Nematic liquid crystals flowing down an incline

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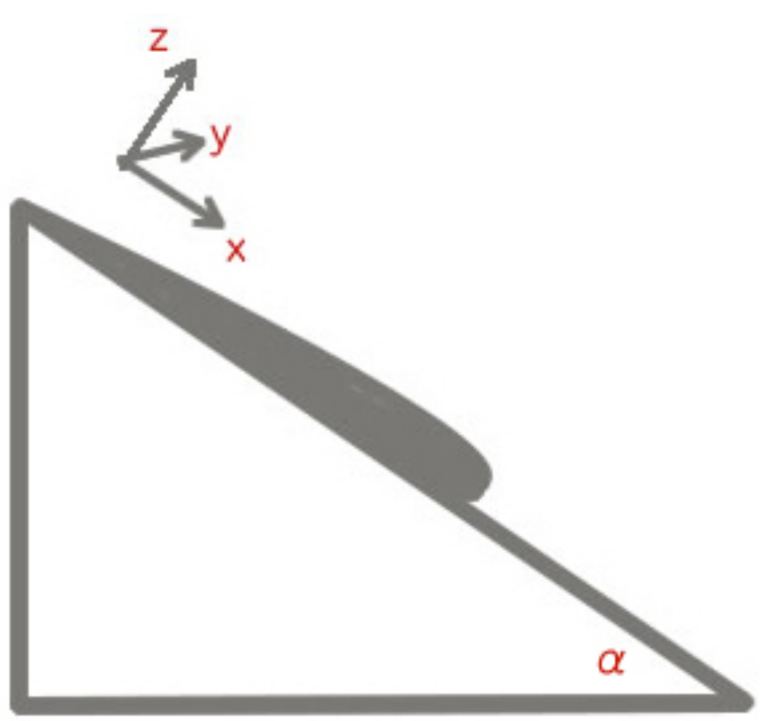
Lubrication theory

- Thin film of a viscous fluid
- lubrication approximation
  - $\epsilon = H / L$ where $H \ll L$
  - velocity gradients in the x,y-directions negligible compared to velocity gradient in z-direction
  - $\epsilon^2 Re \ll 1$, can ignore terms due to inertia
- Lubrication theory simplifies evolution equation

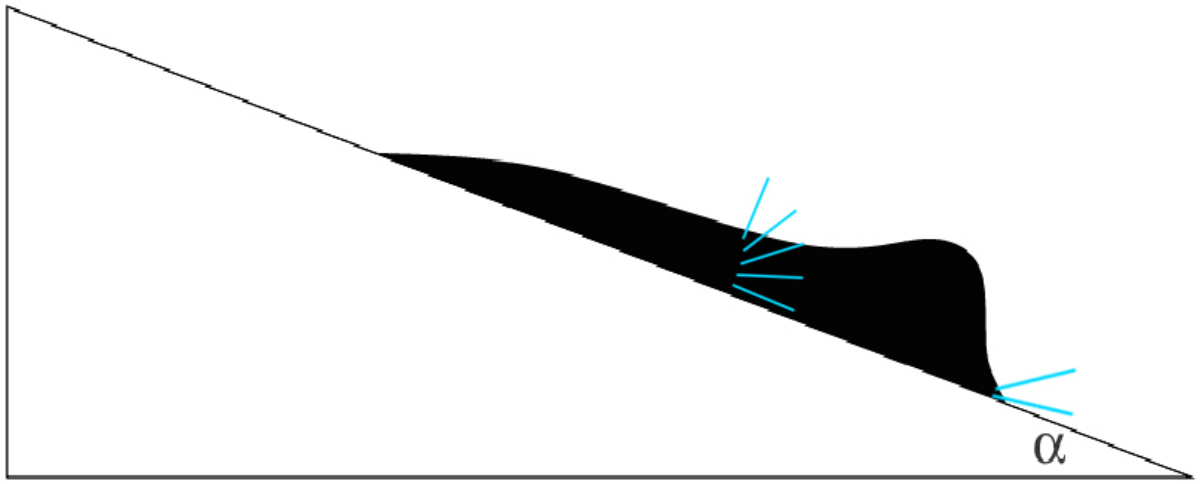


Evolution equation

$$h_t + \nabla\cdot[h^3(C\nabla\nabla^2 h - B\nabla h) + N(m^2 - hmm')\nabla h] + U(h^3)_x = 0$$

$$m = \frac{h^{3/2}}{\beta^{3/2} + h^{3/2}}$$

- lubrication approximation adopted
- prewetting of substrate with precursory layer of thickness $b \ll h$
- weak-anchoring model



h: fluid thickness
C: inverse capillary number
B: Bond number,
N: inverse Ericksen number

