plus one transverse spatial dimension
  - Potential represents waveguide geometry

- Evolution of a Bose-Einstein condensate (BEC)
  - Everyone’s favorite nonlinear playground. A “new” state of matter achieved experimentally in the 1990’s.
  - One, two, or three space dimensions
  - Potential represents magnetic or optical trap
Nonlinear PDEs are hard

If you want to rigorously examine a PDE you can study:

- Existence, Uniqueness, Smoothness
- Linear problems, idealized problems, limiting cases, & perturbations thereof
- Simple exact solutions: traveling waves, stationary states
- Statistical properties, etc.
- Aspects of integrability (unfair)

Some problems of physical interest are display complex dynamics that is difficult to describe using (today's) rigorous analysis. To make progress in applied problems, people use:

- Numerical Simulation
- Simplification & Approximation
- Radical Handwaving

Use to gain intuition needed to begin rigorous study.
Finite-dimensional ODE Models

Common idea in nonlinear waves: construct an approximate solution governed by a small system of ODE.

- Variational methods: use Lagrangian structure
- Galerkin projection
- Other physics-based simplifications

Observation: These ODE's are too often studied by only simple means: enumeration/bifurcations of simplest solutions, + numerics.

Subject of the talk: My attempt (and others') to go further.

Main tool: Exact and approximate symplectic changes of variables

Used to:
- Reduce dimension
- Separate motion on different time scales
- Simplify complicated models
Topic 1: Periodic and chaotic tunneling in a 3-well waveguide

Why three wells?

- Recent work on two-waveguide arrays shows symmetry-breaking bifurcations and an associated wobbling dynamics.
- Three waveguides provide the simplest system in which Hamiltonian Hopf bifurcations, which lead to complex dynamics, are possible.
- Significant interest in many-waveguide arrays. Useful to proceed: Simple Geometry → Complex Geometry, Simple Dynamics → Complex Dynamics
What got me thinking: Double well $V(x) = V_0(x + L) + V_0(x - L)$

Stationary
$\psi(x, t) = \Psi(x)e^{-i\Omega t}$

$$\int_{\mathbb{R}} |\Psi(x)|^2 dx = \|\Psi\|_2^2 = N$$

Time-dependent dynamics

Experiment in Bose-Einstein condensate

- Time dependent dynamics in a single or double well
- Rigorous result: long-time shadowing of ODE solutions by PDE solutions

Spontaneous symmetry breaking above critical intensity that is found analytically.

Kirr, Kevrekidis, Shlizerman, Weinstein 2008

Albiez et al. 2005

Marzuola & Weinstein 2010

Pelinovsky & Phan 2012

RG, Marzuola, Weinstein 2015
What got me thinking: Triple well

3-well potential & eigenfunctions

$$V(x) = V_0(x + L) + V_0(x) + V_0(x - L)$$

Bifurcations of standing waves

(Kapitula/Kevrekidis/Chen SIADS 2006)

Periodic Schrödinger Trimer


$$\frac{d}{dt} \psi_n + C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) + |\psi_n|^2 \psi_n = 0$$

subject to $$\psi_{n+3} = \psi_n$$

“Hamiltonian Hopf Bifurcations”

Numerically-generated chaos
Decompose the solution as

\[
\psi = c_1(t)\Psi_1(x) + c_2(t)\Psi_2(x) + c_3\Psi_3(x) + \eta(x; t)
\]

projection onto eigenmodes

Ignoring contribution of \( \eta(x, t) \) gives finite-dimensional Hamiltonian system with (approximate) Hamiltonian

\[
\bar{H} = \Omega_1 |c_1|^2 + \Omega_2 |c_2|^2 + \Omega_3 |c_3|^2 - A \left[ \frac{3}{2} \left( |c_1|^2 + |c_3|^2 \right)^2 + 2 |c_2|^4 + 4 |c_2|^2 |c_3 - c_1|^2 + \left( |c_1|^2 + |c_3|^2 \right) (c_1 c_3 + \bar{c}_1 \bar{c}_3) + \frac{3}{2} \left( c_1^2 \bar{c}_3^2 + \bar{c}_1^2 c_3^2 \right) + ((c_3 - c_1)^2 \bar{c}_2^2 + (\bar{c}_3 - \bar{c}_1)^2 c_2^2) \right]
\]

