Joseph Zaleski

Math 451H Final Report

Conformal Mapping Methods and ZST Hele Shaw Flow

Introduction

The Hele Shaw problem has been studied using linear stability analysis and numerical methods, but a novel advantage of conformal mapping methods is that exact solutions are obtainable. However much of the theory which deals with conformal mapping is concerned with the zero surface tension problem; as summarized in the end the non-ZST problem is much more intractable. My main goal this semester was to obtain an understanding of why the ZST solutions are not necessarily relevant, and surface tension plays a role in the problem.

The following report makes use of the process from Chapter 2 in Dr. Cumming's thesis.

Basic Equations

Using the process outlined in chapter one of Cummings (1996) we have the basic equations for one phase and Hele-Shaw flow:

$\nabla^2 p = 0 \text{ in } \Omega(t)$ (1), where p is pressure, and Ω is the varying fluid domain.

Surface Tension BC: $p = T\kappa \text{ on } \partial\Omega(t)$ (2), where T is surface tension and κ is curvature

Kinematic BC: $\frac{\partial p}{\partial n} = -v_n \text{ on } \partial\Omega(t)$ (3)

However, under the assumption that curvature is small (that is, the fluid domain $\partial\Omega(t)$ is smooth and nowhere of order 1/T), and surface tension is negligible, we might conjecture that setting the right side of eq. (2) equal to zero is a valid approximation to the problem. Thus we have the zero surface tension boundary condition: $p = 0 \text{ on } \partial\Omega(t)$ (4)

Polubarinova/Galin Method

Next, I outline what I will call the P-G approach. The crucial factor which allows this method to be used is that equation (1) says pressure p is harmonic on the fluid domain. Complex variables theory therefore implies that p is the real or imaginary part of some complex function which is analytic. The key idea to this method is mapping a known region (in this report, we use the unit disc as this region, but for channel flows, the right half plane may be more convenient) into the evolving fluid domain via a time dependant conformal map. We are at the liberty to choose the region and mapping, but for simplicity, mapping the origin to the pressure source is convenient. If the fluid domain is simply connected, the existence of the map is known, by the Riemann mapping theorem (for this reason we also enforce that the map be univalent, which prevents the border of the fluid domain from crossing itself). Furthermore, assuming that the derivative is real and positive gives us a unique map.

I define this map as $w = z(\zeta, t)$, where $\zeta = \xi + i\eta$ are points in the unit disc.

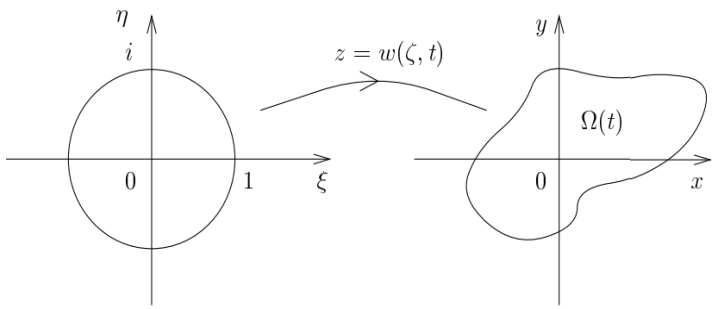


Figure 2.1: The mapping from the unit disc onto the fluid domain.

1 fig 2.1, Cummings (1996)

Next, on the fluid domain in the $z=x+iy$ plane we will define the complex potential of the flow to be:

$$\mathcal{W}(z,t) = f(z,t) + i\,g(z,t),\ s.t.\ \mathcal{W} \text{ analytic within } \Omega(\mathrm{t}) \text{ (except at sink/driving points)}$$
$$\text{and } p = -\Re\{\mathcal{W}(z,t)\} \quad (5)$$

Next, we describe the asymptotic behavior of the problem near singularities in the fluid domain by making assumptions about the behavior at the source or sink of the pressure field (or where fluid is injected or sucked from, in the problem). Common assumptions for the suction or blowing problem include: $p \sim \pm\dfrac{Q}{2\pi}\log(x^2+y^2)^{1/2}$, as $(x,y)\to(0,0)$, where $Q>0$ stands for source/sink strength

Clearly pressure blows up as we near origin, matching the physical nature of the problem.

From this assumption and using the definition of the complex potential, we can now obtain the following relation:

$$\Re\{\mathcal{W}(z)\} = p \sim -\frac{Q}{2\pi}\log r.$$

To satisfy this and have the correct singularity,

we can take: $\mathcal{W}(z) \sim -\dfrac{Q}{2\pi}\log z \ (5)$, since $\log z = \log r \ + \ i\theta$.

