# Free Boundary Models in 

## Viscous Flow

Linda Jane Cummings<br>St. Catherine's College Oxford

Thesis submitted for the degree Doctor of Philosophy

Trinity Term 1996


#### Abstract

The time-dependent free boundary problems of Hele-Shaw flow and slow viscous flow (Stokes flow) are studied, using complex variable methods.

The first chapter introduces the two problems; the mathematical models are presented, and brief literature reviews are given. Chapter 2 is a review of known results for the Hele-Shaw problem, and develops the conformal mapping ideas which are central to the thesis. In chapter 3 , existing work for the Stokes flow problem is reviewed and extended, and new results are presented, principally for the (singularity driven) zero surface tension problem on a bounded flow domain. Chapter 4 discusses an extension of the work of the previous chapter, as applied to sintering problems in the glass industry. Problems on unbounded fluid domains are considered in chapter 5, for both Hele-Shaw and Stokes flow.

Chapter 6 is concerned with singularity-driven Stokes flow, in the limit of small positive surface tension. Established theory of so-called "weak solutions" is reviewed, and applied to a new example.

In chapter 7, the existing "crack" theory of Hele-Shaw flow is presented, and a new, complementary "anticrack" model is developed. Finally, in chapter 8, we summarise and suggest ideas for further work.


## Acknowledgements

I would like to thank my supervisors, Dr S. D. Howison and Dr J. R. Ockendon, for guidance and encouragement over the past three years. I have had several helpful discussions with Professors Yuri Hohlov, John King, and Andrew Lacey, and Dr Peter Howell, to whom I am very grateful. Many thanks to Bruce and Vicki for help with proof-reading, to James and Dan for technical help, and to all the other friends at O.C.I.A.M. (especially DH8 members, past and present) who have made the past three years so enjoyable. I also thank my family and Giles, for their love and support.

Finally, I gratefully acknowledge financial support from the E.P.S.R.C. in the form of an open award, and from Smith System Engineering in the form of a scholarship and travel grant.

## Contents

1 Introduction ..... 3
1.1 Survey of the thesis ..... 3
1.2 The Hele-Shaw problem ..... 4
1.2.1 The basic equations ..... 4
1.3 The Stokes flow problem ..... 6
1.3.1 The basic equations ..... 6
1.4 Literature Reviews and Discussion ..... 7
1.4.1 Hele-Shaw Flow ..... 7
1.4.2 Stokes flow ..... 9
2 Complex variable methods for Hele-Shaw flow ..... 12
2.1 Preliminaries ..... 12
2.2 The Polubarinova/Galin approach ..... 12
2.3 The Schwarz function ..... 14
2.4 A simple example ..... 16
2.5 Richardson's "moments" and the Cauchy transform ..... 17
2.6 Transformation of the dependent variable ..... 20
2.7 Univalency and conformality ..... 23
2.8 Summary ..... 25
3 Complex Variable methods for Stokes flow ..... 27
3.1 Richardson's approach ..... 27
3.2 Reduction to a single equation ..... 31
3.2.1 Another global equation ..... 33
3.3 Method of solution ..... 34
3.4 A simple example ..... 35
3.5 Zero surface tension problems ..... 36
3.6 The conserved quantities ..... 37
3.6.1 Polynomial mapping functions ..... 38
3.6.2 Comparison with the Hele-Shaw problem - 'Richardson's Moments' and other matters ..... 39
3.6.3 Source/sink systems - a warning example ..... 40
3.7 The Schwarz function for the ZST problem ..... 42
3.8 The "moments" for the case $\Phi(0) \neq 0$ ..... 46
3.9 The stress function ..... 47
3.9.1 The "Baiocchi transform" for Stokes flow ..... 48
3.10 Summary ..... 49
4 Applications to the glass industry ..... 51
4.1 Introduction ..... 51
4.2 The theory for a viscous fibre ..... 51
4.3 "Conserved quantities" for fibres ..... 53
4.3.1 Example - the sintering of a bundle of fibres ..... 53
4.3.2 Connectedness considerations ..... 55
4.4 Summary ..... 56
5 Flow in unbounded domains ..... 57
5.1 Introduction ..... 57
5.2 Literature Review ..... 57
5.3 The Hele-Shaw dipole problem ..... 60
5.4 The Stokes flow dipole problem ..... 70
5.4.1 Review of Jeong \& Moffatt's steady solution ..... 70
5.4.2 The time-dependent problem ..... 72
5.5 Steady Stokes flow reconsidered ..... 76
5.6 Summary ..... 82
6 Stokes flow with small surface tension ..... 83
6.1 Review of "weak" solutions ..... 83
6.2 The cubic polynomial map ..... 85
6.2.1 Complex coefficients ..... 92
6.3 Summary ..... 96
7 Crack and Anti-crack solutions to the Hele-Shaw model ..... 97
7.1 Overview of cracks and slits ..... 97
7.2 Introduction to Anticracks ..... 103
7.3 Exact ZST anticrack solutions ..... 106
7.3.1 The "generic" anticrack ..... 106
7.3.2 Solutions with many anticracks ..... 109
7.3.3 Howison's radial "anticrack" solutions ..... 113
7.4 The Schwarz function of an anticrack ..... 114
7.5 Results from formal asymptotics ..... 114
7.6 Paterson's analysis ..... 116
7.6.1 The case $\epsilon \ll T$ ..... 119
7.6.2 The case $1 \gg \epsilon \sim T$ ..... 120
7.6.3 The case $T \ll \epsilon \ll 1$ ..... 121
7.6.4 Conclusions for Paterson's anticracks ..... 121
7.7 Fractal Hele-Shaw ..... 123
7.8 Cracks revisited ..... 125
7.8.1 The case $\epsilon \ll T$ ..... 126
7.8.2 The cases $1 \gg \epsilon \sim T, 1 \gg \epsilon \gg T$ ..... 127
7.9 The "curvature conjecture" ..... 127
7.10 Extremal conformal maps ..... 132
7.11 Summary ..... 135
8 Discussion and further work ..... 137
8.1 Comparison of Hele-Shaw and Stokes flow ..... 137
8.2 Further work ..... 139
A Stability of blobs and bubbles in Stokes flow ..... 140
A. 1 The perturbed circular blob ..... 140
A. 2 The perturbed circular bubble ..... 141
B Results used for the cubic polynomial map ..... 142
C The Stokes flow velocity field in terms of $w(\zeta, t)$ ..... 146

## List of Figures

1.1 The two phase Hele-Shaw problem. ..... 5
2.1 The mapping from the unit disc onto the fluid domain. ..... 13
2.2 The mapping from the right-half plane onto the fluid domain. ..... 14
2.3 The initial and final domains for the quadratic polynomial mapping. ..... 17
2.4 Schematic diagram of a system of sinks $Q_{1}, \ldots Q_{6}$ at points $z_{1}, \ldots z_{6}$ within $\Omega(t)$. ..... 19
2.5 The univalency domain $V$, and phase trajectories, for the "limaçon" example of $\S 2.4$. With a point sink, the phase paths are followed in the direction of the arrows (the nonunivalent region); with a point source, the direction is opposite. Intersection with the boundary of $V$ is associated with the cusped cardioid geometry. ..... 24
4.1 Typical cross-sections generated by the map (4.6) when $n=6$. Picture (a) is the cusped configu- ration, while (b) is the kind of smooth cross-section we might expect to observe in practice. ..... 54
5.1 Schematic diagram showing how a "continuable 5/2-power cusp" solution looks in phase trajectory space within the univalency domain. ..... 58
5.2 The geometry for the problem of a dipole placed off-centre in a circle. ..... 60
5.3 The geometry for the dipole-in-a-half-space problem. ..... 62
5.4 The univalency domain in ( $b, c$ )-space for the mapping function (5.19). ..... 64
5.5 Typical free boundaries generated by points $(b, c)$ on the boundary of the univalency domain(figure 5.4). The dipole is situated at the origin in each case, and is such that the $x$-axis is astreamline in the positive sense. (1) has $a=1, b=1, c=4$, and has a single cusp in the freeboundary; (2) has $a=-1, b=1, c=-5$, and has two cusps in the free boundary, and (3a)has $a=-1, b=4, c=-9$, and shows the free boundary beginning to overlap itself. (3b) is anenlargement of the trapped air bubble in (3a).65
5.6 The phase diagram (within the univalency domain) for the Hele-Shaw dipole problem. ..... 67
5.7 Enlargement of the transient 5/2-power cusp formation. Pictures (a), (b) and (c) illustrate how the free boundary passes through the $5 / 2$-power cusped configuration (a), to a smooth boundary (b), before ultimately blowing up with two $3 / 2$-power cusps (c). ..... 68
5.8 Typical free boundary shapes described by the mapping (5.44). Case (a) has $\alpha=-0.4, \beta=$$-5, \gamma=0.5$, and corresponds to a dipole such that the $x$-axis is a streamline from negative topositive. Case (b) has $\alpha=1, \beta=1.4, \gamma=0.8$, and has the $x$-axis as a streamline from positiveto negative.80
5.9 Typical free boundary shape described by the mapping (5.58). The parameter values used hereare $\alpha=-1, \beta=3.5, \gamma=0.65$. The dipole at the origin is such that the $x$-axis is a streamlinefrom negative to positive.81
6.1 The different regions in the small surface tension "limaçon" problem, using matched asymptotics. ..... 84
6.2 Free boundary shapes described by the map (6.5) for various points ( $b, c$ ) on the boundary $\partial V$of the univalency domain. The values used are: $\left(b_{1}, c_{1}\right)=(0,1),\left(b_{2}, c_{2}\right)=(1,1),\left(b_{3}, c_{3}\right)=$$(4 \sqrt{2} / 3,1),\left(b_{4}, c_{4}\right)=(1.8,0.8461),\left(b_{5}, c_{5}\right)=(8 / 5,3 / 5),\left(b_{6}, c_{6}\right)=(1,0)$, and $\left(b_{7}, c_{7}\right)=(1 / 5,-4 / 5)$.Pictures (3b) and (4b) are magnifications of the nonunivalent region, showing how the free bound-ary begins to overlap itself; the former case is cusped and self-overlapping, while the latter issmooth. The value $a=1$ was used to generate each picture, hence the shapes do not have equalareas.87
6.3 The function $F(b)$ governing evolution on the part $b=1+c$ of $\partial V$. (Note the difference in scales between the two plots.) ..... 90
6.4 The univalency diagram (restricted to the right-half ( $b, c$ )-plane) for the cubic polynomial mapping function. The shaded region corresponds to a nonunivalent map. ..... 91
6.5 The three-dimensional univalency domain $V_{\ddagger} \subset V_{4}$, and its two-dimensional cross-sections $V, V_{\dagger}$ and $V_{o}$. The arrows on $V_{\dagger}$ indicate how the point $\{b=0, c=-1\}$ destabilises ( $c f$. figure 6.4) ..... 95
7.1 The geometry of (a) a finite crack; (b) a semi-infinite crack, along the $x$-axis (driven by a sink at infinity). ..... 98
7.2 Schematic diagram showing how a general slit solution works. ..... 101
7.3 Examples of the radial fingering solutions of [45], together with a photograph of one of Paterson's experiments [73]. ..... 104
7.4 Phase-field computations of the "free boundary" (actually a level set of the phase parameter $\phi$ occurring in the phase field model) for the growth of a seed of solid into a supercooled liquid. This picture was kindly supplied by Dr A. R. Gardiner [23]. ..... 105
7.5 A typical "generic anticrack" solution. The free boundary is shown for times $t=t_{1}, t_{2}, t_{3}$, with $t_{3}>t_{2}>t_{1}$ ..... 108
7.6 A typical solution generated by (7.15), showing 4 well-developed anticracks. Here, $\pi \alpha_{i}=0.5,1,0.8,0.4$,for $i=1,2,3,4$, respectively; $\mu_{i}=-3,-2,0.5,1.5$, and $\lambda_{i}=0.5$ for each $i$.110
7.7 Sketch showing the geometry when we have an array of "fat anticracks" with narrow spacing generated by (7.15). The gaps between the "anticracks" may be viewed as cracks. ..... 111
7.8 Solution of the form (7.15) exhibiting what we interpret as crack and anticrack formation. The values $\alpha_{1}=0.1$ (the anticrack), and $\alpha_{2}=-3$ were used. ..... 112
7.9 The local geometry with a corner of internal angle $\phi$ in the fluid. ..... 115
7.10 Graph showing the relative sizes of the coefficients of the terms $\sin n \theta$ (in the $\epsilon R_{1}$ term of $R$ ) and $\cos 2 n \theta$ (in the $\epsilon^{2} R_{2}$ term of $R$ ) as functions of time, in the $\epsilon \ll T$ régime. The $\sin n \theta$ cofficient is the upper curve. ..... 120
7.11 Evolution of the free boundary of Paterson's expanding bubble for dimensionless times $t=$ $1,1.15,1.3$. This plot applies to the régime in which the amplitude of the perturbations is much greater than $T$, so that the ZST perturbation theory is applicable ..... 122
7.12 Graph showing the cofficients of the terms $\sin n \theta$ (in the $\epsilon R_{1}$ term of $R$ ) and $\cos 2 n \theta$ (in the $\epsilon^{2} R_{2}$ term of $R$ ) as functions of time, for the $\epsilon \gg T$ régime. The $\sin n \theta$ coefficient is the one with initial value 1. ..... 123
7.13 The evolving anti-slit structure at the stage $n=2$, with 2 anti-slits of length $L_{0}, 2$ of length $L_{1}$, and 4 new anti-slits about to form ..... 124
7.14 Evolution of the free boundary of the contracting viscous blob in the régime $\epsilon \gg T$, so that the ZST theory is applicable. The early stages of crack formation are apparent prior to breakdown of the linear theory. ..... 128
7.15 A free boundary with crack and anticrack development ..... 129
7.16 The "Ivantsov" anticrack solution ..... 130
7.17 Anticrack-type structure generated by the map (7.57) with $N=4, \alpha=0.2, b(0)=0.755$, $c(0)=0.887$. We see the onset of cuspidal blow-up, after which we expect continuation by crack or slit evolution towards the point sink ..... 131
7.18 The crack (a) and anticrack (b) geometries generated by maps (7.58) and (7.59) with $\epsilon=0.1$. ..... 132
7.19 The "open sets" interpretation of the slit geometry. ..... 133
7.20 The "open sets" interpretation of the anti-slit geometry. The 'set of boundary points' is not open in the topology of $X$. ..... 134
7.21 The free boundary for a general suction problem (with small surface tension). . . . . . . . . . 136

## Chapter 1

## Introduction

### 1.1 Survey of the thesis

This thesis is concerned with two different free boundary problems: the Hele-Shaw problem, and the problem of two-dimensional slow viscous flow, or Stokes flow, as it is commonly known. Most of the new results we present are for the latter; however, the (complex variable) methods of attack we use for both have many similarities, and since historically the use of such methods for the Hele-Shaw problem predates their use for Stokes flow, we shall review the Hele-Shaw problem first. The complex variable methods used are, in any case, probably more straightforward when applied to Hele-Shaw flow, so this approach has the added advantage of introducing the ideas to the reader as gently as possible.

Our aim throughout the thesis is to present as unified an account as possible; hence wherever practicable we shall keep the same notation for the two problems, this being largely that used by Richardson [82] for the Stokes flow problem. Also in the interests of coherence and continuity, we do not strictly segregate the two problems, but highlight similarities and contrasts between the two as these arise. Since so much more literature exists for the Hele-Shaw problem, such questions of similarities or differences generally arise as we find a result for Stokes flow which has a Hele-Shaw "analogue", or which is quite different from the existing Hele-Shaw result.

The remainder of this chapter is devoted to introducing the two problems, giving a little physical background for each. The governing equations and boundary conditions are derived, and brief literature reviews are presented.

In the next two chapters solution methods for both problems are described, using techniques from complex variable theory. The Hele-Shaw work of chapter 2 is basically review; we present it firstly as necessary background, and secondly because our results for Stokes flow provide interesting analogues, and the possible link between the two problems has, except for the work of Howison \& Richardson [49], been largely ignored in the literature. Chapter 3, which concerns Stokes flow, contains mainly new work. With the exception of $\S \S 3.1$ and 3.3 (which review the work of Richardson [82], but which present a new perspective on it), and $\S 3.9$ (which is based on an idea due to King [58]), the work is original (unless otherwise stated). Work which is closely related to that of chapter 3, but which would break the flow if included there, is presented in chapter 4. This concerns models of slender viscous fibres experiencing traction, and is partly a review of the work of [42], and partly new.

The examples considered in chapter 3 are all for finite domains. Chapter 5 extends the discussion to unbounded fluid domains, to reveal possible complications that can arise with the Stokes flow problem, but not in Hele-Shaw. Apart from the review material in §5.2, and $\S 5.4 .1$, all the work in this chapter is original. Both chapter 3 and chapter 5 are largely (though not exclusively) concerned with the zero-surface tension problems. The limiting case of small, positive surface tension in Stokes flow is the subject of chapter 6, using ideas developed by Howison \& Richardson [49]. After reviewing these ideas, a new example is given, and discussed at some length. From $\S 6.2$ to the end of the chapter is new work.

The perspective shifts somewhat in chapter 7, which discusses "crack" and "anticrack" solutions to the Hele-Shaw model. Section 7.1 reviews the established theory of cracks, while the remainder of the chapter is a blend of review, and original work (the distinction is made clear in the text). The work that is reviewed, however, is presented in a different context in the light of our crack/anticrack theory.

Finally, we discuss our results, and suggest some possible directions for further work in chapter 8.

### 1.2 The Hele-Shaw problem

We begin by giving a short introduction to Hele-Shaw flow, the first of our free boundary problems. A Hele-Shaw cell consists of two rigid parallel plates some small distance (bay) apart, between which is sandwiched one or more (immiscible) Newtonian incompressible viscous fluids which can be injected, sucked out, or subjected to pressure gradients. The problem is to model the flow of the fluid within the cell. It dates back to Hele-Shaw's original paper [31], published in 1898. The emphasis there was on the ability of the Hele-Shaw cell to reproduce faithfully the streamlines for inviscid irrotational flow past obstacles placed in the cell, providing remarkable visual verification of theoretical results. We shall be concerned with the evolution of a fluid domain with a free boundary (adjacent to a zero pressure region), under the action of prescribed pressure gradients. The problem has been extensively (though not continuously) studied since Hele-Shaw's time. The slender geometry of the cell means that the problem is effectively two-dimensional, being independent of the co-ordinate normal to the plane of the cell, which greatly simplifies matters; particularly fortuitous is the consequence that complex variable techniques (such as conformal mapping) can be applied with considerable success. We shall be using complex variable methods almost exclusively throughout the thesis.

The problem is of inherent theoretical interest, but there are various other reasons for wanting to study it: the mathematical model is the same as that for many important physically-occurring moving boundary problems, including flow in porous media [75], filtration [28], pollution of groundwater [76], problems in oil and gas recovery [75], [41], electrochemical machining [61], crystal growth [72], injection moulding, and so on. In particular, it is a special case of the one-phase Stefan model for phase-change [86], the two models coinciding in the limit as the specific heat of the medium tends to zero.

### 1.2.1 The basic equations

Consider first the more general two-phase (or "Muskat") problem of figure 1.1, where the gap between the plates is filled with two fluids of different, constant viscosities $\mu_{1}, \mu_{2}$, occupying regions $\Omega_{1}, \Omega_{2}$ respectively (see for example [22] for a discussion). If $b$ is the gap width and $l$ the linear dimension of the Hele-Shaw cell then under the assumption that $b \ll l$ the Navier-Stokes equations reduce ${ }^{1}$ to

$$
\mathbf{u}_{i}=-\frac{b^{2}}{12 \mu_{i}} \nabla p_{i}, \quad \nabla \cdot \mathbf{u}_{i}=0, \quad i=1,2
$$

where $p_{i}$ is the pressure in fluid $i$ and all quantities depend only on the co-ordinates in the plane of the cell, $(x, y)$, and time, $t$. Hence the pressure is a velocity potential for the flow, and

$$
\nabla^{2} p_{i}=0 \quad \text { in } \Omega_{i}(t), \quad i=1,2 .
$$

There are two conditions holding on the free boundary $\partial \Omega$ between the two fluids. Firstly we have the dynamic boundary condition (DBC), which comes from a force balance at the free boundary, and is usually taken to be

$$
\begin{equation*}
p_{2}-p_{1}=-T \kappa . \tag{1.1}
\end{equation*}
$$

[^0]

Figure 1.1: The two phase Hele-Shaw problem.

Here, $T$ is the surface tension coefficient and $\kappa$ is the curvature in the ( $x, y$ )-plane (positive when the domain $\Omega_{1}$ is convex). This form of the DBC ignores any three-dimensional effects due to the curvature of the free boundary in the plane of the cell. A more accurate condition is given by McLean \& Saffman [65], namely

$$
\begin{equation*}
p_{2}-p_{1}=-T\left(\kappa-\frac{2}{b} \cos \chi\right) \tag{1.2}
\end{equation*}
$$

where $\chi$ is the contact angle between the meniscus and the cell plates at the free boundary. If, as in [65], $\chi$ is assumed to be constant, we again arrive at (1.1) without loss of generality; however if $\chi$ is not constant then (1.2) and (1.1) are very different, so caution is clearly advisable. Even (1.2) is not exact, since it relies on the assumption that the advancing viscous fluid completely expels the receding fluid (i.e. there is no 'wetting' of the plates), which is not the case in general (this is discussed in [74]). Nonetheless, it is usual in the literature to adopt either (1.1), or the simpler "zero surface tension" boundary condition (see below) when solving problems.

We also have the kinematic boundary condition (KBC), encoding the fact that fluid particles which are initially on the boundary must remain there (that is, $\partial \Omega$ is a material curve),

$$
v_{n}=-\frac{b^{2}}{12 \mu_{i}} \frac{\partial p_{i}}{\partial n}, \quad i=1,2
$$

which is derived by equating the normal components of the fluid velocity to the normal velocity $v_{n}$ of the boundary. To close the system we need to specify $\Omega(0)$, and some driving mechanism for the flow; for instance, if we have a point sink of strength $Q>0$ at the origin, the singularity in the pressure is $p \sim(Q / 2 \pi) \log r$ as $r \rightarrow 0$; for a point source of strength $Q$, the sign is reversed in this singular behaviour.

From now on we assume that fluid 2 has negligible viscosity (air, or vacuum). In the limit $\mu_{2} \rightarrow 0$, the solution in region $\Omega_{2}$ tends uniformly to $p_{2}=$ constant, where the value of the constant may vary for different components of $\Omega_{2}$ (we do not yet know that $\Omega_{2}$ is connected). If we take $\Omega_{1}$ to be simply connected, then $\Omega_{2}$ (if it is finite) will be connected, and $p_{2}$ must assume the same constant value throughout $\Omega_{2}$; without loss of generality we take this to be zero. Then, dropping suffices, and making a trivial nondimensionalisation, we arrive at the simpler one-phase model:

$$
\begin{align*}
\nabla^{2} p & =0 \quad \text { in } \Omega(t)  \tag{1.3}\\
p & =T \kappa \quad \text { on } \partial \Omega(t)  \tag{1.4}\\
\frac{\partial p}{\partial n} & =-v_{n} \quad \text { on } \partial \Omega(t) \tag{1.5}
\end{align*}
$$

with prescribed pressure driving mechanism. Without the assumption of simple connectedness we would need boundary conditions analogous to (1.4) on each separate portion of the free boundary, but with extra arbitrary additive constants on the right-hand sides, and with the restriction that $p$ be single-valued. Multiply-connected fluid domains are considered in detail in [84].