For well-separated potential wells, the spectrum has the form

\[
(\Omega_1, \Omega_2, \Omega_3) = (\Omega_2 - \Delta + \epsilon, \Omega_2, \Omega_2 + \Delta + \epsilon)
\]

with \( \epsilon \ll \Delta \ll 1 \)
Symmetry reduction

System conserves squared $L^2$ norm $N$

- Reduces # of degrees of freedom from 3 to 2
- Removes fastest timescale

$$\bar{H}_R = (-\Delta + \epsilon) |z_1|^2 + (\Delta + \epsilon) |z_3|^2 - \frac{AN}{2} \left( z_1^2 + \bar{z}_1^2 + z_3^2 + \bar{z}_3^2 - 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - 4(z_1 \bar{z}_3 + \bar{z}_1 z_3) \right) - \frac{A}{2} \left[ - \frac{1}{2} |z_1|^4 + 2 |z_1|^2 |z_3|^2 - \frac{1}{2} |z_3|^4 + \frac{3}{2} (z_1^2 \bar{z}_3^2 + \bar{z}_1 z_3^2) + \left( |z_1|^2 + |z_3|^2 \right) (5(z_1 \bar{z}_3 + \bar{z}_1 z_3) + 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - z_1^2 - \bar{z}_1^2 - z_3^2 - \bar{z}_3^2) \right].$$

- Relative fixed points in full system $\rightarrow$ fixed points in reduction
- Relative periodic orbits $\rightarrow$ periodic orbits

At $\epsilon = N = 0$, semisimple double frequency $i \Omega = \pm i \Delta$.

When $\epsilon > 0$, non-simple double eigenvalues at $N_{\text{HH1}} \approx \frac{\epsilon}{2A}$ and $N_{\text{HH2}} \approx \frac{\Delta - 2\epsilon}{2A}$, with instability in between.
ODE & PDE simulations

Trivial solution stable

Real($\lambda$)  \nonumber

Real($z_1$)  \nonumber

Poincaré Section  \nonumber

$|\psi(t)|$  \nonumber

(a), N=0.35  \nonumber

Re($\sigma_1$)  \nonumber

$x \times 10^{-3}$  \nonumber

$y \times 10^{-4}$  \nonumber

$X \times 10^{-4}$  \nonumber

$Y$  \nonumber

$t$  \nonumber

$0$  \nonumber

$150$  \nonumber

$0$  \nonumber

$1$  \nonumber

$0.5$  \nonumber

$-0.5$  \nonumber

$0$  \nonumber

$150$  \nonumber

$0$  \nonumber

$1$  \nonumber

$2$  \nonumber

$0$  \nonumber

$2$  \nonumber

$t$  \nonumber

$0$  \nonumber

$150$  \nonumber

$0$  \nonumber

$1$  \nonumber

$2$  \nonumber

$0$  \nonumber

$2$  \nonumber
ODE & PDE simulations

Heteroclinic bursting

Real($z_1$)

Poincaré Section

$|\psi(t)|$
ODE & PDE simulations

Real($\lambda$)

Real($z_1$)

Poincaré Section

$|\psi(t)|$
Reduced Hamiltonian is daunting!

\[ \bar{H}_\text{R} = (-\Delta + \epsilon) |z_1|^2 + (\Delta + \epsilon) |z_3|^2 - AN \left( z_1^2 + \bar{z}_1^2 + z_3^2 + \bar{z}_3^2 - 2(z_1z_3 + \bar{z}_1\bar{z}_3) - 4(z_1\bar{z}_3 + \bar{z}_1z_3) \right) - A \left[ -\frac{1}{2} |z_1|^4 + 2 |z_1|^2 |z_3|^2 - \frac{1}{2} |z_3|^4 + \frac{3}{2} (z_1^2 \bar{z}_3^2 + \bar{z}_1^2 z_3^2) + \left( |z_1|^2 + |z_3|^2 \right) \left( 5(z_1\bar{z}_3 + \bar{z}_1z_3) + 2(z_1z_3 + \bar{z}_1\bar{z}_3) - z_1^2 - \bar{z}_1^2 - z_3^2 - \bar{z}_3^2 \right) \right]. \]