What we next consider below, is complex potential in the ζ plane, and we define this function assuming that the behavior near origin in the ζ plane is identical to that in the z plane:

$\Upsilon(\zeta,t) = \mathcal{W}(z(\zeta,t)) = \frac{-Q}{2\pi}\log\zeta$ (6)

Note that the ZST condition says that $\Re\{\mathcal{W}(z(\zeta,t))\} = 0$ for z on $\partial\Omega$, which corresponds to $\Upsilon(\zeta,t)$ vanishing on the unit disc, $|\zeta| = 1$.
(thus the ZST condition is satisfied in both planes)

Moving on, we write the KBC (3) as:

• KBC:

$n \cdot \nabla p = -v_n$,

where $n = \pm\nabla p / |\nabla p|$ and $v_n = \frac{\partial p}{\partial n} = \mp\frac{\partial p}{\partial t} / |\nabla p|$

$n \cdot \nabla p = -v_n \Rightarrow (\pm\nabla p / |\nabla p|\ ) \cdot \nabla p = -(\mp\frac{\partial p}{\partial t} / |\nabla p|)$

$\Rightarrow \frac{\partial p}{\partial t} = |\nabla p|^2$ on $\partial\Omega(t)$ (i)

To reduce the above expression, on boundary $|\zeta|$=1, we have:

$$|\nabla p|^2 = p_x^2 + p_y^2 = |\mathcal{W}'(z)|^2 = \frac{|\Upsilon'(\zeta)|^2}{|w'(\zeta)|^2} = \frac{\left|\frac{-Q}{2\pi}\right|^2}{|w'(\zeta)|^2} = \frac{Q^2}{4\pi^2|w'(\zeta)|^2} \text{ (ii)}$$

and $p_t = \Re\{\Upsilon_\zeta(\zeta)\zeta_t + \Upsilon_t\} = \frac{-Q}{2\pi}\Re\{\frac{\zeta_t}{\zeta}\}$ (iii)

To calculate ζ_t:

$z = w(\zeta,t) \Rightarrow 0 = w_\zeta\zeta_t + w_t \Rightarrow \zeta_t = \frac{-w_t}{w_\zeta}$ (iv)

Now note that $(i) \Rightarrow (iii) = (ii)$:

$$\frac{-Q}{2\pi}\Re\{\frac{\zeta_t}{\zeta}\} = \frac{Q^2}{4\pi^2|w'(\zeta)|^2} \Rightarrow \Re\{\frac{\zeta_t}{\zeta}\} = -\frac{Q}{2\pi|w'(\zeta)|^2}$$

and therefore $(iv) \Rightarrow \Re\{\frac{-w_t}{w_\zeta\zeta}\} = -\frac{Q}{2\pi|w'(\zeta)|^2}$

Polubarinova-Galin eq: $\Re\{\zeta w'(t)\overline{w_t(\zeta)}\} = -\frac{Q}{2\pi}$ on $|\zeta| = 1$, (7)

To summarize, we now have a relation which mappings must satisfy to be solutions to the ZST problem.

Examples

The P-G equation involves guessing the general form of a map, but a variety of mappings exist.

Ex 1: Take our initial guess for the mapping to be the quadratic map, $z = w(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$, where a_1, a_2 are real valued functions.

Substituting into (7), and taking into account that the P-G eq. applies on $|\zeta| = 1$, or in polar form $\zeta = e^{i\theta}$, we get:

$$w'(t) = a_1(t) + 2a_2(t)\zeta = a_1(t) + 2a_2(t)e^{i\theta},$$

$$w_t(\zeta) = a_{1t}e^{i\theta} + a_{2t}e^{2i\theta} \Rightarrow \overline{w_t(\zeta)} = a_{1t}e^{-i\theta} + a_{2t}e^{-2i\theta}$$

$$\text{P-G: } \Re\{\zeta w'(t)\overline{w_t(\zeta)}\} = -\frac{Q}{2\pi},$$

$$\Re\{e^{i\theta}(a_1(t) + 2a_2(t)e^{i\theta})(a_{1t}e^{-i\theta} + a_{2t}e^{-2i\theta})\} = \Re\{e^{i\theta}(a_1a_{1t}e^{-i\theta} + a_1a_{2t}e^{-2i\theta} + 2a_2a_{1t} + 2a_2a_{2t}e^{-i\theta})\} = -\frac{Q}{2\pi}$$

$$\Rightarrow \Re\{a_1a_{1t} + a_1a_{2t}e^{-i\theta} + 2a_2a_{1t}e^{i\theta} + 2a_2a_{2t}\} = -\frac{Q}{2\pi}$$

$$\Rightarrow \Re\{a_1a_{1t} + 2a_2a_{2t} + a_1a_{2t}\cos(-\theta) + 2a_2a_{1t}\cos\theta + i(...)\} = -\frac{Q}{2\pi}$$