We shall in the main consider situations where $T$ is small, and replace (1.4) by the approximate "zero-surface tension" (henceforth"ZST") condition,

$$
\begin{equation*}
p=0 \quad \text { on } \partial \Omega(t) . \tag{1.6}
\end{equation*}
$$

This may be justifiable provided the curvature of the free boundary is nowhere of the order of $1 / T$ (that is, as long as the boundary is reasonably smooth), and is certainly desirable, since the ZST problem is very much more tractable. Caution is necessary, however, since we have no guarantee that an initially "reasonably smooth" boundary will remain so for the duration of the motion. We defer further discussion of these matters until §1.4.

### 1.3 The Stokes flow problem

We now introduce the second of our two free boundary problems: the problem of two-dimensional slow viscous flow, or Stokes flow, with time-dependent geometry. Despite the undeniably threedimensional nature of most real-world low Reynolds number ${ }^{2}$ flows, the two-dimensional problem is invaluable as an aid to understanding many physical phenomena, either as a preliminary "paradigm" problem, or because the geometry is slender in some sense, so that asymptotic methods may be applied to yield a two-dimensional problem at first order. Being so much simpler than the three-dimensional problem which generally requires heavy computation, it is often worthwhile doing a two-dimensional version of the problem first, providing a sensible one exists.

Real-world situations which can be modelled by Stokes flow are numerous. The dynamics of bubbles and drops trapped within a low Reynolds number flow [1], [77] is one very general example, relevant to many physical processes. The rheology of emulsions, mixing in multi-phase viscous systems, and bubbles trapped within a viscous fluid such as molten glass, are all describable by this model. Fully three-dimensional (unsteady) geometries are difficult to describe mathematically; however, two-dimensional drops and bubbles are easily modelled [78, 80], [96], and whilst clearly physically unrealistic, ${ }^{3}$ such models provide a useful guide before embarking on the full problem. Axisymmetric geometries are also reasonably simple [101], [70], particularly if, as mentioned above, the drop or bubble is slender (such as may occur in an extensional flow), so that asymptotic methods may be used to simplify the problem [7], [43].

The dynamics of two-dimensional viscous blobs (surrounded by inviscid fluid) is also of relevance [38], [82]. This can model viscous sintering, a phenomenon crucial to many physical processes. A review of its applications is given in [100]; a specific example which we shall consider in chapter 4 is the sintering of viscous fibres, such as arises in optical fibre manufacture [42], [85].

Finally, we mention another interesting real-world example which can (at least in certain flow régimes) be modelled by two-dimensional Stokes flow. This is the structure of foams, which may be thought of as thin viscous sheets (the lamellae) joined together along "Plateau borders", which are basically 'tubes' of viscous fluid, and are where most of the liquid of the foam resides.

### 1.3.1 The basic equations

Before making any more general remarks, we derive the equations and boundary conditions which govern slow viscous flow. In this thesis we are considering the two-dimensional motion of a simplyconnected domain of fluid (again denoted by $\Omega(t)$ and taken to lie in the ( $x, y$ )-plane), which we

[^1]assume is dominated by viscous, rather than inertial effects. The Reynolds number of the flow will thus be small, and so we use the Stokes flow equations (see for instance [6] or [71] for the details),
\[

$$
\begin{equation*}
\nabla p=\mu \nabla^{2} \mathbf{u}, \quad \nabla \cdot \mathbf{u}=0 \tag{1.7}
\end{equation*}
$$

\]

holding within $\Omega(t)$. All notation here is as for the Hele-Shaw problem. We also need boundary conditions on the free boundary $\partial \Omega(t)$. There are two stress boundary conditions, or SBC's, (derived from an elementary force balance) the first of which requires the shear stress to be continuous across the free boundary, and the second of which says that the jump in the normal stress (as we pass from the fluid to the air) is given by $T \kappa$, where $T$ is the constant coefficient of surface tension and $\kappa$ is the curvature of the free boundary (measured as before). These two conditions may be written as a single vector equation,

$$
\begin{equation*}
\sigma_{i j} n_{j}=-T \kappa n_{i} \quad i=1,2, \tag{1.8}
\end{equation*}
$$

where $\sigma_{i j}$ is the usual Newtonian stress tensor,

$$
\sigma_{i j}=-p \delta_{i j}+\mu\left(\frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial u_{j}}{\partial x_{i}}\right)
$$

and $\mathbf{n}=\left(n_{i}\right)$ is the outward normal to $\partial \Omega$. If $U$ is some typical flow speed, it is clear from (1.8) that an important parameter of the flow is the Capillary number, $C a=\mu U / T$, which measures the relative effects of viscosity and surface tension. We also have the usual kinematic boundary condition (KBC),

$$
\begin{equation*}
\mathbf{u} \cdot \mathbf{n}=v_{n}, \tag{1.9}
\end{equation*}
$$

$v_{n}$ being the outward normal velocity of the free boundary. Since the flow is two-dimensional and incompressible, there exists a streamfunction $\psi(x, y, t)$ such that

$$
u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x}
$$

To close the problem, any singularities in the flow (such as sources, sinks, dipoles, etc.) must also be specified. Most of our solutions will involve such driving singularities.

Taking the curl of the first of equations (1.7) reveals that $\psi$ must satisfy the biharmonic equation in the flow domain,

$$
\nabla^{4} \psi=0
$$

Like (1.3), this equation has the extremely useful property that its solutions are expressible as functions of complex variables (the so-called Goursat representation of solutions) so that again many powerful results from complex variable theory can be drawn upon. We return to this point in chapter 3.

### 1.4 Literature Reviews and Discussion

### 1.4.1 Hele-Shaw Flow

Any review of Hele-Shaw flow, unless it is to be a thesis in itself, must be highly selective because of the vastness of the existing literature. The problem has been studied using complex variable theory (see for example the work of Richardson [79, 81, 84], or Howison [48] for a review); numerical methods [65], [98], [54, 55]; rigorous existence-uniqueness theory [18], [26, 27], [12], [56] (via a weak formulation of the problem); "phase field" theory [9]; exponential asymptotics [94]; not to mention of course a large body of experimental work (see for instance [87], [59], [73]). The references given
here are very restricted; one could easily give ten or more for each item of the list above. We simply chose a representative few, to give an idea of the wide spread of work that has been done.

Much of the Hele-Shaw literature deals exclusively with the ZST Hele-Shaw problem. As we commented in $\S 1.2 .1$, this is much more tractable than the NZST problem, but there are potential difficulties which it is appropriate here to expand upon.

Firstly, a simple linear stability analysis for a sinusoidally-perturbed planar travelling wave front may be carried out for the ZST, two-phase (Muskat) problem. This reveals that a free boundary advancing into a more/less viscous fluid is always unstable/stable, respectively. With non-zero surface tension (NZST) the same is still true; however, higher wavenumbers (shorter wavelengths) are stabilised. For the one-phase ZST problem the net result is that an advancing/retreating viscous front is stable/unstable respectively, with an analogous result for the NZST problem (see §7.3.2). We refer to these two cases as "blowing"/"suction" (or "injection"/"suction") problems, respectively, there being a fundamental difference between the two.

Secondly, we note that the ZST Hele-Shaw problem is time reversible: if we change the signs of $p$ and $t$ in (1.3), (1.6), (1.5), and reverse the driving singularity, the problem is unchanged. Consider the paradigm problem in which the flow is driven by a single point source of strength $Q>0$ at the origin. If we start from an initially empty fluid domain, the solution is easily seen to be an expanding circle of viscous fluid, with radius $R(t)=\sqrt{Q t / \pi}$. It follows that, for flow driven by a point sink of strength $Q$ at the origin, only if $\Omega(0)$ is a circle centred on the origin will we be able to extract all the fluid from the cell. Any other initial domain must lead to finite-time blow-up of the problem. This is an example of a more general result: the ill-posedness of the ZST Hele-Shaw problem with a retreating free boundary (the "suction" problem). Only those solutions which are time-reversals of well-posed problems with advancing free boundaries ("injection" or "blowing" problems), whose analytic behaviour can be traced back to time $t=-\infty$ or which started from initially empty fluid domains, will avoid finite-time blow-up.

This breakdown of solutions is often via the formation of a cusp in the free boundary (which the theory assumes to be analytic). ${ }^{4}$ For $T \ll 1$ the assumption is that the ZST theory holds good until times very close to breakdown, at which point the high curvature at a single point means that surface tension effects must become important-mathematically, the boundary condition $p=0$ is no longer a valid approximation to $p=T \kappa$ when $\kappa$ becomes large.

The NZST Hele-Shaw problem is notoriously difficult (much more so than the NZST Stokes flow problem, as we will see). This difficulty is tied in with the ill-posedness referred to above; taking the NZST boundary condition (1.4) amounts to a perturbation of the boundary data, which is well known to often have disastrous consequences for an ill-posed problem-even a tiny change in the data is liable to cause a large change in the solution.

A significant body of literature is concerned with the idea of regularising the ill-posed ZST "suction problem". We have already noted that such ZST solutions invariably break down within finite time, often via cusp formation in the free boundary, which is physically unacceptable. The problem must clearly be modified in some way if this breakdown is to be avoided, but hopefully without having to consider the full NZST problem, although this seems the obvious thing to do.

Other types of regularisation which have been studied (none very successfully) include employing a "kinetic undercooling" boundary condition, where the jump in pressure across $\partial \Omega$ is taken to be proportional to the normal velocity $v_{n}$ of $\partial \Omega$, or a "viscosity" type of regularisation ${ }^{5}$, which gives rise to a phase-field model. Both of these ideas are considered in Hohlov et al. [33]. In a series of papers [36, 62, 33], these authors develop a novel kind of possible regularisation for the small surface tension problem. Supposing one has a flow driven by a point sink, then their idea is that ZST theory will apply until times close to blow-up (so an almost cusped configuration has formed), at which point a thin "crack" of air will enter the fluid domain and propagate rapidly towards the sink. Whilst this is happening, the rest of the free boundary remains smooth, and

[^2]hardly moves at all. This has an interesting ZST limit; ZST theory applies until the classical solution breaks down with cusp formation, then a slit (i.e. a crack of zero thickness) propagates into the fluid, its evolution being on a timescale such that the smooth part of the boundary actually remains static. Such solutions are "weak", in the sense that the free boundary is nonanalytic. It is conjectured that the solution would still break down in finite time though, when the crack or slit reaches the sink (the models, and this conjecture are supported by the numerics of [69], [55]). These ideas are discussed further in chapter 7.

Other weak solutions have been found by King et al. [56], who study fluid domains that initially have nonanalytic free boundaries containing corners. Both the suction (ill-posed) and injection (well-posed) problems are considered, and local similarity solutions are constructed near the corners, in a wedge geometry. Surprisingly, their work reveals that solutions to both the injection and suction problems exist which have persistent corners, contrary to the usual conjecture that 'injection always smooths' (that is to say, that the free boundary for $t>0$ will be analytic, even if $\partial \Omega(0)$ is not).

In $\S 1.2 .1$, we commented on the difficult nature of the NZST boundary condition. Remarkably, when one considers how extensively the Hele-Shaw problem has been studied, the NZST problem is still largely intractable, with very few firm analytical results existing. Duchon \& Robert proved the existence of classical solutions to the NZST model, and Escher \& Simonett [26] proved existence and uniqueness for the same problem, with general initial conditions. Chen et al. [12] did the same for the zero specific heat Stefan problem (note that these results are all local in time). Steady explicit solutions have been presented in [24], [99] (although those of [99] are in a highly artificial geometry). Modern computing power, and fast, effective numerical schemes mean that an ever-increasing number of NZST numerical solutions are available (see [54, 55], [65], [69], [98], for instance).

No review of the Hele-Shaw problem would be complete without some mention of the famous Saffman-Taylor "fingering" problem. In 1958 Saffman \& Taylor [87] conducted experiments in which regular, evolving "fingers" of air were observed penetrating a channel of viscous fluid. For small values of the surface tension parameter the width of these fingers was almost exactly half the channel width, and the authors constructed exact travelling-wave solutions [87] (and later, exact time-evolving solutions, [88]) to the ZST problem, which gave free boundary shapes remarkably similar to the (large-time) experimental observations. However, their solutions contained an arbitrary parameter, the finger width $\lambda$. It was believed that the addition of small positive surface tension to the model would resolve this indeterminacy, but all early attempts to do this via perturbation analysis failed, the limit $T \rightarrow 0$ being singular. Numerical results were more satisfactory [65], [98], but the analytical explanation defied researchers until the so-called 'microscopic solvability' hypothesis [13], [34], [91], which claims that the "selection" of a particular value of $\lambda$ is governed by terms in the perturbation expansion which are transcendentally small in the surface tension parameter $T$. Rather than the continuum of solutions found for the ZST problem, solutions in fact exist only for a discrete set of values of $(\lambda-1 / 2)$. For $T \ll 1$, or large Capillary number, $(\lambda-1 / 2)$ approaches zero, in agreement with the observations of [87]. For a comprehensive review of the Saffman-Taylor problem, see [89].

Relatively recent experiments (Kopf-Sill \& Homsy [59] (1987)) show that, under carefully monitored conditions, narrow, evolving fingers may be observed in low surface tension flows. These fingers are stable except at very low surface tension, when they destabilise via dendritic side branching and tip splitting. Such observations may provide evidence for the "crack" theory mentioned above. Radial fingering has also been observed experimentally [73], [12] and families of ZST solutions constructed [45] which give boundary shapes in good agreement with the experiments-we return to these solutions in chapter 7.

### 1.4.2 Stokes flow

We now turn our attention back to our second free boundary problem. The two-dimensional Stokes flow problem has generated a good deal of mathematical interest from the 1960's onwards, the last few years in particular providing a wealth of new results, stimulated primarily by Hopper
[37, 38, 39] and Richardson [82] (where he generalises his steady-state work of [78, 80]). This new spate of activity in the 1990's stems from the independent discovery of the above two authors that many families of exact, time-dependent solutions to the problem can be found in closed form.

This is a remarkable fact, given the apparent awkwardness of the boundary conditions for positive surface tension. It is perhaps even more surprising when we consider how little progress has been made on the corresponding NZST Hele-Shaw problem, which at first glance one feels ought to be the simpler of the two, being governed by only a second order (Laplace) rather than a fourth order (biharmonic) p.d.e.. Of course, steady solutions to the NZST Stokes flow problem have been around for years, many authors having published papers in the 1960's and 1970's (for example, in roughly chronological order, Garabedian [29], Richardson [78, 80], Buckmaster [7], Youngren \& Acrivos [101]).

More recently still, Howison \& Richardson [49] have considered time-evolving problems (for both the NZST and ZST cases) incorporating a driving mechanism at a finite point within the fluid domain. Such problems have already been extensively studied for ZST Hele-Shaw flow, and there are very many results available for comparison in this case. Of particular interest for Stokes flow is the case where the surface tension parameter is small and positive, since experiments with real, high-viscosity fluids (in approximately two-dimensional geometries) demonstrate that the liquid-air free boundary can adopt an almost cusped configuration. In fact to the naked eye the boundary appears to have an actual cusp, with magnification needed to discern the finite curvature at this point-see for example Jeong \& Moffatt [52], or Joseph et al. [53]. We consider Jeong \& Moffatt's work further in chapter 5.

We shall see that, if we approximate this physical situation with the assumption that surface tension is zero we have the situation that arose for Hele-Shaw flow; solutions for the "suction problem" almost invariably break down within finite time. Like the ZST Hele-Shaw problem, the ZST Stokes flow problem is time-reversible, so, as there, we expect contracting viscous blobs to break down within finite time (except in the trivial case of a contracting circular disc with a sink at the origin). This breakdown can occur in the same ways as those listed for Hele-Shaw in footnote (4). The distinction here between "suction" and "injection" is not so clear, however. For the Hele-Shaw problem it is a simple matter to demonstrate the instability of a retreating viscous front (see $\S 7.3 .2$, for example), and both expanding bubbles and contracting blobs are therefore unstable. Stokes flow, on the other hand (in the absence of singularities) is invariant under rigidbody motions (see $\S 3.1$ ), so a travelling wave planar front is neutrally stable, whether advancing or retreating. Contracting or expanding bubbles and blobs can be analysed, however, and it is found that, in contrast with Hele-Shaw, contracting circular blobs and bubbles are unstable, while expanding blobs and bubbles are stable (this is shown in appendix A). For Stokes flow we tend to reserve the term "suction problem" (with its connotations of instability) for the unstable problem driven by a sink at a finite point within the fluid, and not for the stable situation of an expanding bubble with a sink at infinity.

Given the experimental observations cited above, it seems fair to assume that the ZST theory holds good until times very close to breakdown, at which point the high curvature at a single point brings surface tension effects into play, preventing actual breakdown. Analogous to the " $p=0$ on $\partial \Omega$ " approximation becoming invalid for Hele-Shaw, here, the boundary condition $\left[\sigma_{n n}\right]_{\partial \Omega}=0$ is no longer a valid approximation to the condition $\left[\sigma_{n n}\right]_{\partial \Omega}=T \kappa$ when $\kappa$ becomes large. Antanovskii ([2, 3] and several other papers) has also studied cusped configurations in slow flow, using complex variable techniques to obtain steady solutions to the NZST problem.

The introduction of small positive surface tension into the ZST problem (with driving mechanism) may be regarded as a regularisation of this problem, such as we discussed in §1.4.1 for the Hele-Shaw problem, only not so difficult. The $T \rightarrow 0$ limit of this regularisation has been considered, and solutions having persistent cusps in the free boundary have been found to exist (see [49]; also chapter 6). Such solutions may be contrasted with the "slit" limit of the Hele-Shaw "crack" model, indicating perhaps that we do not expect to find the phenomenon of fingering in slow viscous flow. We return to this point in $\S 8.1$.

It may be obvious, but we should point out that it is only for problems with a driving singularity that the ZST problem is nontrivial; if no driving singularity was present then any initial domain
$\Omega(0)$ would be an equilibrium domain in the absence of surface tension. When we do have a driving mechanism, it is often the case in practice that this is the dominant effect in the flow, hence the ZST approximation. The NZST solutions of [49] demonstrate the competing effects of a flow singularity and surface tension.

Despite our opening remarks about the existence of many exact (time evolving) solutions to the NZST problem, the fact remains that the ZST problem is very much simpler, and admits very many exact, closed-form solutions, which would be just too messy to attempt analytically for positive surface tension. Moreover, all is not lost when ZST solutions break down via cusps, since as mentioned above we can use direct asymptotic methods to examine the effect that surface tension will have as we approach a cusped configuration, and the $T=0$ approximation becomes invalid. These observations, together with the independent mathematical interest of our findings, justify our close study of the singularity-driven ZST Stokes flow problem.

For completeness, we also mention work that has been carried out on solutions for bubbles in infinite fluid domains (we shall return to unbounded flow domains in chapter 5). Many of the early steady solutions for Stokes flow were for bubbles, for example the papers of Youngren \& Acrivos [101] (1976), Buckmaster [7] (1972) and Richardson [78, 80] (1968, 1973). More recently, time-dependent analytical solutions have been presented for two-dimensional bubbles (Tanveer \& Vasconcelos [95, 96] (1994, 1995)), and numerical solutions for three-dimensional axisymmetric bubbles (Nie \& Tanveer [70] (1996)). All of these bubble solutions are for the NZST Stokes flow problem; however like the cases mentioned earlier, many of the solutions exhibit "near cusps" in the free boundary, so that ZST theory could be used for times less than the predicted ZST breakdown time. The last three cited works also allow for the interesting possibility that bubbles may "pinch off", with two sides of the bubble meeting in the middle, and consequent change of topology.

## Chapter 2

## Complex variable methods for Hele-Shaw flow

### 2.1 Preliminaries

In this chapter we introduce the idea that will be followed throughout the thesis: the application of complex variable methods to solve the free boundary problem. The work we present is specific to the Hele-Shaw problem, but the general approach is widely applicable, and this chapter will familiarise the reader with key concepts such as conformal mapping, analytic continuation of identities holding on some boundary, univalency concerns, and so on. There are several different possible approaches to the ZST problem; we present a few of the better-known. The aim of this chapter is not to present new work, but to give a review of well-documented methods, and our discussion is largely theoretical, with few examples of the application of the methods. For further examples, the referenced texts are more than adequate. Unless we state otherwise, it should be understood that the work of this chapter pertains to the ZST case, and henceforth the complex variable $z$ is taken to be $x+i y$, where ( $x, y$ ) are co-ordinates in the (two-dimensional) fluid domain.

The crucial factor which allows us to apply such complex variable methods to the Hele-Shaw problem is that the pressure $p$ is harmonic within the fluid domain $\Omega(t)$, so there exists a function $\mathcal{W}(z, t)$ (the complex potential of the flow), analytic within $\Omega(t)$ (except at driving singularities of the flow), such that

$$
\begin{equation*}
p=-\Re\{\mathcal{W}(z, t)\} . \tag{2.1}
\end{equation*}
$$

We will suppress the time dependence of the various functions except where necessary for emphasis.
One of the earliest methods of solution is due to Polubarinova-Kochina [75] and Galin [28], and the method described in the following section is based on their work.

### 2.2 The Polubarinova/Galin approach

The main difficulty in solving free boundary problems is fairly obvious; it is that we do not know, at the outset, the position of the boundary on which we must apply our boundary conditionsit must be determined as part of the solution process. In fact, our investigations are almost solely concerned with this determination of the free boundary. Given that we are using a complex function representation of the pressure field, it seems a sensible thing first to transform to a simple, known domain on which we can solve the field equations. We thus introduce a time-dependent univalent ${ }^{1}$ map $z=w(\zeta, t)$, from the unit disc in $\zeta$-space $(\zeta=\xi+i \eta)$ onto $\Omega(t)$ (figure 2.1).

[^3]

Figure 2.1: The mapping from the unit disc onto the fluid domain.

The existence of such a map is guaranteed by the Riemann mapping theorem (see for example [68]), and the map is uniquely determined if we insist that $w(0, t)=0$ and $w^{\prime}(0, t)$ is real and positive. Since we now have two complex planes to consider, we shall often refer to the $z$-plane as the physical plane. The method is well-illustrated if applied to the problem we have already mentioned, that is, the case where the flow is driven by a single point sink of strength $Q>0$ situated at the origin. In this case the asymptotic behaviour of the pressure near the origin is known, and is identical to the asymptotic behaviour in the $\zeta$-plane (as can be seen by performing a trivial Taylor expansion). The complex potential in the $\zeta$-plane can then be written down as

$$
\begin{equation*}
\Upsilon(\zeta)=\mathcal{W}(z(\zeta))=-\frac{Q}{2 \pi} \log \zeta \tag{2.2}
\end{equation*}
$$

since this has the correct singularity, and its real part vanishes on the unit circle.
The KBC (1.5) can be written in terms of the pressure $p$ as

$$
\begin{equation*}
\frac{\partial p}{\partial t}-|\nabla p|^{2}=0 \quad \text { on } \partial \Omega \tag{2.3}
\end{equation*}
$$

using (1.6) plus the fact that the velocity field $\mathbf{u}=-\nabla p$ in our dimensionless variables. We also have

$$
z=w(\zeta, t) \quad \Rightarrow \quad 0=w^{\prime}(\zeta) \zeta_{t}+w_{t}(\zeta) \quad \Rightarrow \quad \zeta_{t}=-\frac{w_{t}(\zeta)}{w^{\prime}(\zeta)}
$$

Then, using (2.2) and (2.3) and noting that $\zeta \bar{\zeta}=1$ on $|\zeta|=1$, we arrive at

$$
\begin{equation*}
\Re\left\{\zeta w^{\prime}(\zeta) \bar{w}_{t}(1 / \zeta)\right\}=-\frac{Q}{2 \pi} \quad \text { on }|\zeta|=1 \tag{2.4}
\end{equation*}
$$

a result known as the Polubarinova-Galin ( $P-G$ ) equation. In some problems we find it easiest to map from the right-half plane onto the physical domain (figure 2.2). In this case the free boundary is the image of the imaginary axis, $\zeta=i \eta$, under the conformal map, and the boundary condition on the complex potential in the $\zeta$-plane is

$$
\Re(\Upsilon(\zeta))=0 \quad \text { on } \zeta=i \eta
$$

One solution for $\Upsilon$ is clearly then

$$
\begin{equation*}
\Upsilon(\zeta)=A \zeta \tag{2.5}
\end{equation*}
$$

for some real constant $A$ (the negative pressure gradient at infinity in the $\zeta$-plane). The driving mechanism in the physical plane to which this corresponds depends on the particular mapping


Figure 2.2: The mapping from the right-half plane onto the fluid domain.
function proposed; for instance, if $w(\zeta)$ is linear in $\zeta$ at infinity then we have a constant pressure gradient at infinity, exactly as in the $\zeta$-plane, but if $w(\zeta)$ is quadratic in $\zeta$ at infinity (as in the famous Ivantsov parabolic travelling wave solution [51]) then the pressure only has square-root growth at infinity in the physical plane. The P-G equation for this mapping function is readily found to be

$$
\begin{equation*}
\Re\left\{w^{\prime}(\zeta) \bar{w}_{t}(-\zeta)\right\}=A \quad \text { on } \zeta=i \eta, \tag{2.6}
\end{equation*}
$$

in the same way that (2.4) was obtained.
These results enable many exact Hele-Shaw solutions to be constructed, by "guessing" the form of an appropriate mapping function $w(\zeta, t)$ with time-dependent coefficients. Substitution in (2.4) with $\zeta=e^{i \theta}$ (or (2.6) with $\zeta=i \eta$ ) leads to a system of ordinary differential equations for these coefficients, which can (hopefully) be solved, yielding the time-dependent map, and hence the evolution of the fluid domain. The solution thus found will be valid until such time as the mapping function ceases to be univalent on the unit disc.