Goal: understand periodic orbits of \( \bar{H}_\text{R} \) using Hamiltonian Normal Forms

Given a system with Hamiltonian \( H = H_0(z) + \epsilon \tilde{H}(z, \epsilon) \) find a near-identity canonical transformation \( z = \mathcal{F}(y, \epsilon) \) such that the transformed Hamiltonian

\[ K(y, \epsilon) = H \left( \mathcal{F}(y, \epsilon), \epsilon \right) = H_0(y) + \epsilon \tilde{K}(y, \epsilon) \]

is “simpler” than \( H(z, \epsilon) \).
Review: What does “simpler” mean?

- Try to remove terms from $H$ to construct $K$
- Eliminating terms at a given order in $\epsilon, y$ introduces new terms of higher order
- A term can be removed if it lies in the range of the adjoint operator of $\text{ad}_{H_0} = \{\cdot, H_0\}$.
- Invoke Fredholm alternative. Resonant terms in adjoint null space.
Three normal form calculations

- Semisimple -1:1 resonance for $\varepsilon \ll 1$, $N = O(\varepsilon)$
  Gives HH1 at
  $$N_{\text{crit}} = \frac{\varepsilon}{2A} + O(\varepsilon^2)$$

- Nonsemisimple -1:1 resonance at $N_{\text{crit}}$ using a further simplification of above normal form

- Nonsemisimple -1:1 resonance computed numerically at numerical location of HH2
Normal form near semisimple double eigenvalue (Chow/Kim 1988)

\[ H = -\Delta |z_1|^2 + \Delta |z_3|^2 \]

Normal Form

\[ H_{\text{norm}} = -\Delta |z_1|^2 + \Delta |z_3|^2 + \epsilon \left( |z_1|^2 + |z_3|^2 \right) + 2AN(z_1z_3 + \bar{z}_1\bar{z}_3) \]
\[ + A \left[ \frac{1}{2} |z_1|^4 - 2|z_1|^2 |z_3|^2 + \frac{1}{2} |z_3|^4 - 2 \left( |z_1|^2 + |z_3|^2 \right) (z_1z_3 + \bar{z}_1\bar{z}_3) \right] \]

In Canonical Polar Coordinates

\[ H = \Delta (-J_1 + J_3) + \epsilon (J_1 + J_3) + 4AN \sqrt{J_1J_3} \cos (\theta_1 + \theta_3) \]
\[ + A \left( \frac{1}{2} J_1^2 - 2J_1J_3 + \frac{1}{2} J_3^2 - 4\sqrt{J_1J_3}(J_1 + J_3) \cos (\theta_1 + \theta_3) \right) \]

Independent of \((\theta_1 - \theta_3)\) implying the existence of a conserved quantity and the integrability of the Normal Form.

Advantage: Easier to find solution structure in Normal Form.
The system can be further reduced. Periodic orbits \( (J_1) e^{i\Omega t} \) solve:

\[
\sqrt{J_1 J_3} (2\epsilon - A (J_1 + J_3)) + 2A (N (J_1 + J_3) - J_1^2 - 6J_1 J_3 - J_3^2) \cos \Theta = 0
\]

\[
\sqrt{J_1 J_3} (N - J_1 - J_3) \sin \Theta = 0
\]

With \( \Theta = (\theta_1 + \theta_3) \)

\( J_1 \) and \( J_3 \) act as barycentric coordinates on the triangle of admissible solutions showing relative strength of the three modes.
Sequence of bifurcations in Normal Form

\[ 0 < N < \frac{2\epsilon}{5A} \]
\[ \frac{2\epsilon}{5A} < N < \frac{\epsilon}{2A} \]
\[ \frac{\epsilon}{2A} < N < \frac{2\epsilon}{A} \]
\[ \frac{2\epsilon}{A} < N \]

2 Lyapunov families of fixed points + unphysical branch

Unphysical branches cross into physical region

Lyapunov branches “pinch off”

All branches physical

Question: At second bifurcation point HH2, must have Lyapunov families of fixed point. Where do they come from?
Normal form for non-semisimple -1:1 resonances at HH1 and HH2 (Meyer-Schmidt 1974)