Linear stability analysis(LSA) of flat film

- Assume h is independent of y

$$h_t + \partial_x [h^3 (C h_{xxx} - B h_x) + N(m^2 - hmm')h_x] + U(h^3)_x = 0$$

- Perturb profile by a small amplitude

$$h(x,t) = h_o + \epsilon h_1(x,t) + O(\epsilon^2)$$

$$h_{1t} + C h_o^3 h_{1xxxx} - \left(B - NM(h_o)\right) h_o^3 h_{1xx} + 3U h_o^2 h_{1x} = 0$$

$$M(h_o) = \frac{ho^{3/2} - \beta^{3/2}/2}{(ho^{3/2} + \beta^{3/2})^3}$$

LSA of flat film cont.

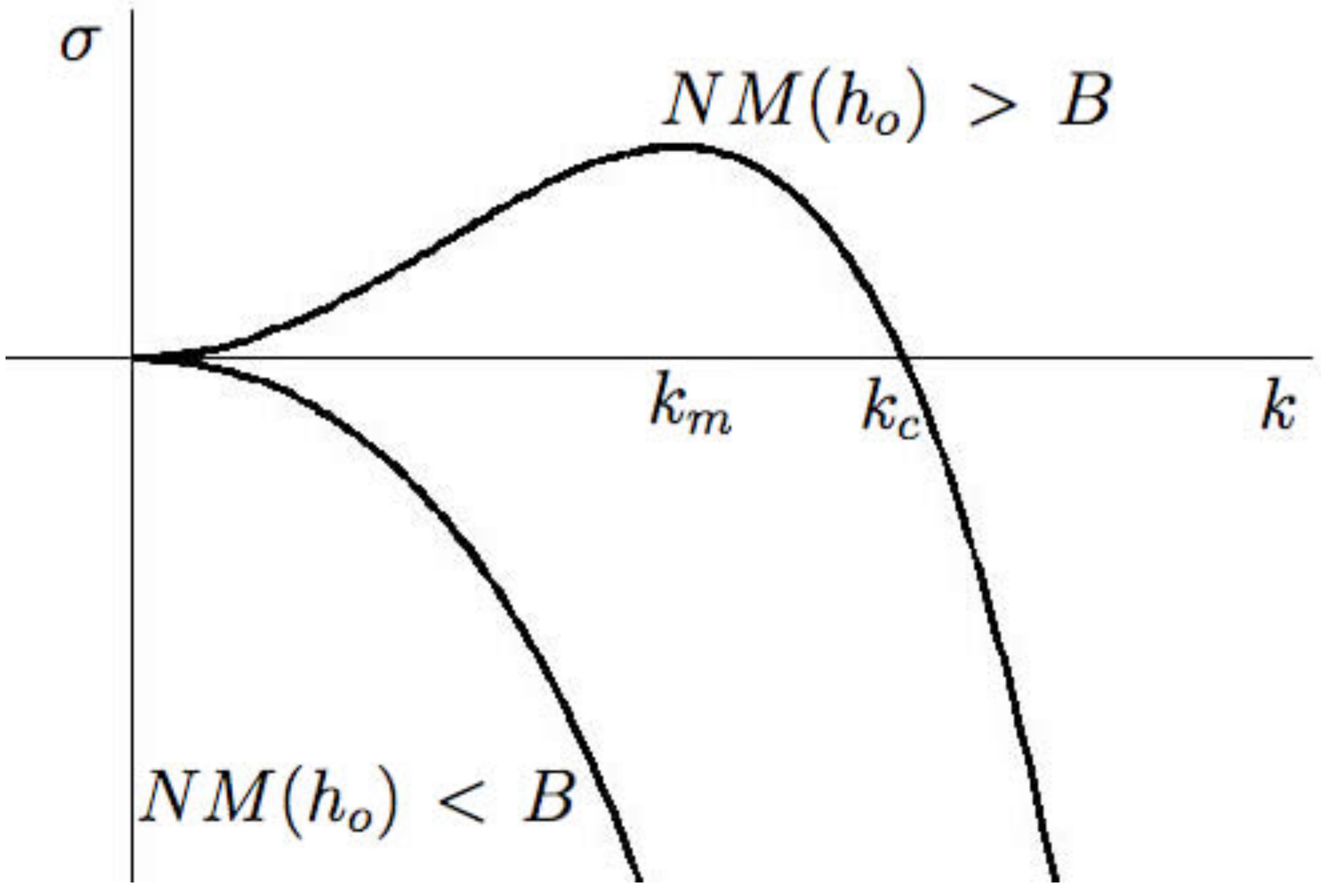
- Obtain dispersion relation by assuming solutions of the form $h_1 \propto e^{\sigma t + ikx}$