An analogous equation arises in our study of the Stokes flow problem (equation (3.13)); there however our approach is to make the equation global by analytically continuing away from the unit circle. We could follow this approach here, but it would be rather cumbersome in practice because such an analytic continuation cannot be general, but must be specific to each case. We first need to propose a specific form for the mapping function, since only then do we know the singularities of the combination in curly brackets within the unit disc (which will be due only to the singularities of $\bar{w}_{t}(1 / \zeta)$, the other parts being analytic on $\left.|\zeta| \leq 1\right)$. These singularities would clearly need to be known, since any equation holding globally would need to have these same singularities on the right-hand side. We do not pursue this point, since the Stokes flow work exploits it much more satisfactorily (we discuss why this is so in §3.6.2). For a detailed discussion of the procedure of analytic continuation the reader is referred to [10] or [20].

### 2.3 The Schwarz function

Equivalent to the analytic continuation of the P-G equation, but much simpler in practice, is the Schwarz function approach which we now outline. The Schwarz function of the free boundary, which exists if and only if the boundary is an analytic curve, is the unique function $g(z, t)$, analytic in some neighbourhood of $\partial \Omega$, such that the equation

$$
\bar{z}=g(z, t)
$$

defines $\partial \Omega$. If we have a Cartesian equation, $F(x, y, t)=0$, for $\partial \Omega$, the Schwarz function may be obtained by substituting for $x=(z+\bar{z}) / 2, y=(z-\bar{z}) /(2 i)$, and solving for $\bar{z}$. Note that
most analytic functions of $z$ will not be Schwarz functions, as they will not satisfy the consistency condition,

$$
z=\overline{g(\overline{g(z, t)}, t)} \equiv \bar{g}(g(z, t), t)
$$

It can be shown (see for instance [17]) that with notation as defined in this chapter, the following identities hold at a point $z$ on $\partial \Omega$ :

$$
\begin{align*}
\frac{\partial z}{\partial s} & =\left(g^{\prime}\right)^{-1 / 2}  \tag{2.7}\\
\kappa & =-i\left[\left(g^{\prime}\right)^{-1 / 2}\right]^{\prime}=\frac{i}{2} \frac{g^{\prime \prime}}{\left(g^{\prime}\right)^{3 / 2}}  \tag{2.8}\\
v_{n} & =-\frac{i}{2} \frac{g_{t}}{\left(g^{\prime}\right)^{1 / 2}} \tag{2.9}
\end{align*}
$$

where $s$ is arclength along $\partial \Omega$ and prime denotes $d(\cdot) / d z$. Then on $\partial \Omega$,

$$
\begin{array}{rlr}
\frac{d \mathcal{W}}{d z} & =\frac{\partial \mathcal{W}}{\partial s} / \frac{\partial z}{\partial s} & \\
& =-\left(\frac{\partial p}{\partial s}+i \frac{\partial p}{\partial n}\right)\left(g^{\prime}\right)^{1 / 2} & \\
\text { since } p=-\Re(\mathcal{W}) \\
& =i v_{n}\left(g^{\prime}\right)^{1 / 2} & \\
& =\frac{1}{2} \frac{\partial g}{\partial t} & \\
\text { using }(1.5),(1.6) \\
& \text { using }(2.9)
\end{array}
$$

Since both sides of this last equality are analytic in some neighbourhood of $\partial \Omega$, we may analytically continue away from $\partial \Omega$ to deduce that it holds wherever both sides exist, that is,

$$
\begin{equation*}
\frac{d \mathcal{W}}{d z} \equiv \frac{1}{2} \frac{\partial g}{\partial t} \tag{2.10}
\end{equation*}
$$

Important conclusions may be drawn from this identity, about the nature of the possible singularities of the Schwarz function. The singularities of $\mathcal{W}(z)$ within $\Omega(t)$ are given as part of the problem specification (the driving singularities of the problem), hence the singularities of $g(z)$ are also specified at such points. It is also possible that $g(z)$ may have other singularities than these within the fluid domain, but such singularities, by (2.10), must remain fixed both in position and strength within the physical domain. The singularities which are external to $\Omega(t)$, on the other hand, may move around, and vary in strength. We note that, since the Schwarz function of an analytic curve must itself be analytic in some neighbourhood of the curve, and since analyticity of the free boundary is a mathematical requirement of our Hele-Shaw theory, it follows that if a singularity of the Schwarz function reaches the free boundary (or vice-versa), this must coincide with solution breakdown. Explicit calculation shows that cuspidal blow-up of the ZST problem is associated with a moving, external singularity of the Schwarz function reaching the free boundary within finite time; in the example to follow ( $\S 2.4$ ), a square-root singularity of $g(z)$ reaches the boundary simultaneously with blow-up. It is also conceivable that blow-up could occur with the free boundary moving inwards and reaching one of the internal singularities; although no explicit examples of this are known, they could, in principle, be constructed as time-reversals of solutions to the well-posed injection problem, with appropriate singularities in the initial data. See for example [48] for further discussion of this point.

A version of equation (2.10) may also be obtained for the NZST problem, making use of identities (2.7)-(2.9), and the boundary conditions (1.4) and (1.5). This has been cited many times (see for example [48]), and is

$$
\begin{equation*}
\frac{d \mathcal{W}}{d z}=\frac{1}{2} \frac{\partial g}{\partial t}-\frac{i T}{2} \frac{d}{d z}\left(\frac{g^{\prime \prime}}{\left(g^{\prime}\right)^{3 / 2}}\right) \tag{2.11}
\end{equation*}
$$

the unpleasant form of the extra surface tension term on the right-hand side here gives warning of how difficult the NZST problem will be. An analogous result can be found if one employs the "kinetic undercooling" regularisation mentioned in §1.4.1, with the boundary condition

$$
p=\epsilon v_{n} \quad \text { on } \partial \Omega,
$$

(for some positive undercooling parameter $\epsilon$ ) replacing (1.4); this is

$$
\frac{d \mathcal{W}}{d z}=\frac{1}{2} \frac{\partial g}{\partial t}-\frac{i \epsilon}{2} \frac{d}{d z}\left(\frac{g_{t}}{\sqrt{g^{\prime}(z)}}\right)
$$

which is considered briefly in [48].
Returning to the ZST problem, in terms of the mapping function $w(\zeta, t)$, we have

$$
g(z)=\bar{z}=\overline{w(\zeta)}=\bar{w}(1 / \zeta), \quad \text { on }|\zeta|=1
$$

The first and last terms in the above are analytic in some neighbourhood of the free boundary (in the $z$ - and $\zeta$-planes respectively); we may then analytically continue away from $|\zeta|=1$ to deduce that they are equal wherever they are defined, hence

$$
\begin{equation*}
g(z) \equiv \bar{w}(1 / \zeta) \tag{2.12}
\end{equation*}
$$

These ideas can be used to provide an alternative method of solution for the problem, which we now outline. For definiteness, and to facilitate comparison of the two approaches, we assume the flow is driven by a single point sink at the origin. We again consider guessing a suitable form for the mapping function to cater for a particular problem, but rather than using the P-G equation, we use (2.12) to evaluate the Schwarz function as a Laurent series in $z$ about $z=0$ (the suction point). (2.10) then yields a Laurent expansion for $d \mathcal{W} / d z$ about $z=0$. However, we know that near $z=0, d \mathcal{W} / d z \sim-Q /(2 \pi z)$, and so we can in principle solve the full problem by equating coefficients in the principal parts of the Laurent expansions-all coefficients must vanish except that of $1 / z$.

### 2.4 A simple example

To illustrate the application of the two solution methods outlined in $\S \S 2.2$ and 2.3 , we present a simple (well-known) example. The simplest nontrivial mapping function to try is the quadratic map,

$$
\begin{equation*}
z=w(\zeta, t)=a_{1}(t) \zeta+a_{2}(t) \zeta^{2} \tag{2.13}
\end{equation*}
$$

with $a_{1}$ and $a_{2}$ taken to be real and positive without loss of generality, ( $a_{1}>0$ by the normalisation condition of $\S 2.2) .{ }^{2}$ For the initial map $w(\zeta, 0)$ to be conformal, we require $\left|a_{1}(0)\right|>2\left|a_{2}(0)\right|$.

First consider the "P-G" method. Writing $\zeta=e^{i \theta}$ in (2.13) and substituting directly into (2.4), we are able to equate terms having the same $\theta$-dependence to obtain the system of equations

$$
\begin{aligned}
a_{1} \dot{a}_{1}+2 a_{2} \dot{a}_{2} & =-\frac{Q}{2 \pi}, \\
a_{1} \dot{a}_{2}+2 \dot{a}_{1} a_{2} & =0,
\end{aligned}
$$

which are easily integrated to give the evolution until the solution breaks down. For this simple case, this happens when $a_{1}\left(t^{*}\right)= \pm 2 a_{2}\left(t^{*}\right)$, at which point a $3 / 2$-power cusp forms in $\partial \Omega$, with $w^{\prime}\left(\mp 1, t^{*}\right)=0$. The free boundary is initially a limaçon, (figure 2.3 (a)) with suction from some point on the axis of symmetry, evolving into a cardioid, (figure 2.3 (b)) at which time the solution breaks down.

[^4]


Figure 2.3: The initial and final domains for the quadratic polynomial mapping.

Using the Schwarz function approach, (2.12) gives

$$
g(w(\zeta))=\frac{a_{1}}{\zeta}+\frac{a_{2}}{\zeta^{2}} .
$$

Inverting $z=w(\zeta, t)$, noting that the origin must map to the origin when choosing the branch of the square root, gives

$$
\zeta=\frac{a_{1}}{2 a_{2}}\left[-1+\left(1+\frac{4 a_{2} z}{a_{1}^{2}}\right)^{1 / 2}\right]
$$

Near $\zeta=0=z$,

$$
\frac{1}{\zeta} \sim \frac{a_{1}}{z}\left(1+\frac{a_{2}}{a_{1}^{2}} z+\cdots\right), \quad \text { and } \quad \frac{1}{\zeta^{2}} \sim \frac{a_{1}^{2}}{z^{2}}\left(1+\frac{2 a_{2}}{a_{1}^{2}} z+\cdots\right)
$$

Hence, near $z=0$,

$$
g(z)=\frac{a_{1}^{2} a_{2}}{z^{2}}+\frac{a_{1}^{2}+2 a_{2}^{2}}{z}+O(1)
$$

and from (2.10) we deduce that

$$
\begin{align*}
\frac{d}{d t}\left(a_{1}^{2} a_{2}\right) & =0  \tag{2.14}\\
\text { and } \frac{d}{d t}\left(a_{1}^{2}+2 a_{2}^{2}\right) & =-\frac{Q}{\pi} \tag{2.15}
\end{align*}
$$

exactly as before, but without having to integrate a system of ordinary differential equations. Of course in this simple case, "spotting" integrals of the o.d.e.'s resulting from the P-G equation is trivial, but for more complicated maps this may not always be the case. Likewise, the Laurent expansions involved in the Schwarz function approach may not always be so painless, but in general, this is the superior method. We shall see further applications of both methods in chapters 5 and 7.

### 2.5 Richardson's "moments" and the Cauchy transform

We now review some useful ideas which are due to Richardson ([79, 81] and subsequent papers). The moments of the (bounded) fluid domain $\Omega(t)$ with respect to the origin are defined by the
formula

$$
\begin{equation*}
M_{k}=\iint_{\Omega} z^{k} d x d y \quad k=0,1,2, \ldots \tag{2.16}
\end{equation*}
$$

and depend only on time $t$. Using Green's theorem in the complex plane, ${ }^{3}$ an alternative representation is

$$
\begin{equation*}
M_{k}(t)=\frac{1}{2 i} \int_{\partial \Omega(t)} z^{k} \bar{z} d z \tag{2.17}
\end{equation*}
$$

Consider first the familiar case of a single sink of strength $Q$ at $z=0$ (a source if $Q<0$ ). We want to know how the moments evolve in time; using (1.5) we see that

$$
\frac{d M_{k}}{d t}=\int_{\partial \Omega} z^{k} v_{n} d s=-\int_{\partial \Omega} z^{k} \frac{\partial p}{\partial n} d s
$$

Now, on the free boundary, if $\alpha$ is the angle made by the tangent to $\partial \Omega$ with the $x$-direction and $s$ denotes arclength along $\partial \Omega$, then

$$
\frac{d \mathcal{W}}{d z}=\frac{\partial \mathcal{W}}{\partial s} / \frac{\partial z}{\partial s}=-e^{-i \alpha}\left(\frac{\partial p}{\partial s}+i \frac{\partial p}{\partial n}\right)=-i e^{-i \alpha} \frac{\partial p}{\partial n}
$$

where we have used the Cauchy-Riemann equations, together with the fact that $\partial p / \partial s=0$ on $\partial \Omega(t)$ (which follows from (1.6)). It then follows, using the relation $d z=e^{i \alpha} d s$ on $\partial \Omega(t)$, that

$$
\begin{equation*}
\frac{d M_{k}}{d t}=-i \int_{\partial \Omega} z^{k} \frac{d \mathcal{W}}{d z} d z \tag{2.18}
\end{equation*}
$$

For this case of the point sink singularity, $d \mathcal{W} / d z$ is analytic in $\Omega(t)$ except at $z=0$, near which $d \mathcal{W} / d z \sim-Q /(2 \pi z)$. Using the Cauchy theorem of complex variable theory on (2.18) to deform the contour of integration $\partial \Omega(t)$ to a small circle about the origin, we see that

$$
\frac{d M_{k}}{d t}= \begin{cases}-Q & k=0  \tag{2.19}\\ 0 & k=1,2, \ldots\end{cases}
$$

The result for $k=0$ is simply an expression of conservation of mass, whilst the $k=1$ equation states that the centre of mass of the fluid domain remains fixed. (This latter result tells us that a necessary condition for complete extraction of all the fluid is that the sink be situated at the centre of mass of $\Omega(0))$. We mention here that the results for the $k=0$ and $k=1$ moments can also be shown to hold for the NZST problem (see for example [69], where a NZST evolution equation for the Cauchy transform (2.21) is formulated, and used to prove this).

Equations (2.19) are readily generalised to the case of suction with rates $Q_{i}(t)$ at points $z_{i}$ $(1 \leq i \leq N)$ in $\Omega(t)$, (figure 2.4) with the result that [81],

$$
\begin{equation*}
\frac{d M_{k}}{d t}=-\sum_{i=1}^{N} Q_{i}(t) z_{i}^{k} \quad k=0,1,2, \ldots \tag{2.20}
\end{equation*}
$$

We may see this by considering the case of a single $\operatorname{sink} Q$ at $z=a$. Then, using (2.18) and deforming the contour $\partial \Omega(t)$ to a small circle about $z=a$, gives

$$
\frac{d M_{k}}{d t}=\lim _{\epsilon \rightarrow 0} \frac{-Q}{2 \pi i} \int_{|z-a|=\epsilon} \frac{z^{k}}{z-a} d z=-Q a^{k}
$$

[^5]

Figure 2.4: Schematic diagram of a system of sinks $Q_{1}, \ldots Q_{6}$ at points $z_{1}, \ldots z_{6}$ within $\Omega(t)$.
by Cauchy's integral formula. Equation (2.20) is then a trivial extension of this result to the case of many (possibly time-dependent) sinks. ${ }^{4}$

We next introduce the closely related concept of the Cauchy transform [81, 83], defined by

$$
\begin{equation*}
\vartheta(x, y, t)=\frac{1}{\pi} \iint_{\Omega} \frac{d x^{\prime} d y^{\prime}}{z-z^{\prime}} \tag{2.21}
\end{equation*}
$$

(where $z^{\prime}=x^{\prime}+i y^{\prime}$ ) the improper integral being understood when $z \in \Omega$. For $z$ outside ('exterior' to) $\Omega$, the right-hand side of (2.21) defines an analytic function of $z$, denoted by $\vartheta_{e}(z)$ (' $e$ ' for 'exterior'); expanding in a Laurent series gives

$$
\begin{equation*}
\vartheta_{e}(z)=\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{M_{k}}{z^{k+1}} \tag{2.22}
\end{equation*}
$$

Writing $A_{i}(t)=\int_{0}^{t} Q_{i}(\tau) d \tau,(2.22)$ and (2.20) together imply

$$
\begin{equation*}
\vartheta_{e}(z, t)=\vartheta_{e}(z, 0)-\frac{1}{\pi} \sum_{i=1}^{N} \frac{A_{i}(t)}{z-z_{i}} . \tag{2.23}
\end{equation*}
$$

The function $\vartheta_{e}(z)$ may be analytically continued inside $\Omega$; this continuation will in general have singularities within $\Omega$. In [83] it is shown that

$$
\vartheta(x, y)= \begin{cases}\bar{z}-\vartheta_{i}(z) & z \in \Omega(t) \\ \vartheta_{e}(z) & z \in \Omega(t)^{c}\end{cases}
$$

for some function $\vartheta_{i}(z)$ analytic inside $\Omega(t)$. From its definition, $\vartheta(x, y)$ is clearly continuous throughout $\mathbb{R}^{2}$ and hence

$$
\vartheta_{e}(z)=\bar{z}-\vartheta_{i}(z) \quad \text { on } \partial \Omega(t)
$$

[^6]In terms of the mapping function, this implies

$$
\vartheta_{e}(w(\zeta))=\overline{w(\zeta)}-\vartheta_{i}(w(\zeta))
$$

and, using $\zeta \bar{\zeta}=1$ on $\partial \Omega$ and analytically continuing, we find

$$
\begin{equation*}
\vartheta_{i}(w(\zeta))+\vartheta_{e}(w(\zeta))=\bar{w}(1 / \zeta) . \tag{2.24}
\end{equation*}
$$

Since $w(\zeta, t)$ is analytic in $|\zeta| \leq 1$, the singularities of the right-hand side lie in $|\zeta|<1$ and, since $\vartheta_{i}(w(\zeta))$ is analytic there, (2.24) tells us that these singularities must be identical with those of $\vartheta_{e}(w(\zeta))$.

Given $\vartheta_{e}(z, t)$ then (for which we only need know $\vartheta_{e}(z, 0)$, by $(2.23)$ ), we are able to write down the general form of the mapping $w(\zeta, t)$, then make a quantitative comparison of singularities in (2.24) to fix $w$ uniquely (bearing in mind the normalisation conditions of $\S 2.2$ ). A detailed example of this procedure is given in §5.3.

The Cauchy transform is also related to the Schwarz function of the free boundary, $g(z, t)$. In general, $g(z)$ will have singularities both inside and outside $\Omega$, and (with subscripts as before) we may decompose it uniquely as

$$
\begin{equation*}
g(z)=g_{e}(z)+g_{i}(z) \tag{2.25}
\end{equation*}
$$

if we insist $g_{e}(z) \rightarrow 0$ as $z \rightarrow \infty$. This decomposition is often obvious, but if not, it can be carried out using the Plemelj formulae (see for instance [10]),

$$
\begin{equation*}
g_{e}(z)=\frac{1}{2 \pi i} \oint_{\partial \Omega} \frac{g(\tau) d \tau}{\tau-z}, \quad \text { for } z \notin \Omega \tag{2.26}
\end{equation*}
$$

with an identical expression for $g_{i}(z)$ when $z \in \Omega$. (These expressions rely on the analyticity of $g(z)$ in a neighbourhood of the free boundary.)

Hence (2.12) may be written

$$
\begin{equation*}
g_{i}(w(\zeta))+g_{e}(w(\zeta))=\bar{w}(1 / \zeta) \tag{2.27}
\end{equation*}
$$

Comparison with (2.24), appealing to the uniqueness of the subscripted functions-clearly, $\vartheta_{e}(z) \rightarrow$ 0 as $z \rightarrow \infty$ from the definition (2.21)-reveals that

$$
\begin{aligned}
& g_{i}(z) \equiv \vartheta_{i}(z) \\
& g_{e}(z) \equiv \vartheta_{e}(z)
\end{aligned}
$$

Hence the time evolution of $g_{e}(z, t)$ is known (exactly as in (2.23)) and the method just described of deducing the right form of the mapping function may again be used if we know the Schwarz function of $\partial \Omega(0)$, and the sinks and sources driving the flow. In fact, it is not necessary to demonstrate the above identities to see this; the same method using the Schwarz function (instead of the Cauchy transform) may be deduced from the identity (2.10) of $\S 2.3$, once we know we can decompose the Schwarz function in the manner of (2.25). We shall consider similar "deductive methods" of finding the mapping function for the Stokes flow problem in §5.4.

### 2.6 Transformation of the dependent variable

Some of the preceding ideas may be linked to an alternative approach, which uses a transformation of the dependent variable. This is often called the Baiocchi transform, after Baiocchi [5], who first introduced it to solve for steady flow in a porous medium through a rectangular dam (which is equivalent to a Hele-Shaw flow, as we remarked in $\S 1.2$ ). The transformation has been applied to Hele-Shaw problems in, for example, [21], [60], [56], [57]; we follow [57]. It is not itself a complex variable method, but we include it in this chapter because it can be related to such methods, and because it is of independent interest. We shall consider an analogous formulation for the Stokes
flow problem in §3.9.1. The transformation is important in that it gives a weak formulation of the problem, enabling non-classical situations to be dealt with (see for instance King et al. [56]).

The approach differs slightly according as to whether the problem is well-posed (the so-called "injection" or "blowing" problem, with an advancing free boundary) or ill-posed (the so-called "suction" problem, with a retreating free boundary). We are able to classify the problems in this way because of the maximum principle for harmonic functions; for the ZST problem with (for example) a point sink, $p=0$ on $\partial \Omega$ must be the global pressure maximum, so the pressure is everywhere negative within $\Omega$, and the free boundary must always retreat. A similar argument applies for the point source problem. A consequence of this "monotonic" behaviour of $\partial \Omega(t)$ is that the free boundary may be represented in the form ${ }^{5}$

$$
t=\sigma(\mathbf{x})
$$

for $\mathbf{x}$ in regions crossed by the free boundary. (For the injection problem this region is $\Omega(t) \backslash \Omega(0)$; for the suction case it is $\Omega(0) \backslash \Omega(t)$.)