In symplectic polar coordinates \((r, \theta, p_r, p_\theta)\), this is:

\[
H = H_0(r, p_r, p_\theta) + \mu^2 \delta H_2(r, p_\theta) + H_4(r, p_\theta)
\]

\[
= \Omega p_\theta + \frac{\sigma}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + \mu^2 \delta \left( a p_\theta + \frac{b}{2} r^2 \right) + \frac{c}{2} p_\theta^2 + \frac{d}{2} p_\theta r^2 + \frac{e}{8} r^4
\]

\[\delta = \pm 1, \mu \ll 1\]

Poincaré-Lindstedt argument: periodic orbits with “amplitude” \(\mu r\) and frequency \(\Omega + \mu \omega_1\) when there is a solution to \(2\omega_1^2 - \sigma e r^2 = 2\delta \sigma \beta\)

Two cases:

Hyperbolic \(\sigma e > 0\)

Elliptic \(\sigma e < 0\)
How this works (quadratic part):

- Let $N = \frac{\epsilon}{2A} + \delta$ and make a linear symplectic change of variables to put $H_0$ in canonical form for HH bifurcation, making

$$H_0 = \Delta \eta_1 \xi_2 - \Delta \eta_2 \xi_1 + \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2}$$

$$H_2 = -2A\delta \eta_1^2 \epsilon - 2A\delta \eta_2^2 \epsilon + \frac{A\delta \xi_1^2}{2\epsilon} + \frac{A\delta \xi_2^2}{2\epsilon}$$

- Let $P_m = \{ \xi_1^{\alpha_1} \xi_2^{\alpha_2} \eta_1^{\beta_1} \eta_2^{\beta_2} | \alpha_1 + \alpha_2 + \beta_2 + \beta_2 = m \}$

- Then $P_2$ is 10-dimensional, $\text{ad}_{H_0}^{(2)}$ the restriction to $P_2$ is a $10 \times 10$ matrix

- Compute orthogonal projection onto $\text{span} \{ \Gamma_1, \Gamma_3 \}$ under the inner product $\langle F, G \rangle = F(\partial_{\bar{x}})G(\bar{x})$

- Now it’s linear algebra, use Mathematica
How this works (quadratic part):

- Actually, the projection of the quadratic part is obvious by inspection.
- The quartic part is harder, involving orthogonal projection in a 35 dimensional space.
- The result:

\[ H = \Delta \Gamma_1 + \Gamma_2 - 4A\delta \Gamma_3 + \frac{2}{3} A\Gamma_1^2 + 7A\epsilon^2 \Gamma_3^2. \]
The bifurcation at HH1

Computations using previous normal form

Increasing $N$ ➝ Numerically Computed Periodic orbits (not normal form)
The bifurcation at HH2

Numerically Computed Periodic orbits

Crossing the second HH point, a new pair of Lyapunov families appear in a bifurcation of elliptic type.
Topic 2: Vortex Interactions in Bose-Einstein Condensates (RHG, Carretero, Kevrekidis 2015)

- Condensate constrained by external magnetic field to be nearly two-dimensional
- When $V_{\text{ext}} = 0$, equations have vortex solutions: localized solution with zero amplitude at center and phase increase of $2\pi$
- Add potential, vortex orbits
Interacting Vortices

Given multiple vortices of vorticity $s_k = \pm 1$, and position $(x_k(t), y_k(t))$, each vortex executes a particle-like dynamics. (No theorems!)

\[
\begin{align*}
\dot{x}_k &= -s_k \omega_y^2 y_k + \frac{B}{2} \sum_{j \neq k} s_j \frac{y_j - y_k}{\rho_{jk}^2}, \\
\dot{y}_k &= s_k \omega_x^2 x_k - \frac{B}{2} \sum_{j \neq k} s_j \frac{x_j - x_k}{\rho_{jk}^2}, \\
\rho_{jk}^2 &= (x_j - x_k)^2 + (y_j - y_k)^2
\end{align*}
\]

Interaction with trap.