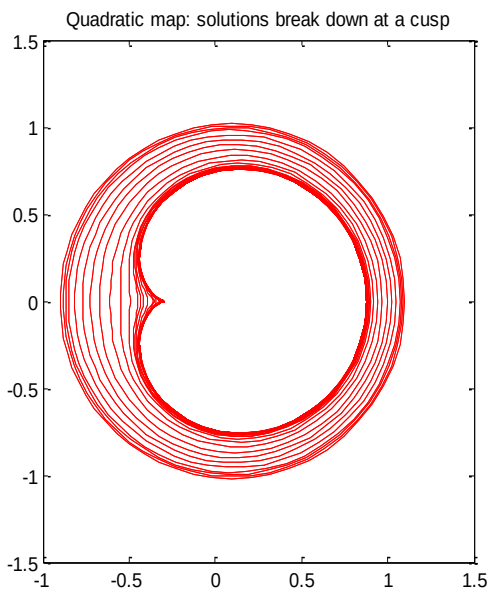
$$\Rightarrow a_1a_{1t} + 2a_2a_{2t} + (a_1a_{2t} + 2a_2a_{1t})\cos\theta = -\frac{Q}{2\pi}$$

And lastly we get the following system of ODEs, by noting that the θ terms must vanish to match the right side:

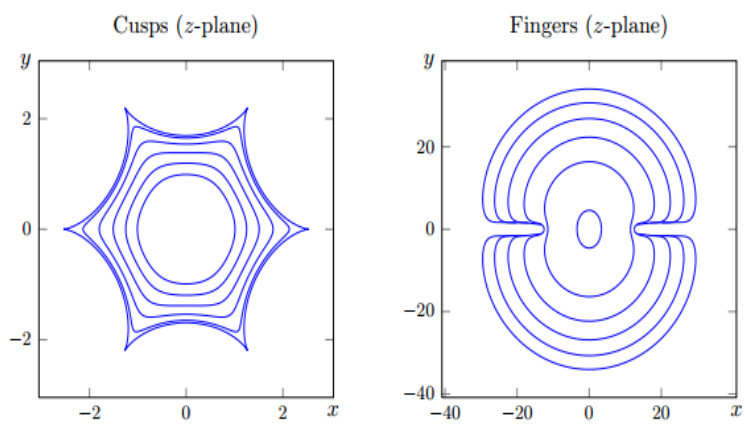
$$a_1a_{1t} + 2a_2a_{2t} = -\frac{Q}{2\pi}$$

$$a_1a_{2t} + 2a_2a_{1t} = 0$$

Using Matlab and random initial conditions, I solved this system (which can be manually integrated):



The previous plot shows behavior which makes the ZST problem not useful – the solutions break down at the cusp point (where the curvature blows up, and the mapping is no longer univalent). It is also worth nothing that various other maps work, making an arbitrary amount of solutions possible, and also making the problem not really seem applicable to the real life case (see figure below, which displays a cusp problem on the left from a polynomial map, and a logarithm map that is a slit/finger model). The real life case also involves surface tension and units, so of course it makes sense that the ZST model may not accurately predict results. The ZST problem is also ill posed, because as stated in one thesis I read, solutions with close initial conditions can either break down in finite or infinite time. (Dallaston, 2013)



2 Dallaston, (2013)

Conclusion

To conclude, surface tension regularizes the Hele-Shaw problem, and makes it relevant – using the method we outlined, it is possible to obtain all sorts of interesting geometric solutions, but in reality these may not be useful. Surface tension also prevents cusps from forming in experiments.

Other complex variables methods for analyzing the Hele Shaw problem (both the ZST and NZST cases), such as the Schwarz function exist, but I have not included them, as they are much more involved. However, they confirm that the NZST problem is generally much less easy to solve than the ZST problem. Further research on the NZST problem is useful, because surface tension is what keeps the problem from being undetermined.

References

Linda Cummings, Phd thesis, Oxford, 1996

Michael Dallaston, Phd thesis, Queensland U. of Technology, 2013

Code

```
function Capstone1
clc
clear all
integrationSpan=[0,0.6]
ICs=[1,0.1]

[t,P]=ode15s(@SystemofEqs,integrationSpan,ICs);
a1=P(:,1);
a2=P(:,2);
s=size(a2)
i=0:2*pi/(s(1)-1):(2*pi);
q=size(i)
circle=transpose(i);
x1=cos(circle);
y1=sin(circle);
for k=1:1:38
plot(a1(k).*x1+a2(k).*(x1.^2-y1.^2),2*a2(k).*x1.*y1+a1(k).*y1,'r')
hold on
end
title 'Quadratic map: solutions break down at a cusp'

 function aprime=SystemofEqs(t,a)

Q=5;

a1=a(1);a2=a(2);
aprime=[(-1*a1*Q/(2*pi))/(a1^2-4*(a2^2));

-2*a2*(-1*Q/(2*pi))/(a1^2-4*(a2^2))];
```