We first define the function $u_{0}(\mathbf{x})$ to be the solution to the (ill-posed) Cauchy problem,

$$
\begin{gather*}
\nabla^{2} u_{0}=1 \quad \text { in } \Omega(0),  \tag{2.28}\\
u_{0}=0=\frac{\partial u_{0}}{\partial n} \quad \text { on } \partial \Omega(0) ;
\end{gather*}
$$

$u_{0}$ must in general have singularities within $\Omega(0)$ since we are imposing two boundary conditions, whereas only one is needed for a well-posed problem. The Baiocchi transform variable $u(\mathbf{x}, t)$ is then defined as follows:

1. Injection Problem:

$$
\begin{gathered}
u=u_{0}+\int_{0}^{t} p(\mathbf{x}, \tau) d \tau \quad \mathbf{x} \in \Omega(0) \\
u=\int_{\sigma(\mathbf{x})}^{t} p(\mathbf{x}, \tau) d \tau \quad \mathbf{x} \in \Omega(t) \backslash \Omega(0)
\end{gathered}
$$

2. Suction Problem:

$$
\begin{equation*}
u=u_{0}+\int_{0}^{t} p(\mathbf{x}, \tau) d \tau \quad \mathbf{x} \in \Omega(t) \tag{2.29}
\end{equation*}
$$

For $\mathbf{x}$ in regions that the free boundary crosses (so that $\sigma(\mathbf{x})$ is defined), $u$ may also be consistently defined as

$$
\begin{equation*}
u=-\int_{t}^{\sigma(\mathbf{x})} p(\mathbf{x}, \tau) d \tau \quad \mathbf{x} \in \Omega(t) \backslash \Omega\left(t^{*}\right) \tag{2.30}
\end{equation*}
$$

where $t^{*}$ is an upper limit for the existence in time of the solution to the suction problem.
In either case we then have the following free boundary problem for $u$ :

$$
\begin{equation*}
\nabla^{2} u=1 \quad \text { in } \Omega(t) \tag{2.31}
\end{equation*}
$$

with

$$
\begin{equation*}
u=0=\frac{\partial u}{\partial n} \quad \text { on } \partial \Omega(t) . \tag{2.32}
\end{equation*}
$$

[^7]Differentiating the definition of $u$ with respect to time, it is clear that for both injection and suction cases,

$$
\begin{equation*}
p=\frac{\partial u}{\partial t} \quad \text { in } \Omega(t) \tag{2.33}
\end{equation*}
$$

The problem (2.31), (2.32) for $u(\mathbf{x}, t)$ must be solved on $\Omega(t)$ with prescribed singularities within $\Omega(t)$, which are of two distinct types:

1. Constant (in space and time) singularities of $u_{0}(\mathbf{x})$, and
2. Those within $\Omega(t)$ which are time integrals of the known (driving) singularities of $p$ (fixed in space).

Hence the basic "recipe" for solving problems is (see [35]):

- Find the singularities of $u_{0} \equiv u(\mathbf{x}, 0)$ by solving the ill-posed Cauchy problem (2.28);
- Find the interior singularities of $u(\mathbf{x}, t)$ using the comments above;
- Solve the free boundary problem (2.31) and (2.32) for $u(\mathbf{x}, t)$ with these singularities.

We can also link $u$ to the Schwarz function $g$ of the free boundary. $\nabla^{2} u=1$ and so the function $f$ defined by

$$
f=u-\frac{1}{4} z \bar{z}
$$

is harmonic, which implies $f=\Re(F)$ for some analytic function $F . F^{\prime}(z) \equiv u_{x}-i u_{y}-\frac{1}{2} \bar{z}$ is then also analytic, away from singularities of $u$. Now, $\nabla u=0$ on the free boundary, so $F^{\prime}(z)=-\frac{1}{2} \bar{z}=-\frac{1}{2} g(z)$ there. But $g$ is analytic in a neighbourhood of $\partial \Omega$ and hence by analytic continuation we may deduce that

$$
\begin{equation*}
u_{x}-i u_{y}=\frac{1}{2}(\bar{z}-g(z)) \tag{2.34}
\end{equation*}
$$

wherever either is defined. Hence, using (2.29) and (2.1) we see that

$$
g(z, t)=g(z, 0)+2 \int_{0}^{t} \frac{d \mathcal{W}}{d z}(z, \tau) d \tau
$$

which is exactly the result (2.10) obtained in $\S 2.3$ by more direct methods. Statements analogous to $(1),(2)$ and (3) hold for the Schwarz function, as was noted there.

Such methods can also be used to treat modifications of the classical Hele-Shaw problem, for instance squeeze films, where the walls of the cell are moved normally relative to each other, or cells with porous plates through which suction can be applied; see for example [35].

It is also interesting that the problem satisfied by the Baiocchi transform variable $u$ is a version of the so-called obstacle problem of the variational calculus, if we impose the extra condition that $u$ be positive everywhere on $\Omega(t)$. This is the (well-posed) problem of determining the contact region when an elastic membrane is stretched over some irregular rigid surface or 'obstacle', so that the membrane is in contact with only part of the obstacle (see for example [22]). The Baiocchi variable $u$ may be identified with the membrane displacement.

It follows [46] that if we have a family of obstacle problems parametrised by, and sufficiently regular in, $t$, we may recover a Hele-Shaw problem by forming $p=\partial u / \partial t$. This is very useful, since the obstacle problem has been widely studied, and there exist many rigorous results which can be carried over for the Hele-Shaw problem. Of particular interest is the question of singularities in the free boundary. For the obstacle problem on a simply-connected domain, it can be shown [90], [64] that the free boundary is a piecewise analytic curve, the only possible singularities being cusps of power $(4 n+1) / 2$, for integer $n$, i.e. having the local behaviour

$$
Y \sim O\left(X^{(4 n+1) / 2}\right)
$$

In $\S 5.3$ we will see an example of a Hele-Shaw flow in which the free boundary evolves (otherwise smoothly) through a singularity of this kind, having a $5 / 2$-power cusp forming in the free boundary which then immediately disappears (see also [46], [50] in this context); presumably similar examples exist with the free boundary evolving through $9 / 2,13 / 2, \ldots$ power cusps, although none has yet been presented. Cusps of power $3 / 2,7 / 2, \ldots$ are not possible singularities of the free boundary in the obstacle problem, and hence represent terminal blow-up in the Hele-Shaw problem.

As a final point, we remark that the restriction $u \geq 0$ on $\Omega(t)$, which the Baiocchi transform variable $u$ is not subject to, is crucial to the well-posedness of the obstacle problem. For the ill-posed Hele-Shaw suction problem it can be shown that, although $u$ must be positive in some neighbourhood of $\partial \Omega(t)$ (indeed, the definition (2.30) requires $u \geq 0$ in the region $\Omega(t) \backslash \Omega\left(t^{*}\right)$ which $\partial \Omega(t)$ crosses, since the pressure is everywhere negative for a suction problem), the time integrals of the negative pressure singularities may lead to $u$ being negative at some points within $\Omega(t)$. If such a "negativity region" appears, it must grow, because, by (2.33), $u$ is a decreasing function of time. Hence it will eventually meet the contracting free boundary, at which point the solution must break down.

### 2.7 Univalency and conformality

We have referred several times in this chapter to the need for univalency and/or conformality of the mapping function $w(\zeta, t)$. We now explain more precisely what we mean by these terms.

> DEFINITION (UNIVALENCY/CONFORMALITY): A single-valued, analytic function $w(\zeta)$ is said to be conformal in a domain $\Omega \subset \mathbb{C}$ if its derivative never vanishes (or becomes unbounded) on $\Omega$. If, in addition, $w$ never takes the same value twice, that is, if $w\left(\zeta_{1}\right) \neq w\left(\zeta_{2}\right)$ for all points $\zeta_{1}$ and $\zeta_{2}$ in $\Omega$, then $w$ is said to be univalent on $\Omega$. It can be shown (see [20]) that conformality at a point $\zeta_{0}$ is equivalent to local univalence at $\zeta_{0}$.

Univalency of the map is necessary on both theoretical and physical grounds. Theoretically, the procedure of analytic continuation, which has been used several times throughout the chapter (and will also be used extensively in the next chapter for the Stokes flow problem), certainly requires conformality of $w(\zeta) .{ }^{6}$ Physically, if the map did assume the same value twice, this would correspond to two distinct particles of fluid occupying the same spot in two-dimensional space, which is not possible. Hence we require univalency according to the definition above. All of the general comments in this section will pertain to both the Hele-Shaw and the Stokes flow problems; where comments are particular to only one of the problems, we make this clear.

The above justifies our statement at the end of the last section, that solutions found by the method outlined there will be tenable "until such time as the mapping function ceases to be univalent on the unit disc". This could happen in several ways: for instance, by a zero of $w^{\prime}(\zeta)$ approaching the boundary $|\zeta|=1$ from outside (corresponding to formation of an inward-pointing cusp or reflex-angled corner in the free boundary); by a singularity of $w^{\prime}(\zeta)$ reaching the boundary (which could imply an obtuse or acute-angled corner, or an outward-pointing cusp), or by loss of the 1-1 character of the map (so that physically the fluid domain intersects itself). In all the examples we consider, univalency can be lost only by cusp formation or by self-overlapping, and the following discussion is restricted to these possibilities.

The map $w$ will be a function of $\zeta$ and $t$, with the time dependence coming in via various parameters in the map; for example if $w$ is a polynomial then these parameters will just be the coefficients of the polynomial. We have already seen one simple example of this type in §2.4. In a general case, if the map contains $N$ complex parameters, then we have $2 N$ real parameters. Assuming that the map satisfies $w(0, t)=0$, we are still free to choose the orientation of the axes within the fluid (usually done by ensuring $w^{\prime}(0, t)>0$ ), which reduces the number of parameters by two, and if we wish, we can eliminate a scaling factor from the problem (a measure of the area

[^8]

Figure 2.5: The univalency domain $V$, and phase trajectories, for the "limaçon" example of $\S 2.4$. With a point sink, the phase paths are followed in the direction of the arrows (the nonunivalent region); with a point source, the direction is opposite. Intersection with the boundary of $V$ is associated with the cusped cardioid geometry.
of the fluid domain) by suitably rescaling time. Thus we will have $2 N-3$ real, time-dependent parameters to consider.

Only for certain values of these parameters will the map be univalent on $|\zeta| \leq 1$. There will be some subset of the $(2 N-3)$-dimensional (or $(2 N-2)$-dimensional, if we do not eliminate the scaling factor) parameter space, $V \subset \mathbb{R}^{2 N-3}$, such that if the parameter values lie within $V$ then we have univalency. We refer to $V$ as a univalency domain. For the map of $\S 2.4$ with $a_{1}>0$ (a sufficiently simple example to make the rescaling of time unnecessary), $V$ is the union of the two domains $a_{1}>2 a_{2}$ (in $a_{2}>0$ ), and $a_{1}>-2 a_{2}$ (in $a_{2}<0$ ). A valid solution to the problem can then be represented as a trajectory within $V$, with solution breakdown occurring when the trajectory reaches the boundary, $\partial V$; here, the solution trajectories within $V$ are given by (2.14) as the curves

$$
a_{1}^{2} a_{2}=k
$$

for constants $k$ (see figure 2.5). The sense in which (and the speed with which) the paths are followed is determined by (2.15).

In general $\partial V$ will comprise different regions, corresponding to the different ways in which univalency can be lost; with obvious notation we can write

$$
\partial V=\partial V_{\text {cusp }} \cup \partial V_{\text {overlap }}
$$

In all the cases we consider, only at the boundaries between $\partial V_{\text {cusp }}$ and $\partial V_{\text {overlap }}$, or at isolated "extremal" points of $\partial V_{\text {overlap }}$, do we get types of cusp other than $3 / 2$-power (although more complicated scenarios can easily be envisaged). In fact, for all the maps in this thesis, the only possibilities are $3 / 2$-power and $5 / 2$-power cusps. Examples of univalency domains arise in $\S 5.3$ (see figures 5.4, 5.6 and 5.7) and in $\S 6.2$ (see figure 6.4); the case of the quadratic map cited above
is not a very good illustration since it affords only the possibility of breakdown via a $3 / 2$-power cusp.

For a general polynomial mapping function it can be shown [32] that blow-up in the Hele-Shaw suction problem can occur only by cusp formation or self-overlapping; by far the most common way for univalency to be lost is via formation of a $3 / 2$-power cusp. Polynomial mapping functions are dense in the set of conformal maps, in the sense that an arbitrary initial boundary shape $\partial \Omega(0)$ can be approximated arbitrarily closely by a polynomial map (and hence, hopefully, the evolution for $t>0$ will be well-described by the solution for this map). For these reasons, $3 / 2-$ power cuspidal blow-up is sometimes referred to as the "generic" situation for ZST Hele-Shaw (or Stokes flow) solution breakdown. It has not been rigorously proved, but it is widely believed that a positive surface tension parameter $T$ (however small), will prevent this generic solution breakdown, for both problems. As a highly-curved (near-cusped) configuration is approached, large surface tension forces are generated, which act to balance the stresses due to the driving mechanism, keeping the boundary smooth at that point. Experiments such as Jeong \& Moffatt's [52] for Stokes flow show, however, that the radius of curvature at the point where the ZST cusp would form may need to be extremely small for a force balance to be achieved. Solution breakdown can certainly still occur by the flow domain beginning to overlap itself, or possibly by the formation of other types of cusp. In particular, solutions of the NZST Stokes flow problem exist which form actual $5 / 2$-power cusps in the free boundary within finite time [85]. Such cusps may be viewed as "geometrically necessary", in that they are a limiting case of a set of solutions which blow up via self-overlapping, for which positive surface tension is not a means of preventing blow-up, in the current theory.

For the NZST Hele-Shaw problem it is thought that solution breakdown may also occur via the free boundary reaching the driving singularity (the "crack" and "slit" theories; see §7.1). For the NZST Stokes flow problem, at least for the case of a point sink singularity, this last suggestion may not be possible-this is discussed in $\S \S 6.3$ and 8.1.

For both problems then (leaving aside the possible complication of the free boundary reaching the singularity, and types of cusp other than 3/2-power), we expect that only in the ZST problem do we only have to worry about breakdown via cusp formation, whilst breakdown due to selfintersection can occur in both the ZST and the NZST problems. 3/2-power cusp formation in the ZST Stokes flow problem can be avoided if, near breakdown time, we abandon the ZST assumption and consider some simple asymptotics (see chapter 6); as we have already noted though, this perturbation of the problem is not so simple for Hele-Shaw flow. Dealing with self-intersection is not easy for either of the ZST problems, since we are then dealing with a multiply-connected fluid domain. To map to, for instance, a doubly-connected domain, we need to map from a doubly connected domain, such as an annulus. Moreover, with the presence of more than one free boundary we have to allow different boundary conditions to hold on each boundary (this complication was noted in $\S 1.2 .1$ before deriving (1.4) and (1.6), and a similar situation arises with the boundary condition (3.4) in $\S 3.1$ for the Stokes flow problem). We do not consider such complications in this thesis; it was mentioned in $\S 1.4 .1$ that Richardson [84] has carried out an extensive study of the multiply-connected Hele-Shaw free boundary problem.

### 2.8 Summary

In this chapter we have reviewed a selection of well-known results for the Hele-Shaw problem, with the dual aims of familiarising the reader with the Hele-Shaw problem, and of presenting a coherent introduction to the application of complex variable methods to the free boundary problem. The order of presentation was roughly chronological, beginning in $\S 2.2$ with the ideas of Polubarinova-Kochina [75] and Galin [28]. This introduced the concept of mapping conformally from some simple geometry onto the unknown fluid domain, an idea which is central to the thesis. The work of [75] and [28] (dating back to 1945) was of immense historical importance for a wide range of free boundary problems (in particular, porous medium and filtration problems, as well as the Hele-Shaw problem). The key "P-G" equations (2.4) and (2.6), satisfied by the conformal
mapping on the unit circle and on the imaginary axis respectively, were derived, which provided one of the earliest methods of finding solutions to the pressure driven Hele-Shaw problem.

More recent concepts were then introduced (the Schwarz function, Richardson's moments, the Cauchy transform, the Baiocchi transform), and possible applications to the solution of the free boundary problem were discussed. These concepts were shown to be closely linked; Richardson's moments are the coefficients of the principal part of the Laurent expansion of the Cauchy transform, and the Cauchy transform is essentially equivalent to the Schwarz function of the free boundary. The Baiocchi transform is related to the Schwarz function through (2.34). The link between the Schwarz function and the complex potential of the flow was also demonstrated ( $\S 2.3$ ), a much deeper result than the P-G equation, since it holds not just on the free boundary, but globally.

We concluded the chapter with a discussion of the univalency of the mapping function $w(\zeta, t)$, which is crucial if the theory outlined in the previous sections is to be valid.

This chapter lays down the groundwork for the rest of the thesis. Similar complex variable methods are to be employed throughout, and we find many results for the Stokes flow problem in chapter 3 which parallel those listed above. On the other hand, some of the Stokes flow results may be contrasted with the corresponding Hele-Shaw results; these are discussed as and when they arise, and summarised in $\S 8.1$.

## Chapter 3

## Complex Variable methods for Stokes flow

In the previous chapter we gave an overview of some of the ways in which complex variable methods may be applied to solve the Hele-Shaw problem. The important concept of time-dependent conformal maps from some known, simple domain onto the fluid domain was introduced, so that the problem reduces to solving for the time-dependent coefficients of this map. We saw how useful the process of analytic continuation can be in transforming boundary conditions to global equations (an idea which will be used even more extensively throughout this chapter). We defined the Schwarz function of the free boundary, found the relation with the mapping function, and showed how (at least for the zero-surface tension problem) it is simply related to the complex potential of the flow. An infinite set of conserved quantities was found to exist for the case of flow driven by a single point sink at the origin, and the analogous result for more than one sink was given. Also, deductive methods for finding the correct functional form for the conformal map were described.

A natural question to ask now, since we are using similar complex variable methods to attack the Stokes flow problem, is: "can we find Stokes flow analogues of any of these Hele-Shaw results?" Thus, in this chapter, we are mainly concerned with the ZST, singularity-driven, Stokes flow model. Provided the Capillary number of the flow is large (that is, the surface tension parameter is small), the driving singularity may be supposed to dominate the evolution, and the ZST model should provide a good approximation to the motion. The two examples of driving mechanisms we consider are a point source (or sink) within the flow, and a dipole. As for the Hele-Shaw problem, the ZST model has the advantage of being much easier to deal with analytically, but also the disadvantage that solutions of interest often blow up within finite time.

### 3.1 Richardson's approach

The idea of applying complex variable methods to two-dimensional slow viscous flow is not a new one. The formulation we use closely follows that of Richardson [82], and we use almost identical notation. ${ }^{1}$ Since Stokes flow is quasistatic, we suppress time dependence in the notation, except where needed for emphasis. As mentioned in $\S 1.3 .1$, biharmonic functions may be expressed in terms of complex-valued functions, using the Goursat representation (see for example [10]). Writing $z=x+i y$, the biharmonic streamfunction can thus be written as

$$
\begin{equation*}
\psi(x, y)=-\Im\{\bar{z} \phi(z)+\chi(z)\} \tag{3.1}
\end{equation*}
$$

for functions $\phi, \chi$ analytic on the flow domain except at driving singularities of the flow. All physical quantities of interest can be written in terms of these "Goursat functions" $\phi$ and $\chi$, for

[^9]instance the (complex) velocity field is easily verified to be
\[

$$
\begin{equation*}
u+i v=\phi(z)-z \overline{\phi^{\prime}(z)}-\overline{\chi^{\prime}(z)} \tag{3.2}
\end{equation*}
$$

\]

and the pressure is

$$
\begin{equation*}
p=-4 \mu \Re\left\{\phi^{\prime}(z)\right\} \tag{3.3}
\end{equation*}
$$

The two SBC's (1.8) can be combined to give a single complex boundary condition. We note that if $s$ is arclength along $\partial \Omega$, then the normal $\mathbf{n}$ to the fluid domain is given by $\mathbf{n}=(d y / d s,-d x / d s)$, and also that if $\alpha$ is the angle made by the tangent to $\partial \Omega$ with the $x$-axis, then the curvature $\kappa$ is given by $d \alpha / d s$. Hence on $\partial \Omega$,

$$
\begin{aligned}
\frac{d z}{d s} & =e^{i \alpha} \\
\Rightarrow \quad \frac{d^{2} z}{d s^{2}} & =i \kappa e^{i \alpha}=i \kappa \frac{d z}{d s}
\end{aligned}
$$

Then, writing (1.8) as separate components, adding the second to $i$ times the first, and taking the complex conjugate, yields

$$
-4 \mu i \frac{\partial^{2} \psi}{\partial \bar{z}^{2}} \frac{d \bar{z}}{d s}+p \frac{d z}{d s}=T \kappa \frac{d z}{d s}
$$

so we can substitute for $\kappa(d z / d s)$ from above, and for the pressure from (3.3) and finally integrate with respect to arclength $s$ along $\partial \Omega$ to obtain

$$
\begin{equation*}
\phi(z)+z \overline{\phi^{\prime}(z)}+\overline{\chi^{\prime}(z)}=\frac{T i}{2 \mu} \frac{d z}{d s} \quad \text { on } \partial \Omega(t) \tag{3.4}
\end{equation*}
$$

It should be pointed out that, although we stated earlier that we were considering $\Omega(t)$ to be simply connected, (3.4) above is the first point at which we have used this assumption. In performing the integration along $\partial \Omega$, we dropped an arbitrary constant of integration. Provided we have only one free boundary present, this is justifiable; however, if we have more than one free boundary (i.e. a multiply connected domain), we will have independent arbitrary constants of integration for each one, only one of which can then be taken to be zero without loss of generality. This assumption greatly simplifies the following analysis.

We again introduce a time-dependent conformal map, $z=w(\zeta, t)$, from the unit disc in $\zeta$ space onto $\Omega(t)$, which is uniquely determined if we impose the usual normalisation conditions, $w(0, t)=0$ and $w^{\prime}(0, t)>0$ (see figure 2.1).

Richardson [82] derives two key equations ((2.18) and the unlabelled equation preceding (2.19) in his paper) governing the flow evolution. To simply present these without justification would be unduly confusing, so we give a condensed account of the derivation of that paper, with full acknowledgement.