For anisotropic trap

$\omega_x \neq \omega_y$

Pairwise interactions between vortices
Vortex Dipoles in Isotropic and Anisotropic Traps

For isotropic trap $\omega_x = \omega_y$
and counter-rotating vortices $s_1 = -s_2$ the system is integrable
And there exists a family of circular periodic orbits:

- $\rho_+ > \rho_-$: Clockwise rotation
- $\rho_+ = \rho_-$: No rotation, arbitrary phase, a “resonance”
- $\rho_+ < \rho_-$: Counterclockwise rotation

For anisotropic trap $\omega_x \neq \omega_y$
only two stationary states survive
(Stockhofe et al 2011)
Question: what are time-dependent dynamics?
Understanding this symmetry-breaking bifurcation

Assume weak anisotropy

\[
\begin{align*}
\omega_x &= 1 \\
\omega_y &= 1 + \epsilon
\end{align*}
\]

In canonical polar coordinates, this system has Hamiltonian

\[
H = H_0 + \epsilon H_1
\]

\[
H_0 = \frac{B}{4} \log \left( J_1 + J_2 - 2 \sqrt{J_1} \sqrt{J_2} \cos(\phi_1 + \phi_2) \right) - J_1 - J_2;
\]

\[
H_1 = -J_1 \sin^2(\phi_1) - J_2 \sin^2(\phi_2).
\]
Reducing the Hamiltonian

The leading-order Hamiltonian depends on angles only through the combination \( \phi_1 + \phi_2 \), so we make the change of variables:

\[
\theta_1 = -\phi_1 + \phi_2, \quad \theta_2 = \phi_1 + \phi_2, \quad \rho_1 = \frac{-J_1 + J_2}{2}, \quad \rho_2 = \frac{J_1 + J_2}{2},
\]

Giving the Hamiltonian

\[
H = H_0(\theta_1, \rho_1, \rho_2) + \epsilon H_0(\theta_1, \theta_2, \rho_1, \rho_2)
\]

\[
H_0 = \frac{B}{4} \log \left( \rho_2 - \sqrt{\rho_2^2 - \rho_1^2 \cos \theta_2} \right) - 2\rho_2;
\]

\[
H_1 = -\rho_1 \sin \theta_1 \sin \theta_2 + \rho_2 (1 - \cos \theta_1 \cos \theta_2)
\]

Leading-order phase space

Fixed point=
(degenerate)
circular orbit

\( \rho_1 = 0 \)

\( \rho_1 \neq 0 \)
A numerical simulation: chaotic dynamics near the center

**Fast dynamics in**
$(\theta_2, \rho_2)$

**Slow dynamics in**
$(\theta_1, \rho_1)$

Vortices remain roughly opposite each other, and their average distance from center doesn’t change much.

Long spells of clockwise or counterclockwise motion with abrupt jumps. Direction of rotation determined by $\rho_1$, which measures which vortex closer to minimum of potential.
Explaining this dynamics by further Hamiltonian reduction

When $\epsilon > 0$, the fixed point

$$(\theta_2^*, \rho_2^*) = \left(\pi, \left(\rho_1^2 + B^2/64\right)^{1/2}\right)$$

becomes slowly-varying. We change of variables to fix it at the origin. Near there we find dynamics:

$$\ddot{\Theta}_1 - \epsilon \sin \Theta_1 - \frac{8\epsilon}{B} r_2 \sin \Theta_1 = 0;$$

$$\ddot{r}_2 + 2r_2 + \frac{B\epsilon}{8} \cos \Theta_1 = 0.$$ 

This is a classical problem going back to Poincaré! (and Camassa!) We can analyze it to death (but out of time)
the comments about the previous figure apply here. We have not plotted the solutions with (A and G), the two vortices execute orbits that are identical up to a 180°-interval on which this is defined is about 4.9 space with a width of 0.01, we are able to include librations (such as Ca and D) and rotations (such as 4.8), namely, the ODE periodic orbit system of this article is prohibited. The depicted orbits correspond, respectively, to most of the ODE periodic solutions with the larger value of 4.9, specifically, an odd number of intersections, which include panels similar to those shown in the previous figure.

Figure 10.
Ricardo’s Problem + anisotropy

Question: How does dynamics of co-rotating vortex pair change in anisotropic potential?

Is there an interesting interaction between the symmetry-breaking bifurcation and the symmetry-breaking of the medium?

Answer: It appears “yes” 😊.

Parabolic Resonance Chaos!
Thanks!

For re/preprints http://web.njit.edu/~goodman