$$\sigma = -Ch_o^3 k^4 + \left(B - NM(h_o)\right) h_o^3 k^2 - i3Uh_o^2 k$$

- Surface tension responsible for stabilizing system for perturbations of short wavelengths

$$k_m = \sqrt{\frac{NM(h_o) - B}{2C}}$$

$$\sigma_m = \frac{(NM(h_o) - B)^2}{4C} h_o^3$$



σ: growth-rate
k: wavenumber
k_m: fastest growing wavenumber

Traveling-wave solutions for 2D equation

$$h_t + \partial_x[h^3(Ch_{xxx} - Bh_x) + N(m^2 - hmm')h_x] + U(h^3)_x = 0$$

$$m(h) = \frac{h^{3/2}}{\beta^{3/2}}$$

$$h \to b \text{ as } x \to -\infty \quad h \to 1 \text{ as } x \to \infty$$

$$h_x \to 0 \text{ as } x \to -\infty \quad h_x \to 0 \text{ as } x \to \infty$$

- simplify analysis by assuming $\beta \gg h$
- Neumann BCs used since far behind & far in front of the contact line, fluid thickness approximately constant

Traveling-wave solutions cont.

- Look at solution in a moving reference frame
- make a change of variables $\xi = x - Vt$ where V is the wave speed
- $\therefore$substituting $h_o(\xi) = h(x,t)$ and integrating

$$-Vh_o + Ch_o^3 h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3}\right) h_o^3 h_{o\xi} + U(h_o^3)_\xi = d$$

- Applying the BCs

$$d = -b(1+b)$$

$$V = U(1 + b + b^2)$$

LSA for 3D evolution equation

$$h_t + \nabla\cdot[h^3(C\nabla\nabla^2 h - B\nabla h) + N(m^2 - hmm')\nabla h] + U(h^3)_x = 0$$

- constant flux-driven case
- fluid is being injected into the film
  - infinite volume
- initially, flow in transverse direction fairly stable
- apply perturbations to leading order equation

$$h(x,y,t) = h_o(\xi) + \epsilon\phi(\xi)e^{\sigma t + iky} + O(\epsilon^2)$$

LSA for 3D evolution equation cont.

$$-\sigma\phi = -Vg_{\xi} + C\left[k^4 h_o^3\phi - k^2\left((h_o^3\phi_{\xi})_{\xi} + h_o^3\phi_{\xi\xi}\right) + (h_o^3\phi_{\xi\xi\xi} + 3h_o^3 h_{o\xi\xi\xi}\phi)_{\xi}\right]$$

$$+\left(B + \frac{N}{2\beta^3}\right)\left(k^2 h_o^3\phi - (h_o^3\phi_{\xi} + 3h_o^2 h_{o\xi}\phi)_{\xi}\right) + 3U(h_o^2\phi)_{\xi}$$

- σ, ϕ depend only on even powers of *k*
- looking at the solution in the limit of a small wavenumber

$$\phi = \phi_o + k^2\phi_1 + O(k^4) \qquad \sigma = \sigma_o + k^2\sigma_1 + O(k^4)$$

- we modified the position of the contact line by the perturbation, so BCs need to linearized accordingly giving $\phi_o(\xi) = h_{o\xi}(\xi)$

LSA for 3D evolution equation cont.

- leading-order equation

$$-\sigma_o h_{o\xi} = \left[-Vh_o + Ch_o^3 h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3} \right) h_o^3 h_{o\xi} + U(h_o^3)_\xi \right]_{\xi\xi}$$

- right hand side is our leading order equation, therefore $\sigma_o = 0$

LSA for 3D evolution equation cont.

- $O(k^2)$ equation

$$-\sigma_1 h_{o\xi} = -V\phi_{1\xi} + C\left(-(h_o^3 h_{o\xi\xi})_\xi - h_o^3 h_{o\xi\xi\xi} + (h_o^3 \phi_{1\xi\xi\xi})_\xi + 3(ho^2 h_{o\xi\xi\xi}\phi_1)_\xi\right)$$
$$+\left(B + \frac{N}{2\beta^3}\right)\left(h_o^3 h_{o\xi} - (h_o^3 \phi_1)_{\xi\xi}\right) + 3U(h_{o\xi}\phi_1)_\xi$$

- We integrate and apply the BCs
- Finally,

$$\sigma \approx \frac{k^2}{1-b}\int_{-\infty}^{0} Ch_o^3 h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3}\right)h_o^3 h_{o\xi} d\xi$$