If we define

$$
\begin{equation*}
\Phi(\zeta):=\phi(w(\zeta)), \quad \mathcal{X}(\zeta):=\chi(w(\zeta)) \tag{3.5}
\end{equation*}
$$

then transformation of the left-hand side of (3.4) to the $\zeta$-plane is straightforward. For the righthand side we need an expression for the complex (anticlockwise) tangent vector $d z / d s$ in terms of $\zeta$; for this we note that on $\partial \Omega$ (so $\zeta=e^{i \theta}$ for some $\theta \in(0,2 \pi)$ ) we have

$$
\begin{equation*}
\frac{d z}{d s}=w^{\prime}\left(e^{i \theta}\right) \frac{d\left(e^{i \theta}\right)}{d s}=i \zeta w^{\prime}(\zeta) \frac{d \theta}{d s}=i \zeta \frac{w^{\prime}(\zeta)}{\left|w^{\prime}(\zeta)\right|}=i \zeta\left\{w^{\prime}(\zeta) / \overline{w^{\prime}(\zeta)}\right\}^{1 / 2} \tag{3.6}
\end{equation*}
$$

in the third equality here we used the facts that $|d z / d s|=1$ and that $d \theta / d s$ is real and positive for the anticlockwise tangent. We shall want to analytically continue (3.4) in the $\zeta$-plane, a process which is complicated by the presence of the square-root branch-point on the right hand side of
(3.6). The troublesome term may be split up into parts analytic inside and outside the unit disc, writing

$$
\begin{equation*}
\frac{1}{\left[w^{\prime}(\zeta) \bar{w}^{\prime}(1 / \zeta)\right]^{1 / 2}}=\frac{1}{\left|w^{\prime}(\zeta)\right|}=f_{+}(\zeta)-f_{-}(\zeta) \tag{3.7}
\end{equation*}
$$

where $f_{+}(\zeta)$ and $f_{-}(\zeta)$ are functions analytic on $|\zeta| \leq 1,|\zeta| \geq 1$ respectively. This decomposition is unique if we also insist $f_{-}(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$. The functions $f_{ \pm}(\zeta)$ have explicit representations obtained via the Plemelj formulae ( $c f(2.26)$; see [10]),

$$
\begin{equation*}
f_{ \pm}(\zeta)=\frac{1}{2 \pi i} \oint_{|\zeta|=1} \frac{1}{\left|w^{\prime}(\tau)\right|} \frac{d \tau}{(\tau-\zeta)} \tag{3.8}
\end{equation*}
$$

for $|\zeta|<1$ and $|\zeta|>1$ respectively, and $f_{+}(0)$ is easily seen to be real. These identities rely on the assumption that $w^{\prime}(\zeta)$ is nonvanishing in a neighbourhood of $\partial \Omega$ (conformality). A simple argument reveals that the following identities are satisfied:

$$
\begin{equation*}
f_{+}(\zeta)=f_{+}(0)-\bar{f}_{-}(1 / \zeta) \quad \text { and } \quad f_{-}(\zeta)=f_{+}(0)-\bar{f}_{+}(1 / \zeta) \tag{3.9}
\end{equation*}
$$

We now have all the information we need to formulate the $\operatorname{SBC}(3.4)$ in the $\zeta$-plane. On the right-hand side we use $(3.6),(3.7)$ and (3.9), with the result (after a trivial rearrangement, and use of the identity $\zeta=1 / \bar{\zeta}$ on $|\zeta|=1$ ) that

$$
\begin{equation*}
\Phi(\zeta)+\frac{T}{2 \mu} f_{+}(\zeta) \zeta w^{\prime}(\zeta)=-w(\zeta) \frac{\bar{\Phi}^{\prime}(1 / \zeta)}{\bar{w}^{\prime}(1 / \zeta)}-\frac{\overline{\mathcal{X}}^{\prime}(1 / \zeta)}{\bar{w}^{\prime}(1 / \zeta)}+\frac{T}{2 \mu} f_{-}(\zeta) \zeta w^{\prime}(\zeta) \tag{3.10}
\end{equation*}
$$

this is the first of the "key equations" and will hold not just on $|\zeta|=1$, but also elsewhere by analytic continuation.

We also need to formulate the $\mathrm{KBC}(1.9)$ in the $\zeta$-plane. Firstly, note that (3.2) and (3.4) combine to give

$$
\begin{equation*}
(u+i v)_{\partial \Omega}=2 \phi(z)-\frac{i T}{2 \mu} \frac{d z}{d s} \tag{3.11}
\end{equation*}
$$

Consideration of the motion of a general particle at the point $w\left(e^{i \theta(t)}, t\right)$ on the boundary gives

$$
(u+i v)_{\partial \Omega}=i \zeta w^{\prime}(\zeta, t) \frac{d \theta}{d t}+\frac{\partial w}{\partial t}(\zeta, t)
$$

equating these two expressions, we find that

$$
\begin{equation*}
\frac{1}{\zeta w^{\prime}(\zeta, t)}\left\{2 \Phi(\zeta)-\frac{\partial w}{\partial t}(\zeta, t)\right\}+\frac{T}{2 \mu} \frac{1}{\left\{w^{\prime}(\zeta) \bar{w}^{\prime}(1 / \zeta)\right\}^{1 / 2}}=i \frac{d \theta}{d t} \quad \text { on }|\zeta|=1 \tag{3.12}
\end{equation*}
$$

Clearly, the real part of the left-hand side must then vanish on $|\zeta|=1$. Using (3.7) and (3.9) this can be rewritten as

$$
\begin{equation*}
\Re\left\{\frac{1}{\zeta w^{\prime}(\zeta, t)}\left[2 \Phi(\zeta)-\frac{\partial w}{\partial t}(\zeta, t)\right]+\frac{T}{\mu} f_{+}(\zeta)\right\}=\frac{T}{2 \mu} f_{+}(0) \quad \text { on }|\zeta|=1 \tag{3.13}
\end{equation*}
$$

which is the second of our key equations, and is analogous to the Galin equation (2.4) of $\S 2.2$.
Richardson specified $\phi(0)=0$ (so $\Phi(0)=0$ too), a convenient choice which ensures a unique solution to the problem (providing, of course, one exists). In this case, $[\cdot]$ in (3.13) has a simple zero at $\zeta=0$, as does its denominator $\left(w^{\prime}(0) \neq 0\right.$ since the map is conformal). Hence the combination in curly brackets in (3.13) is analytic on the unit disc, and since it is also real at $\zeta=0$, we may continue analytically, removing the " $\Re$ " from the left-hand side to get an equation which holds wherever the quantities are defined ((2.19) in [82]). A trivial rearrangement then gives

$$
\begin{equation*}
2 \Phi(\zeta)-\frac{\partial w}{\partial t}(\zeta, t)=\frac{T}{2 \mu}\left[f_{+}(0)-2 f_{+}(\zeta)\right] \zeta w^{\prime}(\zeta, t) \tag{3.14}
\end{equation*}
$$

In many cases this assumption is reasonable, giving physically acceptable solutions; however in cases where we have a driving singularity at the origin we may need to allow $\Phi(0)$ to be finite but nonzero, or possibly even infinite, and the above procedure is not so simple.

Consider a Stokes flow driven solely by surface tension, with no driving singularity in the flow domain. It can be shown [82] that, where a solution exists, the equations and boundary conditions for the problem specify the solution only up to an arbitrary rigid-body motion, i.e. there is nonuniqueness of the solution. A family of possible solutions to the problem then exists, and can be obtained from any one solution by adding on arbitrary translations and/or rotations to the velocity field.

To illustrate, suppose we have a Goursat solution pair $\left(\phi_{1}(z), \chi_{1}(z)\right)$, satisfying all physical requirements for a given problem; in particular, we expect that the total momentum of the fluid domain should be zero. ${ }^{2}$ Suppose also that $\phi_{1}(0)=A(t) \neq 0$, where we assume $A \in \mathbb{R}$ for simplicity. Consider the second Goursat pair,

$$
\begin{aligned}
\phi_{2}(z) & =\phi_{1}(z)-A \\
\chi_{2}(z) & =\chi_{1}(z)+A z .
\end{aligned}
$$

Clearly, the pressure fields for these two Goursat pairs are identical by (3.3), and it is easily checked that if the first pair satisfies the force balance condition (3.4), then so does the second pair. However, the velocity fields differ according to

$$
u_{2}+i v_{2}=u_{1}+i v_{1}-2 A
$$

in obvious notation.
The pair $\left(\phi_{2}(z), \chi_{2}(z)\right)$ has the feature that $\phi_{2}(0)=0$, which as we shall see, simplifies the solution procedure considerably (in particular, a polynomial mapping function will yield a solution if and only if this condition holds); however if the first solution is the physically realistic one, then this second solution will have a nonzero net momentum (along the $x$-axis, in the case $A \in \mathbb{R}$ ). If there is no driving singularity in the flow, this is irrelevant; we may solve for the easier case $\phi_{2}(0)=0$, and subtract off the appropriate velocity contribution a posteriori, if necessary. However, if we do have a fixed driving mechanism such as a source or sink in the flow, then doing this gives rise to a solution which is still contrived, since it will have a singularity translating in some specified way within the flow domain.

If we are only interested in solutions to the mathematical model then this is of little consequence, since mathematically, the solutions are perfectly valid. However, if we are solving with a particular physical situation in mind, it is important to solve for the correct Goursat pair; we cannot then adjust the final form of the solution. Of course, if it were the case that $\phi(0)=0$ always gave the "correct" solution this would not be an issue, but this is not so, as we shall see in §5.4.2. Similar remarks apply to the nonuniqueness of solutions up to rigid-body rotation; however, in all the situations we consider, net angular momentum vanishes automatically, and this is not a problem. We therefore consider the theory for the more general case $\phi(0) \neq 0$.

If $\Phi(0)=A(t)$ then the combination in curly brackets in (3.13) is no longer analytic on the unit disc, but has a simple pole at the origin. The "analytic continuation" for this case must therefore have a simple pole on the right-hand side, and it is relatively easy to write down the global equation (after multiplying through by $\left.\zeta w^{\prime}(\zeta)\right)$ as

$$
\begin{align*}
2 \Phi(\zeta)-\frac{\partial w}{\partial t}(\zeta, t)=\frac{T}{2 \mu}\left[f_{+}(0)-\right. & \left.2 f_{+}(\zeta)\right] \zeta w^{\prime}(\zeta, t) \\
& +\frac{2 w^{\prime}(\zeta)}{w^{\prime}(0)}\left(A-\bar{A} \zeta^{2}\right) \tag{3.15}
\end{align*}
$$

[^10]Note that, for this case, the expression (3.2) reveals that $\Phi(0)$ represents a uniform stream superimposed on the flow at the origin. Taking $A=\bar{A}$ in the above then corresponds to choosing a direction for this uniform stream (along the $x$-axis, here), which will often be justifiable on symmetry grounds. If $\Phi(\zeta)$ has a simple pole at the origin, so that

$$
\Phi(\zeta) \sim \frac{B}{\zeta}+A+O(\zeta) \quad \text { as } \zeta \rightarrow 0
$$

then if $A$ and $B$ are both real the analogous analytic continuation of the KBC is

$$
\begin{array}{r}
2 \Phi(\zeta)-\frac{\partial w}{\partial t}(\zeta, t)=\frac{T}{2 \mu}\left[f_{+}(0)-2 f_{+}(\zeta)\right] \zeta w^{\prime}(\zeta, t)+\frac{2 B w^{\prime}(\zeta)}{\zeta w^{\prime}(0)}\left(1-\zeta^{4}\right)+ \\
2 w^{\prime}(\zeta)\left(1-\zeta^{2}\right)\left(\frac{A}{w^{\prime}(0)}-\frac{B w^{\prime \prime}(0)}{w^{\prime}(0)^{2}}\right) \tag{3.16}
\end{array}
$$

with a somewhat more unwieldy expression if $A$ and $B$ are complex. We could clearly go on to consider higher-order singularities of $\Phi(\zeta)$ at the origin, but we do not do this; for the work of this thesis it is enough to note that we are able to deal with such situations, should the need arise. Note that by means of equations (3.10) and (3.14) (or (3.15), or (3.16)), we are, in principle, able to express all physical quantities in terms only of the mapping function $w(\zeta, t)$. Such an expression for the velocity field, which is sometimes useful, is given in appendix C.

An alternative perspective is given by writing the momentum in integral form. If $\mathbf{P}=\left(P_{1}, P_{2}\right)$ is the dimensionless momentum of the fluid domain then we have

$$
\begin{aligned}
P:=P_{1}+i P_{2} & =\iint_{\Omega}(u+i v) d x d y \\
& =\frac{1}{2 i} \int_{\partial \Omega} \bar{z}(u+i v) d z
\end{aligned}
$$

using the complex form of Green's theorem (see footnote (3) in chapter 2). Now, $(u+i v)_{\partial \Omega}$ is given by (3.11) and so

$$
P=\frac{1}{2 i} \int_{\partial \Omega}\left(2 \phi(z)-\frac{i T}{2 \mu} \frac{d z}{d s}\right) \bar{z} d z
$$

which, formulated in the $\zeta$-plane, becomes

$$
\begin{equation*}
P=\frac{1}{2 i} \int_{|\zeta|=1}\left(2 \Phi(\zeta)+\frac{T}{2 \mu} \zeta\left(\frac{w^{\prime}(\zeta)}{\bar{w}^{\prime}(1 / \zeta)}\right)^{1 / 2}\right) w^{\prime}(\zeta) \bar{w}(1 / \zeta) d \zeta \tag{3.17}
\end{equation*}
$$

We do not expect surface tension to alter momentum considerations, and we will be studying the ZST problem in detail in any case, so we consider the ZST version of this momentum expression. The integrand is such that we can apply the Residue Theorem, once we know the singularities within the unit disc. We assume that $\Phi(\zeta)$ is regular except possibly at the origin (since at present we are allowing a driving singularity only there). Hence the possible singularities are at $\zeta=0$, and at the singularities of $\bar{w}(1 / \zeta)$ (which will be the inverse complex conjugates of the singularities of $w(\zeta)$ in $|\zeta| \geq 1)$. From this it is clear that momentum conservation is intricately tied up with the behaviour of $\Phi(\zeta)$ at the origin.

### 3.2 Reduction to a single equation

We would like to simplify the problem to a single functional equation, which may also permit us to see what kinds of mapping functions $w(\zeta, t)$ will give solutions in particular situations. To this end, we note that the analytic continuation of the KBC (either (3.14), (3.15), or (3.16), depending on what kind of asymptotic behaviour we want $\Phi(\zeta)$ to have near $\zeta=0$ ) gives $\Phi(\zeta)$ in terms of the
mapping function. We can then substitute for $\Phi(\zeta)$ in (3.10) and rearrange slightly (replacing $\zeta$ by $1 / \zeta$ and taking the complex conjugate) to get a single equation for $\mathcal{X}^{\prime}(\zeta)$ which holds globally. We refer back to equations (3.1) and (3.5) for the definitions of $\Phi(\zeta)$ and $\mathcal{X}(\zeta)$. For the case in which $\Phi(0)=0$ the result is especially simple, being

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]+2 \mathcal{X}^{\prime}(\zeta)=\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right] \tag{3.18}
\end{equation*}
$$

(the function $f_{+}(\zeta)$ is defined by (3.7) and (3.8)); this equation is equivalent to equation (21) of [38]. If $\Phi(0)=A(t)$ (real and nonzero) the result is

$$
\begin{array}{r}
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]+2 \mathcal{X}^{\prime}(\zeta)+\frac{2 A}{w^{\prime}(0)} \frac{\partial}{\partial \zeta}\left[\left(1-\zeta^{2}\right) w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]= \\
\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right] \tag{3.19}
\end{array}
$$

Clearly, these equations ${ }^{3}$ are much simpler if we take $T=0$, and in this case we find some rather interesting results, which we discuss in subsequent sections of this chapter. For the general case, one first has to specify the singularity in the flow, which we take to be at the origin. From the expressions for the velocity and streamfunction in terms of $\phi(z)$ and $\chi(z)$, we then know what the behaviour of $\Phi(\zeta)$ and $\mathcal{X}(\zeta)$ near $\zeta=0$ must be. If $\Phi(\zeta)$ is bounded at the origin then we have to decide whether or not we are able to insist $\Phi(0)=0$, since this will change the governing equations. In [82] and [49], the net momentum of the flows is discussed. This certainly ought to be conserved, and for most physically-realistic flows it should be zero, which, as commented earlier, is not always the case with the assumption $\Phi(0)=0$. We could consider a blob of viscous fluid with nonzero net momentum, but this will usually be an artificial situation. In general, the velocity field in the neighbourhood of a singularity at the origin will have the form

$$
u+i v=[\text { prescribed singularity }]+[O(1) \text { uniform stream }]+O(z),
$$

and in [49], it is conjectured that a possible alternative condition to impose is that the $O(1)$ uniform stream in this expression should vanish, for a physically realisable flow.

By way of illustration, suppose we have a point sink of strength $Q>0$ at the origin. It is easy to see (by considering the local velocity field or streamfunction) that the behaviour of $\phi(z)$ and $\chi^{\prime}(z)$ must be

$$
\begin{equation*}
\phi(z)=\Phi(0)+O(z), \quad \chi^{\prime}(z)=\frac{Q}{2 \pi z}+\lambda+O(z), \quad \text { as } z \rightarrow 0 \tag{3.20}
\end{equation*}
$$

(since $\Phi(0) \equiv \phi(0))$ where $\lambda$ is some $O(1)$ quantity. Then near $z=0$,

$$
u-i v=-\frac{Q}{2 \pi z}+[\overline{\Phi(0)}-\lambda]+O(z)
$$

so $\Phi(0)$ is only one of two terms in this uniform stream superimposed on the point sink. Assuming the conjecture of [49] to be true, then, it is not immediately obvious in any given situation whether we have sufficient freedom to take it to be zero. We could also consider the velocity field in the neighbourhood of other types of driving singularities such as vortex dipoles. This gives the same inconclusive result; a pure vortex dipole of strength $M$ at the origin (with no auxiliary source or sink) requires

$$
\phi(z)=\Phi(0)+O(z), \quad \chi^{\prime}(z)=-\frac{M}{z^{2}}+\lambda+O(z), \quad \text { as } z \rightarrow 0
$$

[^11]for some (different) $O(1)$ quantity $\lambda$, and so the velocity field near $z=0$ is
$$
u-i v=\frac{M}{z^{2}}+[\overline{\Phi(0)}-\lambda]+O(z)
$$

In either case, with the above conjecture, solutions with $\Phi(0)=0$ can only be found if $\lambda=0$ in the local expansion for $\chi^{\prime}(z)$.

### 3.2.1 Another global equation

We can derive another global equation using a slightly different approach, which is similar to that adopted by Jeong \& Moffatt [52] for the steady problem. The derivation relies on writing the KBC in a different way to that of $\S 3.1$.

Using $u_{t}(z), u_{n}(z)$ to denote the tangential and normal components of the fluid velocity ${ }^{4}$ (both real), at a point $z$ on $\partial \Omega$, and $(u, v)$ to denote the usual $(x, y)$ components of velocity, we have

$$
\left.(u+i v)\right|_{\partial \Omega}=\left(u_{t}-i u_{n}\right) \frac{d z}{d s}
$$

Hence from (3.2) we see that

$$
\begin{equation*}
\phi(z)-z \overline{\phi^{\prime}(z)}-\overline{\chi^{\prime}(z)}=\left(u_{t}(z)-i u_{n}(z)\right) \frac{d z}{d s} \quad \text { on } \partial \Omega, \tag{3.21}
\end{equation*}
$$

and this boundary condition holds together with the force balance condition (3.4),

$$
\phi(z)+z \overline{\phi^{\prime}(z)}+\overline{\chi^{\prime}(z)}=\frac{T i}{2 \mu} \frac{d z}{d s} \quad \text { on } \partial \Omega(t) .
$$

Rewriting these conditions in terms of $\zeta$ (this was done in $\S 3.1$ for the force balance condition) and using the expression (3.6) for the complex tangent $d z / d s$, they become:

$$
\begin{align*}
& \Phi(\zeta)-w(\zeta) \frac{\overline{\Phi^{\prime}(\zeta)}}{\overline{w^{\prime}(\zeta)}}-\frac{\overline{\mathcal{X}^{\prime}(\zeta)}}{\overline{w^{\prime}(\zeta)}}=i \zeta\left(U_{t}(\zeta)-i U_{n}(\zeta)\right) \frac{w^{\prime}(\zeta)}{\left|w^{\prime}(\zeta)\right|}  \tag{3.22}\\
& \Phi(\zeta)+w(\zeta) \frac{\overline{\Phi^{\prime}(\zeta)}}{\overline{w^{\prime}(\zeta)}}+\frac{\overline{\mathcal{X}^{\prime}(\zeta)}}{\overline{w^{\prime}(\zeta)}}=-\frac{T}{2 \mu} \zeta \frac{w^{\prime}(\zeta)}{\left|w^{\prime}(\zeta)\right|} \tag{3.23}
\end{align*}
$$

both holding on $|\zeta|=1$. We use the notation $U_{t}(\zeta), U_{n}(\zeta)$ to denote the the tangential and normal components of the fluid velocity in the $\zeta$-plane. Adding (3.22) and (3.23) gives

$$
\frac{2 \Phi(\zeta)}{\zeta w^{\prime}(\zeta)}=\frac{1}{\left|w^{\prime}(\zeta)\right|}\left(U_{n}(\zeta)+i U_{t}(\zeta)-\frac{T}{2 \mu}\right) \quad \text { on }|\zeta|=1
$$

We also have equation (3.12) from $\S 3.1$,

$$
\frac{1}{\zeta w^{\prime}(\zeta, t)}\left\{2 \Phi(\zeta)-\frac{\partial w}{\partial t}(\zeta, t)\right\}+\frac{T}{2 \mu} \frac{1}{\left|w^{\prime}(\zeta)\right|}=i \frac{d \theta}{d t} \quad \text { on }|\zeta|=1
$$

Comparing these two equations we see that

$$
\begin{align*}
\frac{U_{n}}{\left|w^{\prime}(\zeta)\right|} & =\Re\left(\frac{w_{t}(\zeta)}{\zeta w^{\prime}(\zeta)}\right)  \tag{3.24}\\
\frac{U_{t}}{\left|w^{\prime}(\zeta)\right|} & =\Im\left(\frac{w_{t}(\zeta)}{\zeta w^{\prime}(\zeta)}\right)+\frac{d \theta}{d t} \tag{3.25}
\end{align*}
$$

[^12]Addition of equations (3.21) and (3.4) yields

$$
2 \Im\left[\phi(z) \frac{d \bar{z}}{d s}\right]=\frac{T}{2 \mu}-u_{n}
$$

whilst (3.4) alone gives

$$
\phi(z) \frac{d \bar{z}}{d s}+z \overline{\phi^{\prime}(z)} \frac{d \bar{z}}{d s}+\overline{\chi^{\prime}(z)} \frac{d \bar{z}}{d s}=\frac{i T}{2 \mu} .
$$

Elimination of the quantity $T /(2 \mu)$ between these two equations gives

$$
\begin{aligned}
\phi(z) \frac{d \bar{z}}{d s}+\bar{z} \phi^{\prime}(z) \frac{d z}{d s}+\chi^{\prime}(z) \frac{d z}{d s} & =-i u_{n} \\
\Rightarrow \frac{d}{d s}(\bar{z} \phi(z)+\chi(z)) & =-i u_{n}
\end{aligned}
$$

Clearly, for the steady problem, in which $u_{n} \equiv 0$, this will just reduce to the "streamline condition" that both the streamfunction and its biharmonic conjugate (the Airy stress function; see §3.9) may be taken to be constant on the free boundary. For the time-dependent problem, we may recast this equation in terms of $\zeta$ using the chain rule for the derivative with respect to $\zeta$, and equation (3.24); we find, after some rearrangement, the condition:

$$
\begin{equation*}
\frac{\partial}{\partial \zeta}\left(\mathcal{X}(\zeta)+\bar{w}(1 / \zeta)\left[\Phi(\zeta)-\frac{1}{2} w_{t}(\zeta)\right]\right)+\frac{1}{2} \frac{\partial}{\partial t}\left(w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right)=0 \tag{3.26}
\end{equation*}
$$

holding on $|\zeta|=1$, and elsewhere, by analytic continuation. This equation may be compared with those derived earlier, namely (3.18)

$$
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]+2 \mathcal{X}^{\prime}(\zeta)=\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right]
$$

(in the case $\Phi(0)=0$ ), and (3.19)

$$
\begin{array}{r}
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]+2 \mathcal{X}^{\prime}(\zeta)+\frac{2 A}{w^{\prime}(0)} \frac{\partial}{\partial \zeta}\left[\left(1-\zeta^{2}\right) w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]= \\
\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right]
\end{array}
$$

(in the case $\Phi(0)=A(t)$, real and nonzero). The main point to note about (3.26) is that it is independent of the surface tension parameter $T$, so has the same form for both ZST and NZST problems; however, it does contain both the unknown Goursat functions, which makes it less convenient to work with. It is a time-dependent generalisation of the "streamline condition" for the steady problem. If we wish to use (3.26) to solve problems, we must use it in conjunction with the relevant expression for $\Phi(\zeta)$ (examples of which are given by (3.14), (3.15), (3.16) in $\S 3.1)$. Note that $\Phi(\zeta)$ only appears in the particular combination $\Phi_{*}(\zeta):=\Phi(\zeta)-w_{t}(\zeta) / 2$, which simplifies things a little. We shall return to this form of the equations in chapter 5.

### 3.3 Method of solution

We now reconsider equations (3.18) or (3.19). The function $f_{+}(\zeta)$ is defined in (3.8), albeit awkwardly, in terms of the mapping function $w(\zeta)$, so once we have proposed a form for this map (typically a rational function of $\zeta$ with time-dependent coefficients), the only unknown in the evolution equation is the function $\mathcal{X}(\zeta)$. For a particular problem we know exactly what the singularities of $\mathcal{X}(\zeta)$ within the unit disc are (the "driving singularities", some examples of which were given above); elsewhere on the disc $\mathcal{X}(\zeta)$ must be analytic. This is what enables us to solve the problem; we must match singularities in equation (3.18) or (3.19).

Suppose we rewrite the relevant equation, placing $\mathcal{X}^{\prime}(\zeta)$ on the left-hand side and everything which depends only on $w(\zeta)$ on the right. Firstly, we must ensure that we choose a functional form for $w(\zeta)$ which will give a singularity of the right order (and in the right place) to match with that in $\mathcal{X}^{\prime}(\zeta)$ on the left-hand side. We consider systematic procedures for doing this later on, but often trial and error, or an "educated guess" is good enough. The presence of the messily-defined $f_{+}(\zeta)$ on the right-hand side is not a problem at this stage, since it is analytic on the unit disc.

Once we have decided on a particular form for $w(\zeta)$, we have the task of evaluating the righthand side of the equation asymptotically, near each of its singularities. Any singularities for which there is no "match" on the left-hand side must be eliminated by setting the coefficient to zero; if there is a match, the relevant coefficients must be equated.

Providing we have chosen a suitable form for the mapping function, this procedure should yield a well-determined system of o.d.e.'s for the time-dependent coefficients in the map. Solving these equations then gives the solution for the conformal map, and hence the evolution of the fluid domain $\Omega(t)$, which is valid until such time as the map ceases to be univalent on the unit disc (we refer back to the comments of $\S 2.7$ ).

### 3.4 A simple example

We illustrate the technique with a very simple solution, the ZST Hele-Shaw version of which was given in $\S 2.4$. This is one of a family of solutions presented by Howison \& Richardson [49], though we use slightly different notation. The mapping function used is

$$
\begin{equation*}
w(\zeta, t)=a_{1} \zeta+a_{2} \zeta^{2} \tag{3.27}
\end{equation*}
$$

by choosing axes suitably we may assume $a_{1}$ and $a_{2}$ to be real and positive, as explained in §2.4. Parameter values satisfying $a_{1} \geq 2 a_{2}$ give maps univalent on the unit disc (which describe limaçon-shaped free boundaries); univalency is lost when $a_{1}\left(t^{*}\right)=2 a_{2}\left(t^{*}\right)$, with the free boundary becoming a cardioid with a $3 / 2$-power cusp at the point $z^{*}=w\left(-1, t^{*}\right)$. The flow is driven by a single point sink at the origin, so that the Goursat functions have the asymptotic behaviour of (3.20). We take $\Phi(0)=0$ : it can be easily seen that this is a necessary condition for a solution of the form (3.27). Then the equation governing the evolution is (3.18), with $\Phi(\zeta)$ given by (3.14). Near the origin, equation (3.18) becomes

$$
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]-\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right]=-\frac{Q}{\pi \zeta}+O(1)
$$

Asymptotic evaluation of the left-hand side gives singularities of orders $1 / \zeta$ and $1 / \zeta^{2}$, the coefficients of which must be equated to $-Q / \pi$ and zero, respectively. We find

$$
w^{\prime}(\zeta) \bar{w}(1 / \zeta)=\frac{a_{1} a_{2}}{\zeta^{2}}+\frac{1}{\zeta}\left(a_{1}^{2}+2 a_{2}^{2}\right)+O(1)
$$

and

$$
\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)=\frac{a_{1} a_{2}}{\zeta} f_{+}(0)+O(1)
$$

hence the resulting equations are:

$$
\begin{array}{r}
\frac{d}{d t}\left(a_{1}^{2}+2 a_{2}^{2}\right)=-\frac{Q}{\pi} \\
\frac{d}{d t}\left(a_{1} a_{2}\right)=-\frac{T}{2 \mu} a_{1} a_{2} f_{+}(0)
\end{array}
$$

The function $f_{+}(0)$ is found (by direct integration in (3.8)) to be

$$
f_{+}(0)=\frac{2}{\pi a_{1}} K\left(\frac{2 a_{2}}{a_{1}}\right)
$$

where $K(\cdot)$ denotes the complete elliptic integral of the first kind (see [8], [30], or appendix B). The evolution is determined by solving these equations for $a_{1}(t)$ and $a_{2}(t)$. When $T>0$ the solution does not break down, and all the fluid is extracted from the domain. Note that the solution procedure does not require the determination of the Goursat functions $\Phi(\zeta)$ and $\mathcal{X}(\zeta)$, but should we wish to find them we can do so using (3.14) and (3.18). Likewise, we can find physical quantities such as the pressure and the velocity fields using expressions (3.3) (formulated in the $\zeta$-plane), and (3.41) or (C.1).

This particular solution is an example of the kind discussed in $\S 3.1$, in that it has a constant, nonzero component of momentum in the $x$-direction, although the sink is fixed within the flow. However, as stated there, it is still mathematically tenable, even if physically dubious.

Polynomial solutions are considered further in §3.6.1, and the work of [49], from which this example is taken, is reviewed in $\S 6.1$.

### 3.5 Zero surface tension problems

The zero surface tension (ZST) model is appropriate when the surface tension coefficient is small, and where we have a driving singularity in the flow which dominates the motion (the Capillary number, introduced in $\S 1.3 .1$, is large). This singularity might be of a very general kind, but in much of the theory we shall, for definiteness, assume that we have a single point sink in the flow. We recall here the comment of $\S 1.4 .2$ that the ZST Stokes flow is time-reversible; for the case in hand this means that if we let the flow evolve for some time under the action of the point sink (but not so long that solution breakdown occurs!), stop the motion and replace the sink by a source of equal strength, the motion will be exactly reversed. ${ }^{5}$ A consequence of this fact is that (for classical solutions to the problem) unless our initial domain is a circle, with the sink at its centre, complete extraction of the fluid cannot occur. Solution breakdown will inevitably be an issue then, for all except trivial cases of this problem.

We assume that the singularity is situated at the origin, $z=0$, so that it is also fixed at the origin in the $\zeta$-plane. For a point sink of strength $Q>0$ the local behaviour of $\phi$ and $\chi$ is given by (3.20), hence that of $\Phi(\zeta)$ and $\mathcal{X}^{\prime}(\zeta)$ is given by

$$
\begin{equation*}
\Phi(\zeta)=\Phi(0)+O(\zeta), \quad \mathcal{X}^{\prime}(\zeta)=\frac{Q}{2 \pi \zeta}+\left(\lambda w^{\prime}(0)+\frac{Q w^{\prime \prime}(0)}{4 \pi w^{\prime}(0)}\right)+O(\zeta) \tag{3.28}
\end{equation*}
$$

as $\zeta \rightarrow 0$. With the assumption $\Phi(0)=0$, by (3.18) the governing equation is simply

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]=-2 \mathcal{X}^{\prime}(\zeta) \tag{3.29}
\end{equation*}
$$

and the function $\Phi(\zeta)$ is given in terms of the mapping function by the relevant analytic continuation of (3.13),

$$
\begin{equation*}
\frac{\partial w}{\partial t}(\zeta, t)=2 \Phi(\zeta) \tag{3.30}
\end{equation*}
$$

we consider this simplest case first. The right-hand side of (3.29) is analytic on the unit disc, save for the simple pole at the origin specified in (3.28). We begin by deriving an interesting result of a very general nature; the existence of an infinite set of conserved quantities for the problem. For completeness and for future reference we shall also give the result for the case of non-zero surface tension, although in this case the quantities are not conserved, but evolve according to a complicated system of nonlinear o.d.e.'s.

[^13]
### 3.6 The conserved quantities

Consider the quantities $C_{k}(t)$ defined by

$$
\begin{align*}
C_{k}(t): & =\iint_{\Omega} \zeta^{k} d x d y \quad(k \geq 0) \\
& =\frac{1}{2 i} \int_{\partial \Omega} \zeta^{k} \bar{z} d z \\
& =\frac{1}{2 i} \int_{|\zeta|=1} \zeta^{k} w^{\prime}(\zeta) \bar{w}(1 / \zeta) d \zeta \tag{3.31}
\end{align*}
$$

Then, from (3.29), we see that

$$
\begin{aligned}
2 i \frac{d C_{k}}{d t}=\frac{d}{d t}\left[\int_{|\zeta|=1} \zeta^{k} w^{\prime}(\zeta) \bar{w}(1 / \zeta) d \zeta\right] & =\int_{|\zeta|=1} \zeta^{k} \frac{\partial}{\partial t}\left(w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right) d \zeta \\
& =-2 \int_{|\zeta|=1} \zeta^{k} \mathcal{X}^{\prime}(\zeta) d \zeta \\
& =-4 \pi i \operatorname{Res}\left[\zeta^{k} \mathcal{X}^{\prime}(\zeta)\right]_{\zeta=0}
\end{aligned}
$$

With a single point sink/source at the origin, we have the asymptotic behaviour of (3.28), and the above equations reduce to

$$
\frac{d C_{k}}{d t}= \begin{cases}-Q & k=0  \tag{3.32}\\ 0 & k=1,2, \ldots\end{cases}
$$

thus revealing the conserved quantities. The first of these clearly represents conservation of mass, since we have

$$
\text { Area of } \Omega=\iint_{\Omega} d x d y \equiv C_{0}
$$

This system may be modified to deal with other singularities at the origin, for example multipoles (the analogous Hele-Shaw multipole problem was considered by Entov et al. in [24]). For instance, if we have a dipole singularity at the origin (having the $x$-axis as streamline), so that the local behaviour is $\mathcal{X}(\zeta)=M /\left(\zeta w^{\prime}(0)\right)+O(1)$, and $\Phi(\zeta)=O(1)$, then the corresponding system of equations is easily seen to be

$$
\frac{d C_{k}}{d t}= \begin{cases}\frac{2 \pi M}{w^{\prime}(0)} & k=1 \\ 0 & k=0, k \geq 2\end{cases}
$$

We note also that the above readily generalises from $\zeta^{k}$ (in the definition of $C_{k}$ ) to arbitrary functions $h(\zeta)$ analytic on the unit disc, the result being

$$
\begin{align*}
C_{[h]}(t)=\iint_{\Omega} h(\zeta) d x d y & =\frac{1}{2 i} \int_{|\zeta|=1} h(\zeta) w^{\prime}(\zeta) \bar{w}(1 / \zeta) d \zeta \\
\Rightarrow \frac{d}{d t} C_{[h]} & =-Q h(0) \tag{3.33}
\end{align*}
$$

this can be useful for some initial and boundary value problems.
As promised, we now give the analogous result for the NZST problem. The procedure is exactly the same, except we use (3.18) rather than (3.29), with the extra term on the right-hand
side. Integration by parts is used to deal with this term, giving the system of equations for this case as

$$
\frac{d C_{k}}{d t}= \begin{cases}-Q & (k=0)  \tag{3.34}\\ -\frac{k T}{\mu}\left[\frac{f_{+}(0)}{2} C_{k}+\sum_{r=1}^{\infty} \frac{f_{+}^{(r)}(0)}{r!} C_{k+r}\right] & (k \geq 1)\end{cases}
$$

The first equation is the same as for the ZST problem, this being the mass conservation result. The $f_{+}^{(r)}(0)$ are obtained from (3.8) and are nonlinear functions of the coefficients of $w(\zeta, t)$. Note that if time is rescaled via

$$
\tau=\frac{T}{\mu} t
$$

then the system (3.34) becomes

$$
\frac{d C_{k}}{d \tau}= \begin{cases}-\frac{\mu}{T} Q & (k=0) \\ -k\left[\frac{f_{+}(0)}{2} C_{k}+\sum_{r=1}^{\infty} \frac{f_{+}^{(r)}(0)}{r!} C_{k+r}\right] & (k \geq 1)\end{cases}
$$

Whenever $T \neq 0$ then, it can be scaled out of the problem, provided we also rescale the sink strength $Q$.

Further progress on this NZST problem for the general mapping function looks decidedly unpromising and we do not pursue it further, although we recall that, as mentioned earlier, the NZST problem proves surprisingly tractable in certain individual cases.

### 3.6.1 Polynomial mapping functions

We illustrate our results with the map

$$
\begin{equation*}
w(\zeta)=\sum_{r=1}^{N} a_{r}(t) \zeta^{r} \tag{3.35}
\end{equation*}
$$

This will clearly give a solution to both the ZST and NZST problems, by (3.32) and (3.34), since only the first $N$ of the $C_{k}(t)$ are nonzero; moreover, the degree of the polynomial map must remain the same throughout for both problems, as a consequence of the invariants $C_{k} \equiv 0(k \geq N)$, and $C_{N-1} \neq 0$. Bearing in mind the discussion of $\S 2.7$, the coefficients $a_{r}(t)$ here must be subject to various constraints to ensure univalency of $w(\zeta)$, but we can only be specific about these in special simple cases, for example $N \leq 3$ (see [50], [15]) or if $a_{r}=0$ for $r \neq 1, N$ (see [49]); for the general polynomial they are too difficult. Assuming we have a univalent map then, ${ }^{6}$ we may evaluate the $C_{k}$ directly from the definition (3.31),

$$
\begin{equation*}
C_{k}=\pi \sum_{n=1}^{N-k} n a_{n} \bar{a}_{n+k} \quad 0 \leq k \leq N-1, \tag{3.36}
\end{equation*}
$$

all other $C_{k}$ being identically zero. The nonzero invariants $C_{k}$, and $C_{0}(0)$, are determined by the initial conditions. Equations (3.32), when integrated, constitute a set of nonlinear simultaneous equations for the coefficients $a_{r}(t)$ which may be solved by starting with the last

[^14]$\left(C_{N-1}=\right.$ constant $)$ and working backwards. The evolution is then fully determined until such time as the mapping (3.35) ceases to be univalent.

In $\S 3.4$ we solved for the mapping function (3.35) in the case $N=2$. The results of this section give the same evolution equations much more quickly; in the NZST case equations (3.34) give

$$
\begin{aligned}
\frac{d C_{0}}{d t} & =-Q \\
\frac{d C_{1}}{d t} & =-\frac{T}{2 \mu} f_{+}(0) C_{1}
\end{aligned}
$$

with $C_{0}$ and $C_{1}$ given by (3.36) as

$$
C_{0}=\pi\left(\left|a_{1}\right|^{2}+2\left|a_{2}\right|^{2}\right), \quad C_{1}=\pi a_{1} \bar{a}_{2}
$$

this is exactly as we found in $\S 3.4$ (where we assumed real coefficients, without loss of generality).
Tanveer \& Vasconcelos [96] considered polynomial solutions for the complementary NZST problem of a bubble in an unbounded expanse of fluid. Setting $T=0$ in their analysis, one can recover conserved quantities similar to those above. The NZST polynomial solution for the special case in which $a_{r}=0$ for $r \neq 1, N$ has been found exactly by Howison \& Richardson [49]; we shall later use ideas from that paper to consider the NZST case for a cubic polynomial map in the limit $T \rightarrow 0$.

### 3.6.2 Comparison with the Hele-Shaw problem - 'Richardson's Moments' and other matters

The results of this section bear a strong resemblance to the theory of 'Richardson's moments' in the ZST Hele-Shaw problem, which we described in $\S 2.5$. The similarities and differences of the Stokes flow and the Hele-Shaw problems were first remarked upon in [49], and the results of this chapter add weight to their observations.

Recall from $\S 2.5$ that the Hele-Shaw moments of the fluid domain are defined by the formula

$$
M_{k}=\iint_{\Omega} z^{k} d x d y=\frac{1}{2 i} \int_{\partial \Omega} z^{k} \bar{z} d z \quad k=0,1,2, \ldots
$$

for the case we have been considering here, namely flows driven by a single sink at the origin, they are conserved in a manner identical to our $C_{k}$,

$$
\frac{d M_{k}}{d t}= \begin{cases}-Q & k=0, \\ 0 & k=1,2, \ldots\end{cases}
$$

The $C_{k}$ are defined by the integrals

$$
\begin{equation*}
C_{k}=\iint_{\Omega} \zeta(z)^{k} d x d y=\frac{1}{2 i} \int_{\partial \Omega} \zeta(z)^{k} \bar{z} d z \quad k=0,1,2, \ldots ; \tag{3.37}
\end{equation*}
$$

the similarity of these expressions with the definitions of the $M_{k}$ is striking. The apparent simplicity of the expression (3.37) is misleading, however, since it presupposes knowledge of the inverse conformal map, $\zeta=\zeta(z)$. The derivation of the conservation laws required that we reformulate the integrals in the $\zeta$-plane, as

$$
C_{k}=\frac{1}{2 i} \int_{\partial \Omega} \zeta^{k} \bar{w}(1 / \zeta) w^{\prime}(\zeta) d \zeta
$$

which although superficially more complicated, is actually more convenient to work with.

In addition, a version of the Stokes flow evolution equation (3.18) may be obtained for the Hele-Shaw problem, which in dimensionless form is

$$
\begin{align*}
& \frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]-\frac{\partial}{\partial \zeta}\left[2 \Upsilon(\zeta)+w_{t}(\zeta) \bar{w}(1 / \zeta)\right]= \\
& \quad-T \frac{\partial}{\partial \zeta}\left[\frac{2 \zeta w^{\prime}(\zeta) \bar{w}^{\prime}(1 / \zeta)+w^{\prime}(\zeta) \bar{w}^{\prime \prime}(1 / \zeta)+\zeta^{2} w^{\prime \prime}(\zeta) \bar{w}^{\prime}(1 / \zeta)}{\zeta\left(w^{\prime}(\zeta) \bar{w}^{\prime}(1 / \zeta)\right)^{3 / 2}}\right] \tag{3.38}
\end{align*}
$$

where $\Upsilon(\zeta)$ is the complex potential for the flow (so $\Upsilon=-p+i \psi$ ), recalling the definitions of $\S 2.2 .{ }^{7}$ The form of the singularity on the right-hand side of this equation, as compared with (3.18), explains why the NZST Hele-Shaw problem is so much less tractable than the NZST Stokes flow problem, despite being governed by only a second-order (rather than a fourth-order) p.d.e.-in general, if one assumes a specific form for the mapping function $w(\zeta)$ at time $t=0$ (one which works for the ZST problem; usually rational, or rational-logarithmic), it is no longer guaranteed that the same functional form persists for $t>0$. For the Stokes flow problem, at least with the assumption $\Phi(0)=0$, we do have this guarantee (see [82]); but in any case this can be seen, more or less, just by looking at the form of equation (3.18). To put it another way, for the Stokes flow problem, if a particular mapping function gives a solution to the ZST problem, this same map will also give a solution to the NZST problem (recall the comment made for the polynomial maps in $\S 3.6 .1$ ); this is very definitely not the case for the Hele-Shaw problem.

Multiplying the ZST version of (3.38) by $w(\zeta)^{k}$ and integrating around the unit circle yields the moment conservation result (when we transform the result to an integral in the $z$-plane). Writing the equations in this way highlights the differences, as well as the similarities, between the ZST problems, and their methods of solution. Recall that the Schwarz function for an analytic curve $\gamma$ is the unique (locally analytic) function $g(z)$ such that the equation $\bar{z}=g(z)$ defines $\gamma$, with the identity

$$
g(z)=g(w(\zeta))=\bar{w}(1 / \zeta)
$$

holding. Considering the ZST version of (3.18), we see that since $\mathcal{X}(\zeta)$ must be analytic on the flow domain except at driving singularities, the (non-driving) singularities of the Schwarz function within $\Omega(t)$ for Stokes flow must remain fixed in the $\zeta$-plane, whereas for Hele-Shaw flow we have seen that they remain fixed in the physical plane. For Hele-Shaw, it is best to work in the physical plane wherever possible, hence we use equation (2.11) rather than (3.38), while the equations of Stokes flow are easiest to deal with when formulated in the $\zeta$-plane. These observations tie in with the above integral expressions (over the domains in physical space for Hele-Shaw, and in $\zeta$-space for Stokes flow) for the conserved quantities.

### 3.6.3 Source/sink systems - a warning example

Given the results for Hele-Shaw, an "obvious" question to consider next is whether the results of $\S 3.6$ might be extended to systems of sources and sinks distributed throughout the flow domain (refer back to figure 2.4 for a typical geometry). It is instructive to do this, as it reveals complications which can arise with the solution method of $\S 3.3$. We may as well take one of the singularities to be situated at the origin, and suppose the others to be at fixed points $z_{k}$ in $\Omega(t)$ $(1 \leq k \leq N)$. Since we have already stipulated $w(0)=0, w^{\prime}(0)>0$, we assume the preimages of these points under our conformal map to be time dependent, $z_{k}=w\left(\alpha_{k}(t)\right)$, say. The local behaviour of $\mathcal{X}^{\prime}(\zeta)$ at such a point will then be

$$
\begin{equation*}
\mathcal{X}^{\prime}(\zeta)=\frac{Q_{k}}{2 \pi\left(\zeta-\alpha_{k}(t)\right)}+O(1) \quad \text { as } \zeta \rightarrow \alpha_{k}(t) \tag{3.39}
\end{equation*}
$$

[^15]where $Q_{k}<,>0$ indicates that we have a source/sink (respectively) of strength $\left|Q_{k}\right|$. We shall refer to "sinks" throughout, for simplicity, but it is understood that we have a source if $Q$ is negative.

Before we attempt to derive invariants etc., we first pause to think about the restrictions equation (3.29) will impose on the choice of mapping function $w(\zeta)$. Consider this equation near one of the sinks $Q_{k}($ not $z=0)$. If we are to have a balance of terms there ( $\left.\S 3.3\right)$, then by (3.39) we must have (integrating with respect to time)

$$
\begin{equation*}
w^{\prime}(\zeta) \bar{w}(1 / \zeta)=\lambda_{k} \log \left(\zeta-\alpha_{k}(t)\right)+O(1) \tag{3.40}
\end{equation*}
$$

for some $\lambda_{k}$, which must be constant if we are to avoid a logarithmic singularity in $\mathcal{X}^{\prime}(\zeta)$, which should not be present if we have a pure sink at $z_{k}$. Even if we relax this assumption somewhat, such a singularity is not physically acceptable, as can be seen by considering the expression for the velocity field in terms of $\zeta$ 's (see (3.2)),

$$
\begin{equation*}
u-i v=\overline{\Phi(\zeta)}-\overline{w(\zeta)} \frac{\Phi^{\prime}(\zeta)}{w^{\prime}(\zeta)}-\frac{\mathcal{X}^{\prime}(\zeta)}{w^{\prime}(\zeta)} \tag{3.41}
\end{equation*}
$$

The first two terms on the left-hand side are regular $\left(\Phi(\zeta)=w_{t}(\zeta) / 2\right.$, so is regular everywhere), as is $1 / w^{\prime}(\zeta)$ in the third term, hence the only singularities here are those of $\mathcal{X}^{\prime}(\zeta)$. Logarithmic singularities in $\mathcal{X}^{\prime}(\zeta)$ will never be allowable within $|\zeta| \leq 1$ then, since they give rise to a multivalued velocity field.

Returning to (3.40), since $w(\zeta)$ is univalent on the unit disc, the singularity on the righthand side must come from $\bar{w}(1 / \zeta)$ on the left-hand side, so that the mapping function must have logarithmic branch-points outside the unit disc (at points $1 / \bar{\alpha}_{k}$, to be specific; branch cuts can then be taken from these points to infinity). The preimage of $z=0$ is time-independent, so we do not need a logarithmic singularity in $\bar{w}(1 / \zeta)$ at the origin. Indeed, the presence of such a singularity would be unacceptable, since it would imply a logarithmic singularity in $w$ itself at the origin.

Possible solutions are thus very restricted by the presence of additional sinks. Seeking the simplest options, we first attempt a solution with one central sink $Q$ and the others placed symmetrically about it, and of equal strengths $Q_{1}$. The above analysis strongly suggests trying a map of the form

$$
w(\zeta)=a \zeta+b \sum_{k=1}^{k=N} \varpi^{k} \log \left(1-\gamma \zeta \varpi^{-k}\right)
$$

where $\varpi=e^{2 \pi i / N}$, and $0<\gamma<1$, which has the necessary singularities. The non-central sinks here must be positioned at $z_{k}=w\left(\gamma \varpi^{k}\right)$ (so $\left.\alpha_{k} \equiv \gamma \varpi^{k}\right)$. Matching singularities in (3.29) at $\zeta=0$ according to the procedure outlined in $\S 3.3$ yields the o.d.e.

$$
\begin{equation*}
\frac{d}{d t}[a(a-\gamma b N)]=-\frac{Q}{\pi} \tag{3.42}
\end{equation*}
$$

When matching at $\zeta=\gamma \varpi^{k}$, the left-hand side of (3.29) has the local behaviour

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right] & =\frac{-b}{\left(\zeta-\gamma \varpi^{k}\right)}\left(a-\frac{\gamma N b}{1-\gamma^{2}}\right) \frac{d \gamma}{d t} \\
& +\varpi^{-k} \frac{d}{d t}\left[b\left(a-\frac{\gamma N b}{1-\gamma^{2}}\right)\right] \log \left(\zeta-\gamma \varpi^{k}\right)+O(1)
\end{aligned}
$$

so matching the simple poles and eliminating the logarithm (recall the earlier discussion about constant $\lambda_{k}$ in (3.40)) gives two more equations,

$$
\begin{align*}
b \frac{d \gamma}{d t}\left(a-\frac{\gamma N b}{1-\gamma^{2}}\right) & =-\frac{Q_{1}}{\pi}  \tag{3.43}\\
\text { and } \quad b\left(a-\frac{\gamma N b}{1-\gamma^{2}}\right) & =\text { constant. } \tag{3.44}
\end{align*}
$$

This would all be fine, were it not for the fact that we have not yet required that the sinks be fixed in the physical plane. This imposes an extra condition,

$$
w(\gamma)=\text { constant }
$$

giving a total of four independent equations for the three unknown functions $a(t), b(t), \gamma(t)$.
One might expect that the difficulty could be overcome if, instead of assuming time-dependent preimages $\alpha_{k}(t)$, we tried to fix them in the $\zeta$-plane. Similar arguments (for the problem with symmetry) then lead us to a map with simple poles at $\zeta=\varpi^{k} / \gamma$ (instead of the logarithmic singularities), where now $\gamma$ must be constant, so we have one fewer unknowns. In this case we get one equation from matching at the sink at $z=0$, one from matching at any of the sinks $z=z_{k}$, but again we have the further condition that the sinks be fixed in the physical plane, so we have a total of three independent equations for only two unknowns, $a(t), b(t)$, and the system is still overdetermined. The best we can do is to impose conditions (3.42), (3.43) and (3.44), and allow the sinks to move in a manner dictated by their solution, which is not very satisfactory.

This "overdeterminedness" of the system of o.d.e.'s which arises when we try to allow more than one sink is a problem, since we can see that it is not a feature of the particular geometry we assumed in this example, but will arise quite generally whenever we have more than one singularity. Similar problems are encountered in chapter 5 in our discussion of problems on unbounded domains; there, we attempt to circumvent the difficulty by allowing $\Phi(0) \neq 0$.

### 3.7 The Schwarz function for the ZST problem

Recall the results of $\S 2.5$, where the Hele-Shaw moment constants were linked to the Cauchy transform of the fluid domain (and hence to the singular part of the Schwarz function of the boundary), and a systematic method was presented of finding the correct form of the mapping function to solve the ZST Hele-Shaw problem with a particular driving mechanism. We now consider whether similar results might exist for the ZST Stokes flow problem.

The Schwarz function is known to be analytic in some neighbourhood of the free boundary, so we may write $g(z)=h^{\prime}(z)$ for some function $h$, which will be analytic in the same neighbourhood of $\partial \Omega$. Then defining $H(\zeta)=h(w(\zeta))$ we find

$$
\begin{equation*}
H^{\prime}(\zeta)=\frac{d}{d \zeta}(h(w(\zeta)))=g(w(\zeta)) w^{\prime}(\zeta)=w^{\prime}(\zeta) \bar{w}(1 / \zeta) \tag{3.45}
\end{equation*}
$$

using the functional identity (2.12), $g(w(\zeta))=\bar{w}(1 / \zeta)$. Comparison of this expression (3.45) for $H^{\prime}(\zeta)$ with the definition (3.31) of the quantities $C_{k}(t)$ immediately reveals the $C_{k}$ to be the coefficients of the principal part of the Laurent expansion of $H^{\prime}(\zeta, t)$ about $\zeta=0$,

$$
H^{\prime}(\zeta, t)=\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{C_{k}}{\zeta^{k+1}}+(\text { regular at } \zeta=0)
$$

this will hold regardless of any assumption about the behaviour of $\Phi(\zeta)$ at the origin. Using the decomposition (2.27) on the Schwarz function, and analogously on the function $h(z)$, then writing $H_{e}(\zeta)=h_{e}(w(\zeta))$, so that $H_{e}$ contains all the singularities of $H$ within the unit disc, we have

$$
\begin{align*}
H_{e}^{\prime}(\zeta, t) & =\frac{1}{\pi} \sum_{0}^{\infty} \frac{C_{k}}{\zeta^{k+1}} \\
\Rightarrow H_{e}(\zeta, t) & =\frac{C_{0}}{\pi} \log \zeta-\frac{1}{\pi} \sum_{1}^{\infty} \frac{C_{k}}{k \zeta^{k}} . \tag{3.46}
\end{align*}
$$

We use the notation $h_{e}$ and $H_{e}$ to conform with the Hele-Shaw work of $\S 2.5$, but with slight reluctance, since the subscript " $e$ " was introduced to denote analyticity in the exterior of the fluid
domain. Although this is always true for the singular part of the Schwarz function $g_{e}(z)$, the function $h_{e}(z)$ has a logarithmic singularity at the origin, and hence also at infinity. ${ }^{8}$

Suppose we have the case in which $\Phi(0)=A(t)$ (nonzero and real). The equation governing the evolution is then

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]+2 \mathcal{X}^{\prime}(\zeta)+\frac{2 A}{w^{\prime}(0)} \frac{\partial}{\partial \zeta}\left[\left(1-\zeta^{2}\right) w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]=0 \tag{3.47}
\end{equation*}
$$

and $\Phi(\zeta)$ is given by

$$
\begin{equation*}
\Phi(\zeta)=\frac{1}{2} w_{t}(\zeta)+\frac{A}{w^{\prime}(0)} w^{\prime}(\zeta)\left(1-\zeta^{2}\right) \tag{3.48}
\end{equation*}
$$

using equations (3.19) and (3.15), with $T=0$. Recalling (3.45), it follows from (3.47) that

$$
-\mathcal{X}^{\prime}(\zeta)=\frac{1}{2} \frac{\partial^{2} H}{\partial t \partial \zeta}+\frac{A}{w^{\prime}(0)} \frac{\partial}{\partial \zeta}\left[\left(1-\zeta^{2}\right) H^{\prime}(\zeta)\right]
$$

which may be integrated once with respect to $\zeta$, giving

$$
\begin{equation*}
\frac{1}{2} \frac{\partial H}{\partial t}+\frac{A}{w^{\prime}(0)}\left(1-\zeta^{2}\right) \frac{\partial H}{\partial \zeta}=-\mathcal{X}(\zeta) \tag{3.49}
\end{equation*}
$$

So, we have a partial differential equation which must be satisfied globally by the primitive of the Schwarz function, in the $\zeta$-plane.

We want to take the singular part of (3.49) within the unit disc, to get a p.d.e. for $H_{e}(\zeta, t)$. When doing this, we must remember to subtract off the regular terms arising from the term $-\left(A \zeta^{2} / w^{\prime}(0)\right) \partial H_{e} / \partial \zeta$ on the left-hand side. The result is

$$
\begin{equation*}
\frac{1}{2} \frac{\partial H_{e}}{\partial t}+\frac{A}{w^{\prime}(0)}\left(1-\zeta^{2}\right) \frac{\partial H_{e}}{\partial \zeta}=-\frac{A}{\pi w^{\prime}(0)}\left(C_{0} \zeta+C_{1}\right)-\mathcal{X}_{\mathrm{sing}}(\zeta) \tag{3.50}
\end{equation*}
$$

or, defining the scaled time variable $\tau$ by

$$
\begin{equation*}
\frac{d \tau}{d t}=\frac{2 A(t)}{w^{\prime}(0, t)} \tag{3.51}
\end{equation*}
$$

(the constant of integration taken to be zero, so that the time origins coincide),

$$
\begin{equation*}
\frac{\partial H_{e}}{\partial \tau}+\left(1-\zeta^{2}\right) \frac{\partial H_{e}}{\partial \zeta}=-\frac{1}{\pi}\left(C_{0} \zeta+C_{1}\right)-\frac{w^{\prime}(0)}{A} \mathcal{X}_{\operatorname{sing}}(\zeta) \tag{3.52}
\end{equation*}
$$

In the above, $\mathcal{X}_{\text {sing }}(\zeta)$ denotes the strictly singular part of $\mathcal{X}(\zeta)$ within the unit disc, which will be known precisely once we have specified the driving singularity. For instance, with the point sink at the origin, $\mathcal{X}_{\text {sing }}(\zeta)=Q /(2 \pi) \log \zeta$, whilst a dipole of strength $M$ at the origin, having the $x$-axis as a streamline, gives $\mathcal{X}_{\text {sing }}(\zeta)=M /\left(w^{\prime}(0) \zeta\right)$.

Define the function

$$
\begin{equation*}
\mathcal{F}(\zeta, \tau)=\sum_{1}^{\infty} \frac{C_{k}}{k \zeta^{k}} \tag{3.53}
\end{equation*}
$$

Then

$$
\begin{equation*}
H_{e}(\zeta, \tau)=\frac{C_{0}}{\pi} \log \zeta-\frac{1}{\pi} \mathcal{F}(\zeta, \tau) \tag{3.54}
\end{equation*}
$$

[^16]and (3.52) becomes
\[

$$
\begin{equation*}
\frac{\partial \mathcal{F}}{\partial \tau}+\left(1-\zeta^{2}\right) \frac{\partial \mathcal{F}}{\partial \zeta}=\frac{C_{0}}{\zeta}+C_{1}+\frac{\pi w^{\prime}(0)}{A} \hat{\mathcal{X}}_{\text {sing }}(\zeta) \tag{3.55}
\end{equation*}
$$

\]

where $\hat{\mathcal{X}}_{\text {sing }}(\zeta):=\mathcal{X}_{\text {sing }}(\zeta)-Q /(2 \pi) \log \zeta$, so we have subtracted off any point sink behaviour. If, for example, we have a flow driven only by a point sink, then $\hat{\mathcal{X}}_{\operatorname{sing}}(\zeta) \equiv 0$, and the p.d.e. for $\mathcal{F}$ is just

$$
\begin{equation*}
\frac{\partial \mathcal{F}}{\partial \tau}+\left(1-\zeta^{2}\right) \frac{\partial \mathcal{F}}{\partial \zeta}=\frac{C_{0}}{\zeta}+C_{1} \tag{3.56}
\end{equation*}
$$

while if we have a dipole of strength $M$ at the origin (and no point sink) driving the flow, then (3.55) becomes

$$
\begin{equation*}
\frac{\partial \mathcal{F}}{\partial \tau}+\left(1-\zeta^{2}\right) \frac{\partial \mathcal{F}}{\partial \zeta}=\left(C_{0}+\frac{M \pi}{A}\right) \frac{1}{\zeta}+C_{1} \tag{3.57}
\end{equation*}
$$

which is essentially the same equation. In equation (3.56), $C_{0}(\tau)$ will be changing in accordance with mass conservation for a point sink, while in (3.57) $C_{0}$ will just be a positive constant equal to the area of the fluid domain. We can solve such p.d.e.'s with relative ease. Consider the case in which $\mathcal{F}(\zeta, \tau)$ satisfies (3.56). We can simplify this equation by subtracting from $\mathcal{F}$ the quantity $\int^{\tau} C_{1}\left(\tau^{\prime}\right) d \tau^{\prime}$, so that we need only solve

$$
\frac{\partial \hat{\mathcal{F}}}{\partial \tau}+\left(1-\zeta^{2}\right) \frac{\partial \hat{\mathcal{F}}}{\partial \zeta}=\frac{C_{0}(\tau)}{\zeta}
$$

The equations of the characteristic projections of this p.d.e. in the $(\zeta, \tau)$-plane are

$$
\zeta=\tanh (\tau+\alpha)
$$

the different characteristics being given by varying the parameter $\alpha$. Equivalently, the combination

$$
\alpha=\tanh ^{-1} \zeta-\tau
$$

is constant along a characteristic. On characteristics,

$$
\frac{d \hat{\mathcal{F}}}{d \tau}=\frac{C_{0}(\tau)}{\tanh (\tau+\alpha)}
$$

and so finally the solution to the p.d.e. (3.56) is:

$$
\begin{align*}
\mathcal{F}(\zeta, \tau)=\int_{0}^{\tau} \frac{C_{0}\left(\tau^{\prime}\right) d \tau^{\prime}}{\tanh \left(\tau^{\prime}-\tau+\tanh ^{-1} \zeta\right)} & +f\left(\tanh ^{-1} \zeta-\tau\right) \\
& +\int_{0}^{\tau} C_{1}\left(\tau^{\prime}\right) d \tau^{\prime} \tag{3.58}
\end{align*}
$$

where $f$ is some arbitrary function which depends on the initial conditions imposed. The solution of (3.57) also follows immediately from this, if we just replace $C_{0}$ by $\left(C_{0}+M \pi / A\right)$ in (3.58).

For the Hele-Shaw problem we are able to work out how the singularities of the Schwarz function $g(z)$ vary in time (and space), and, in certain situations, deduce the form of the mapping function from the functional identity (2.27). This procedure was outlined in $\S 2.5$, and a detailed example will be given in $\S 5.3$. For Stokes flow, the singular part of the Schwarz function is given by

$$
G_{e}(\zeta)=g_{e}(w(\zeta))=h_{e}^{\prime}(w(\zeta))=\frac{H_{e}^{\prime}(\zeta)}{w^{\prime}(\zeta)}
$$

We know also from the identity (2.27) that

$$
\begin{equation*}
g_{i}(w(\zeta))+G_{e}(\zeta)=\bar{w}(1 / \zeta) \tag{3.59}
\end{equation*}
$$

and that the first term on the left-hand side here is analytic within the unit disc. Hence the singularities of $\bar{w}(1 / \zeta)$ in $|\zeta| \geq 1$ must be exactly those of $G_{e}(\zeta)$, which (in principle) will tell us the general form of the mapping function we must try if we wish to obtain a solution. In practice, we find it easier to make use of the equivalent relation (3.45), which implies that the singular part of the combination $w^{\prime}(\zeta) \bar{w}(1 / \zeta)$ within the unit disc is given by

$$
\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]_{\operatorname{sing}}=H_{e}^{\prime}(\zeta)
$$

We have $H_{e}(\zeta)$ from the solution of (3.55), and the relation (3.54). The above equation then tells us the singularities of $\bar{w}(1 / \zeta)$ within the unit disc, since the mapping function itself is analytic there.

A few remarks are in order before we move on. Firstly, if we do use the method outlined above to find the form of the mapping function for a particular geometry, we must remember that we are working with a scaled time variable, and so instead of using the governing equations in the form (3.19) and (3.15) to determine how the parameters of the map evolve in time, we must first rescale time and use them in the form

$$
\begin{equation*}
\frac{w^{\prime}(0)}{A(\tau)} \mathcal{X}^{\prime}(\zeta)+\frac{\partial}{\partial \tau}\left(w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right)+\frac{\partial}{\partial \zeta}\left[\left(1-\zeta^{2}\right) w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]=0 \tag{3.60}
\end{equation*}
$$

and

$$
\begin{equation*}
\Phi(\zeta, \tau)=\frac{A(\tau)}{w^{\prime}(0)}\left\{\frac{\partial w}{\partial \tau}(\zeta)+w^{\prime}(\zeta)\left(1-\zeta^{2}\right)\right\} \tag{3.61}
\end{equation*}
$$

Secondly, we have so far in this section ignored the simpler case in which $\Phi(0)=A \equiv 0$. In this case, the p.d.e. satisfied by $H_{e}(\zeta)$ (now in terms of the original time variable, $t$ ) is just

$$
\begin{equation*}
-2 \mathcal{X}_{\mathrm{sing}}(\zeta)=\frac{\partial H_{e}}{\partial t} \tag{3.62}
\end{equation*}
$$

For the case of a dipole singularity at the origin, this has solution

$$
\begin{equation*}
H_{e}(\zeta, t)=H_{e}(\zeta, 0)-\frac{2 M \theta(t)}{\zeta} \tag{3.63}
\end{equation*}
$$

where $\theta(t)$ is defined by

$$
\theta(t):=\int_{0}^{t} \frac{d t^{\prime}}{w^{\prime}\left(0, t^{\prime}\right)}
$$

With a single point sink driving the flow the equation is even simpler since the left-hand side is fully known; the solution in this case is

$$
\begin{equation*}
H_{e}(\zeta, t)=H_{e}(\zeta, 0)-\frac{Q t}{\pi} \log \zeta \tag{3.64}
\end{equation*}
$$

Explicit solutions for $H_{e}(\zeta, t)$ are much easier to deal with now, hence working out the form of $w(\zeta)$ needed for a particular geometry is a simpler task.

The example we consider at length in $\S 5.4$ is the same for both cases $A \neq 0, A=0$; it is the problem of a vortex dipole placed off-centre in an initially circular fluid domain (this is also the example we give in $\S 5.3$ illustrating the analogous procedure for the Hele-Shaw problem, and was motivated by a very similar Hele-Shaw problem solved by Richardson [79]). We present it because it highlights a problem which can arise with Stokes flow solutions, which does not occur with Hele-Shaw problems: we can obtain solutions to the mathematical problem with relative
ease, but they are not always physically realistic. Thus, in our example, we find that the solution for the case $A=0$ is unlikely on physical grounds (with the large-domain limit having the dipole singularity moving relative to the fluid mass in some specified way), which is what leads us to consider the more complicated case.

Such behaviour means that we should, in general, treat Stokes flow solutions with circumspection. In §5.4.2, we "solve" a finite domain problem, and then take the large domain limit. It is only when we do this that the unphysical nature of the solution becomes apparent; for the finite domain case it is not obvious that there is a problem. We might call such solutions "formal", since they are certainly solutions to the mathematical problem, but are unlikely to be observed in practice.

### 3.8 The "moments" for the case $\Phi(0) \neq 0$

We now consider the evolution equations satisfied by the (ZST) Stokes flow "moments" in the case that $\Phi(0)=A$ is nonzero (but bounded). Clearly, they will no longer be conserved in this case. The governing equations are then (3.47) and (3.48); for simplicity, we assume the flow to be driven by a single point sink at the origin. Following the procedure of $\S 3.6$, we multiply equation (3.47) through by $\zeta^{k}$ and integrate around the unit disc. Using integration by parts on the extra term containing the factor $A$ (which in general will depend on time, since $A(t):=\Phi(0, t)$ ), and with the $C_{k}(t)$ defined by (3.31), it is readily seen that the equations satisfied are

$$
\frac{d C_{k}}{d t}= \begin{cases}-Q & k=0  \tag{3.65}\\ \frac{2 A k}{w^{\prime}(0)}\left(C_{k-1}-C_{k+1}\right) & k=1,2, \ldots\end{cases}
$$

so in general we have a system of coupled differential equations to solve for the $C_{k}$. An immediate consequence of these equations is that polynomial solutions no longer exist. If $w(\zeta)$ is a polynomial of degree $N$, then recalling (3.36) we have $C_{k}(t) \equiv 0$ for $k \geq N$. It then follows from the $k=N$ equation of (3.65) that $C_{N-1} \equiv 0$, and working back through the system in this way we see that all the $C_{k}$ will have to vanish identically, and there can be no such solution. Hence, if there exist solutions to this problem, they must be for non-terminating power series mapping functions.

The system (3.65) is simpler when written in terms of the time variable $\tau$ introduced in (3.51) of $\S 3.7$, since we then have

$$
\frac{d C_{k}}{d \tau}= \begin{cases}-\frac{Q w^{\prime}(0)}{2 A} & k=0,  \tag{3.66}\\ k\left(C_{k-1}-C_{k+1}\right) & k=1,2, \ldots\end{cases}
$$

The function $\mathcal{F}(\zeta, \tau)$ introduced (and solved for) in (3.53) of $\S 3.7$ is a kind of generating function for the ( $k \geq 1$ ) "moments", and we could equally well have derived equation (3.56) by multiplying the equations (3.66) (for $k \geq 1$ ) through by $\zeta^{-(k+1)} / k$ and summing them. To recover the quantities $C_{k}(\tau)$, we need the derivatives (evaluated at $\zeta=0$ ) of the "usual" generating function, defined by

$$
\begin{equation*}
\mathcal{F}_{1}(\zeta, \tau)=\sum_{1}^{\infty} \frac{C_{k}}{k} \zeta^{k} \equiv \mathcal{F}\left(\frac{1}{\zeta}, \tau\right) \tag{3.67}
\end{equation*}
$$

The equation satisfied by $\mathcal{F}_{1}$ follows from (3.56) as

$$
\frac{\partial \mathcal{F}_{1}}{\partial \tau}+\left(1-\zeta^{2}\right) \frac{\partial \mathcal{F}_{1}}{\partial \zeta}=C_{0} \zeta+C_{1}
$$

so that the solution is immediate from (3.58) as

$$
\mathcal{F}_{1}(\zeta, \tau)=\int_{0}^{\tau} C_{0}\left(\tau^{\prime}\right) \tanh \left(\tau^{\prime}-\tau+\tanh ^{-1} \zeta\right) d \tau^{\prime}+f_{1}\left(\tanh ^{-1} \zeta-\tau\right)
$$

$$
\begin{equation*}
+\int_{0}^{\tau} C_{1}\left(\tau^{\prime}\right) d \tau^{\prime} \tag{3.68}
\end{equation*}
$$

for some function $f_{1}$ which depends on the initial conditions. Again, the solution for $\mathcal{F}_{1}$ when we have a dipole singularity at the origin (instead of the point sink) is obtained from this by replacing $C_{0}\left(\tau^{\prime}\right)$ by $\left(C_{0}+M \pi / A\left(\tau^{\prime}\right)\right)$ in (3.68). Equating the constant term on the right-hand side to zero gives a "consistency condition" which must be satisfied, namely

$$
\begin{equation*}
0=\int_{0}^{\tau} C_{0}\left(\tau^{\prime}\right) \tanh \left(\tau^{\prime}-\tau\right) d \tau^{\prime}+f_{1}(-\tau)+\int_{0}^{\tau} C_{1}\left(\tau^{\prime}\right) d \tau^{\prime} \tag{3.69}
\end{equation*}
$$

equating the coefficients of $\zeta$ gives

$$
\begin{equation*}
C_{1}(\tau)=\int_{0}^{\tau} C_{0}\left(\tau^{\prime}\right) \operatorname{sech}^{2}\left(\tau^{\prime}-\tau\right) d \tau^{\prime}+f_{1}^{\prime}(-\tau) \tag{3.70}
\end{equation*}
$$

which is clearly equivalent to (3.69). This equation, and the $k=0$ equation of (3.66), provide constraints on $C_{0}(\tau), C_{1}(\tau)$, and $A(\tau)$.

### 3.9 The stress function

It has recently been noted by King [58] that it is possible to define a kind of 'Baiocchi transform' for the Stokes flow problem (recall the analogous transformation for the Hele-Shaw problem, defined in $\S 2.6$ ). Here, the dependent variable we transform is the Airy stress function, which we now introduce. Definitions vary slightly in the literature, but we define the stress function, $\mathcal{A}$, to be a biharmonic conjugate of the streamfunction $\psi$, so that in the Goursat representation we have

$$
\begin{equation*}
\mathcal{A}+i \psi=-[\bar{z} \phi(z)+\chi(z)] . \tag{3.71}
\end{equation*}
$$

Alternatively, in terms of the stress tensor $\left(\sigma_{i j}\right)$ the equations satisfied by $\mathcal{A}$ are

$$
\begin{aligned}
\sigma_{11} & =-p+2 \mu u_{x}=-2 \mu \mathcal{A}_{y y} \\
\sigma_{12} & =\sigma_{21}=\mu\left(u_{y}+v_{x}\right)=2 \mu \mathcal{A}_{x y} \\
\sigma_{22} & =-p+2 \mu v_{y}=-2 \mu \mathcal{A}_{x x}
\end{aligned}
$$

which shows that $p=\mu \nabla^{2} \mathcal{A}$. The pressure is harmonic for Stokes flow, so it follows that $\mathcal{A}$ must be biharmonic. It is straightforward to work from the stress tensor definition to the form in (3.71). To find the boundary conditions satisfied by $\mathcal{A}$ we use the Goursat form. Firstly, the chain rule gives

$$
\frac{\partial \mathcal{A}}{\partial s}=\frac{\partial \mathcal{A}}{\partial z} \frac{\partial z}{\partial s}+\frac{\partial \mathcal{A}}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial s}
$$

from which we see, using (3.71), that on $\partial \Omega$,

$$
\begin{aligned}
-2 \frac{\partial \mathcal{A}}{\partial s} & =\frac{d z}{d s} \overline{\left(z \overline{\phi^{\prime}(z)}+\overline{\chi^{\prime}(z)}+\phi(z)\right)}+\overline{\frac{d z}{d s}}\left(z \overline{\phi^{\prime}(z)}+\overline{\chi^{\prime}(z)}+\phi(z)\right) \\
& =\frac{d z}{d s} \overline{\left(\frac{i T}{2 \mu} \frac{d z}{d s}\right)}+\overline{\frac{d z}{d s}}\left(\frac{i T}{2 \mu} \frac{d z}{d s}\right) \quad \text { using (3.4) } \\
& =0
\end{aligned}
$$

Hence we may integrate along $\partial \Omega$ to deduce that

$$
\left.\mathcal{A}\right|_{\partial \Omega}=0
$$

without loss of generality. We may evaluate $\partial \mathcal{A} / \partial n$ on $\partial \Omega$ similarly, noting that, since $d z / d s$ is the complex tangent, and $d z / d n$ is the complex normal to $\partial \Omega$, we will have $d z / d n=-i d z / d s$ and $d \bar{z} / d n=i(d \bar{z} / d s)$. Then:

$$
\begin{aligned}
-2 \frac{\partial \mathcal{A}}{\partial n} & =-i \frac{d z}{d s} \overline{\left(\frac{i T}{2 \mu} \frac{d z}{d s}\right)}+i \frac{\overline{d z}}{d s}\left(\frac{i T}{2 \mu} \frac{d z}{d s}\right) \\
& =-\frac{T}{2 \mu}-\frac{T}{2 \mu} \quad \text { since }|d z / d s|=1
\end{aligned}
$$

Hence within $\Omega(t)$ and away from singularities, $\mathcal{A}$ satisfies the problem

$$
\nabla^{4} \mathcal{A}=0
$$

with boundary conditions

$$
\mathcal{A}=0, \quad \frac{\partial \mathcal{A}}{\partial n}=\frac{T}{2 \mu} \quad \text { on } \partial \Omega(t)
$$

The KBC (1.9) will also hold; however this has no "nice" interpretation in terms of the Airy stress function.

### 3.9.1 The "Baiocchi transform" for Stokes flow

We are now ready to define the "Baiocchi transform" for the Stokes flow problem. For simplicity, we restrict ourselves to the ZST problem, so that both boundary conditions for the stress function are homogeneous. For the Hele-Shaw problem (see $\S 2.6$ ) we were able to appeal to the maximum principle for harmonic functions to deduce that the free boundary behaviour is monotone, and hence write the free boundary in the form $t=\sigma(\mathbf{x})$; however no such maximum principle exists for the biharmonic equation, so we cannot do this here. We shall see that the desired transform variable is just the time integral of the stress function, but we find it necessary to use a complex variable approach, in some sense the reverse of the direct method adopted for the Hele-Shaw problem.

Define the variable $u$ by the formula

$$
\begin{equation*}
u=\frac{1}{4}(z \bar{z}-h(z)-\bar{h}(\bar{z})), \tag{3.72}
\end{equation*}
$$

where, as in $\S 3.7, h(z)$ is a primitive of the Schwarz function $g(z)$, so that

$$
g(z)=h^{\prime}(z)
$$

note that equation (3.72) is entirely equivalent to (2.34) derived for the Hele-Shaw problem in $\S 2.6$. Since $g$ is analytic (except at isolated singularities), it is immediate that $u$ satisfies the Poisson equation,

$$
\nabla^{2} u=1 \quad \text { in } \Omega(t)
$$

To find the boundary conditions satisfied by $u$, note that since $\partial \Omega$ is defined by the relation $\bar{z}=g(z)$,

$$
\begin{aligned}
\frac{\partial u}{\partial z} & =\frac{1}{4}\left(\bar{z}-h^{\prime}(z)\right)=0 \quad \text { on } \partial \Omega(t) \\
\frac{\partial u}{\partial \bar{z}} & =\frac{1}{4}\left(z-\overline{h^{\prime}(z)}\right)=0 \quad \text { on } \partial \Omega(t) \\
& \Rightarrow \quad \nabla u=0 \quad \text { on } \partial \Omega(t)
\end{aligned}
$$

Hence $u=0$ on $\partial \Omega$, if we choose the constant in $h$ appropriately, and $u$ satisfies the problem

$$
\begin{array}{rlr}
\nabla^{2} u & =1 & \text { in } \Omega(t) \\
u=0 & =\frac{\partial u}{\partial n} & \text { on } \partial \Omega(t)
\end{array}
$$

Note that the derivation so far is entirely independent of the dynamics of the problem; we could, if we wished, define the Hele-Shaw (or indeed any two-dimensional free boundary problem) Baiocchi transform from the starting-point (3.72).

We now assume slow viscous flow, and demonstrate the relationship with the stress function in the special case of zero surface tension, with the assumption $\Phi(0)=0$. Recalling the relation (2.12) between the Schwarz function and the mapping function, we see that

$$
\begin{aligned}
\frac{\partial u}{\partial \zeta} & =\frac{w^{\prime}(\zeta)}{4}(\overline{w(\zeta)}-\bar{w}(1 / \zeta)) \\
\Rightarrow \frac{\partial^{2} u}{\partial t \partial \zeta} & =\frac{1}{4} \frac{\partial}{\partial t}\left(w^{\prime}(\zeta) \overline{w(\zeta)}\right)-\frac{1}{4} \frac{\partial}{\partial t}\left(w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right) \\
& =\frac{1}{4} \frac{\partial}{\partial t}\left(w^{\prime}(\zeta) \overline{w(\zeta)}\right)+\frac{1}{2} \mathcal{X}^{\prime}(\zeta)
\end{aligned}
$$

using (3.29) in the last step. Next, transferring (3.71) to the $\zeta$-plane gives

$$
-2 \mathcal{A}=\overline{w(\zeta)} \Phi(\zeta)+\mathcal{X}(\zeta)+w(\zeta) \overline{\Phi(\zeta)}+\overline{\mathcal{X}(\zeta)}
$$

differentiating this with respect to $\zeta$ and using (3.30) then yields

$$
\begin{aligned}
-4 \frac{\partial \mathcal{A}}{\partial \zeta} & =\overline{w(\zeta)} w_{t}^{\prime}(\zeta)+2 \mathcal{X}^{\prime}(\zeta)+w^{\prime}(\zeta) \overline{w_{t}(\zeta)} \\
& \equiv 4 \frac{\partial^{2} u}{\partial t \partial \zeta}
\end{aligned}
$$

Integrating with respect to $\zeta$ then,

$$
\mathcal{A}=-\frac{\partial u}{\partial t}+\lambda(t)
$$

for some function of time, $\lambda$. But we know that both $\mathcal{A}$ and $u$ vanish on $\partial \Omega$, and therefore also on $|\zeta|=1$, so that $\lambda(t)$ must in fact be zero. Hence finally,

$$
\begin{equation*}
u=-\int^{t} \mathcal{A}(\zeta, \tau) d \tau \tag{3.73}
\end{equation*}
$$

(c.f. (2.30)) and we see that $u$ is a "Baiocchi transform" of the stress function, in the $\zeta$-plane.

### 3.10 Summary

This chapter is rather long, and contains many different ideas, which it is helpful to summarise before moving on to new things. We began in $\S 3.1$ by reviewing the work of [82], deriving the equations, holding in the $\zeta$-plane, which govern slow viscous flow (either surface tension driven, or singularity driven). In this and subsequent sections we extended the work of [82], reducing the problem to a single functional evolution equation which holds globally in the $\zeta$-plane.

For the point-sink driven ZST problem (when the Goursat function $\phi$ is assumed to vanish at the origin) an infinite set of conserved quantities of the motion was found in $\S 3.6$, which are analogous to Richardson's moments for the Hele-Shaw problem ( $\S 2.5$ ). The principal difference between the Hele-Shaw moments and the Stokes "moments" is that the former are defined by integrals over (or around) the fluid domain itself, while the latter are best defined in terms of
integrals over (or around) the unit disc, which is (so to speak) the fluid domain in the $\zeta$-plane. Consequently, although the Hele-Shaw moments have a clear physical interpretation, this is not so for the Stokes moments (with the exception of the mass conservation result). Underlying these results is the fact that the internal singularities of the Schwarz function remain fixed within the fluid domain for Hele-Shaw flow, while for Stokes flow (with the assumption $\phi(0)=0$ ) they remain fixed within the unit disc. In fact, Stokes flow is almost always best dealt with in the $\zeta$-plane, while it is often the case that Hele-Shaw flow is more tractable working within the physical plane.

A consequence of this is that, while we are easily able to generalise the Hele-Shaw results to more than one fixed driving singularity, this is not so for Stokes flow (§3.6.3) -in Hele-Shaw, the preimages of the singularities can move around in the $\zeta$-plane so long as they remain fixed in the $z$-plane; in Stokes flow, with $\phi(0)=0$, they must remain fixed in both the $z$-plane and the $\zeta$-plane. The moments were also solved for in the case $\phi(0) \neq 0(\S 3.8)$, which is a situation we consider further in chapter 5.

In $\S 3.7$ we considered the Schwarz function for the ZST Stokes flow problem in some detail, and saw how it is related to the Stokes flow moments. A p.d.e. governing the evolution of its singularities was formulated, and a method outlined for deducing the form of the mapping function required for a given problem.

We concluded in $\S \S 3.9$ and 3.9 .1 with another result (due to King [58]) which has a HeleShaw analogue; we defined a "Baiocchi transform" $u$ for the Stokes flow problem. The original definition (3.72) was in terms of the Schwarz function, but we then established that $u$ is in fact the time integral of the Airy stress function, evaluated in the $\zeta$-plane (3.73). We did not present any examples of the use of this transformed variable to solve problems; our interest is purely mathematical, in that it demonstrates yet another parallel between the Hele-Shaw and slow viscous flow problems.

## Chapter 4

## Applications to the glass industry

### 4.1 Introduction

In this short chapter, we digress to discuss an extension of the theory of chapter 3 to models of fibre drawing. We shall return to 'ordinary' Stokes flow in chapter 5; this chapter may be skipped without loss of continuity. The situation we have in mind is of one or more long viscous fibres, which are being stretched from either end, and possibly also twisted, such as may occur during optical fibre manufacture. In real-life problems we do not expect this stretching and/or twisting to dominate the motion, hence surface tension effects are important, and we include them.

The analysis of slender fibres under tension (and hence in extensional flow) relies on expansions in inverse powers of the large aspect ratio (the "slenderness parameter", $\epsilon$ ). (The régime of interest, in which surface tension is present at leading-order in the cross-flow problem, is when the Capillary number is of order $1 / \epsilon$. Surface tension is not important in the flow along the fibre.) Broadly speaking, for a thin fibre in extensional flow, the leading-order flow in any cross-section normal to the centre-line is two-dimensional Stokes flow, but with a non-zero fluid divergence due to the extensional component of the velocity. However, this may be dealt with by subtracting off an eigensolution, and the techniques of chapter 3 augmented to describe the new flow-field.

We now summarise the model and the complex variable formulation following Howell ([42], chapter 4 ). We then simplify the equations as in $\S 3.2$, and interpret them in terms of the "moments" introduced in $\S 3.6$. The results are essentially equivalent to those obtained for the strictly two-dimensional problem of chapter 3 , but with the convective derivative along the fibre, $\partial(\cdot) / \partial t+\partial(u(\cdot)) / \partial x$, replacing $\partial(\cdot) / \partial t$. A new solution, illustrating the theory, is given in $\S 4.3 .1$.

### 4.2 The theory for a viscous fibre

Consider the situation for a single viscous fibre, under tension along its length (so that it is nearly straight), and with velocity field $\mathbf{u}=(u, v, \varpi)$ within the fibre. Following [42] we change notation slightly, choosing axes such that the fibre is roughly aligned with the $x$-axis, and the fibre crosssection lies in the ( $y, z$ )-plane. The Reynolds number based on the flow along the fibre is assumed to be $O(1)$ so that the starting point is the full Navier-Stokes equations; however, the co-ordinates in the fibre cross-section are scaled with $\epsilon$, the small slenderness parameter. The components of velocity and the pressure are then expanded as power series in $\epsilon$, and the leading-order flow along the fibre is seen to be extensional, that is, $u_{0}=u_{0}(x, t)$ (the subscript " 0 " denoting leading-order). As mentioned above, the flow in the cross-section, $\left(v_{0}, \varpi_{0}\right)$, is not divergence-free, as the velocity component $u_{0}$ acts as a distributed mass sink (or source).

If we assume $u_{0}(x, t)$ to be known, then this "cross-flow" problem for $\left(v_{0}, \varpi_{0}\right)$ effectively decouples from the flow along the fibre, and an eigensolution of the leading order zero-surface tension problem can be found, which has exactly the right non-zero divergence. This "ZST eigensolution" can then be subtracted from the leading order cross-flow problem, and the problem
for the "residual" leading order cross-flow ( $\tilde{v}, \tilde{\varpi}$ ) (now divergence-free), and the residual pressure, may be considered. It must be remembered that the cross-section in which we are working is a function of both $x$ and $t$, which we denote by $\Omega(x, t)$. Henceforth we drop subscripts, on the understanding that we are considering only the leading-order problem for a finite fibre, with the full solution a power series in the slenderness parameter $\epsilon$. The length of the fibre is thus implicitly assumed to be infinite. (In a real problem, boundary conditions would be imposed at the ends of the fibre, specifying the "pulling" velocity $u$ there; hence here we expect to be able to specify the behaviour of $u$ as $x \rightarrow \pm \infty$.)
[42] uses an adaptation of the techniques of [82], presented in §3.1, working with the streamfunction $\psi$ for the tilded flow field,

$$
\tilde{v}=\frac{\partial \psi}{\partial z}, \quad \tilde{\varpi}=-\frac{\partial \psi}{\partial y}
$$

Using the familiar Goursat representation, with $\mathcal{Z}=y+i z$, we have

$$
\psi=-\Im\{\overline{\mathcal{Z}} \phi(\mathcal{Z})+\chi(\mathcal{Z})\}
$$

for functions $\phi$ and $\chi$ analytic within $\Omega(x, t)$. As usual we then map the unit disc onto $\Omega(x, t)$, after first eliminating rotation and translation of $\Omega(x, t)$, via

$$
\mathcal{Z}=\mathcal{Z}^{*}(x, t)+w(\zeta, x, t) e^{-i \alpha(x, t)}
$$

where $\mathcal{Z}^{*}(x, t)$ is the centreline of the viscous fibre, and $\alpha(x, t)$ represents the rotation. The usual normalisation assumption, $w(0, x, t)=0$ for all $x$ and $t$, can then be made, and since the functions $\phi$ and $\chi$ are regular throughout the fluid domain, the assumption of [82] that $\phi(0, x, t)=0$ can now be imposed without loss of generality. The analysis of [82] can then be followed through almost exactly as in $\S 3.1$, and analogues of equations (3.10) and (3.13) found. These are

$$
\begin{align*}
e^{i \alpha} \Phi(\zeta)+\frac{T}{2 \mu} f_{+}(\zeta) \zeta w^{\prime}(\zeta)= & -\left(\mathcal{Z}^{*}+w(\zeta) e^{-i \alpha}\right) \frac{\bar{\Phi}^{\prime}(1 / \zeta)}{\bar{w}^{\prime}(1 / \zeta)} \\
& -e^{i \alpha} \frac{\overline{\mathcal{X}}^{\prime}(1 / \zeta)}{\bar{w}^{\prime}(1 / \zeta)}+\frac{T}{2 \mu} f_{-}(\zeta) \zeta w^{\prime}(\zeta) \tag{4.1}
\end{align*}
$$

holding on $|\zeta|=1$ (and elsewhere, by analytic continuation), and

$$
\begin{align*}
\Re\left\{\frac{1}{\zeta w^{\prime}(\zeta)}\left[2 \Phi(\zeta) e^{i \alpha}-\left(w_{t}(\zeta)+u w_{x}(\zeta)+\frac{1}{2} u_{x} w(\zeta)\right)\right]\right. & \left.+\frac{T}{\mu} f_{+}(\zeta)\right\} \\
& =\frac{T}{2 \mu} f_{+}(0) \tag{4.2}
\end{align*}
$$

holding on $|\zeta|=1$. In these equations the dependence of the various functions on $x$ and $t$ has been dropped explicitly, but is understood. All notation is exactly as in $\S 3.1$ (in particular, the functions $f_{ \pm}(\zeta)$ satisfy (3.8) and (3.9)) and $u$ is the leading-order extensional flow along the fibre. Since we may assume $\Phi(0)=0$, we are able to continue (4.2) analytically by simply "removing the $\Re "$ from the left hand side, to get a global equation (which is identical to (3.14) if there is no $x$-dependence and if $\alpha=0$ ).

All analysis has so far been as in [42]. We may now, as in $\S 3.2$, depart from this approach, and derive a single functional evolution equation for the mapping function, which holds globally. If we first replace $\zeta$ in (4.1) by $1 / \zeta$ and take the complex conjugate, we can substitute in (4.1) for $\Phi(\zeta)$ from (4.2) to get an equation for $\mathcal{X}^{\prime}(\zeta)$ in terms of quantities depending only on the mapping function. When doing this, it is helpful to define the differential operators $E:=\partial_{t}+u \partial_{x}+u_{x} / 2$, and $D:=\partial_{t}+u \partial_{x}+u_{x}$, where the $u_{x}$ terms are understood to only multiply whatever function the operator is acting on (so $D$ is the usual convective derivative). The general result is rather complicated, being

$$
\begin{align*}
D\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right] & +e^{-i \alpha} \overline{\mathcal{Z}}^{*} \frac{\partial}{\partial \zeta}\left[E(w(\zeta))+\frac{T}{2 \mu} \zeta w^{\prime}(\zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right] \\
& +2 e^{-i \alpha} \mathcal{X}^{\prime}(\zeta)=\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right] \tag{4.3}
\end{align*}
$$

but is analogous to (3.18) if we consider the term in $\overline{\mathcal{Z}}^{*}$ (which is analytic on the unit disc, as well as being a perfect differential), and the term in $\mathcal{X}^{\prime}(\zeta)$, together, as a kind of "modified" $\mathcal{X}^{\prime}(\zeta)$.

In any case, there are obvious simplifications which can be made. For instance, if we assume that there is no lateral motion of the fibre, so that its centreline is exactly aligned with the $x$-axis and $\mathcal{Z}^{*}=0$, then the second term on the left-hand side is eliminated. If in addition there is no twist applied to the fibre $(\alpha=0)$, then we are only stretching it (via the term $u(x, t)$ ), and the evolution equation for the cross-section is exactly analogous to (3.18), with the operator $D$ replacing the $\partial(\cdot) / \partial t$,

$$
\begin{equation*}
D\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]+2 \mathcal{X}^{\prime}(\zeta)=\frac{T}{2 \mu} \frac{\partial}{\partial \zeta}\left[\zeta w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right] \tag{4.4}
\end{equation*}
$$

## 4.3 "Conserved quantities" for fibres

A point to note about equations (4.3) and (4.4) is that, if we were to consider the ZST versions, then the analysis of $\S 3.6$ would follow through to give the infinite system of conservation laws,

$$
\begin{array}{rlr}
D\left(C_{k}\right) & =0 & k=0,1,2, \ldots \\
\Rightarrow \quad \frac{\partial C_{k}}{\partial t}+\frac{\partial}{\partial x}\left(u C_{k}\right) & =0 & k=0,1,2, \ldots
\end{array}
$$

for quantities $C_{k}(x, t)$ defined exactly as in (3.31). This just says that these "moments" are convected with the flow along the fibre in this simple case, as we would expect, and is analogous to the usual two-dimensional result that ZST Stokes flow is completely trivial in the absence of driving singularities. For the NZST problem, we can write down the analogue of equations (3.34), which will hold here if we set $Q=0$ and replace $d(\cdot) / d t$ by $D$, that is,

$$
\frac{\partial C_{k}}{\partial t}+\frac{\partial}{\partial x}\left(u C_{k}\right)= \begin{cases}0 & (k=0) \\ -\frac{k T}{\mu}\left[\frac{f_{+}(0)}{2} C_{k}+\sum_{r=1}^{\infty} \frac{f_{+}^{(r)}(0)}{r!} C_{k+r}\right] & (k \geq 1)\end{cases}
$$

We emphasise that these equations still hold for the case in which we have twist, and/or lateral motion of the centreline, since the terms in equation (4.3) which represent these effects are regular on the unit disc, and so vanish upon integrating around the unit circle. The $k=0$ equation here immediately reveals the general mass conservation result, ${ }^{1}$ since $C_{0}(x, t)$ is exactly the crosssectional area of $\Omega(x, t)$.

### 4.3.1 Example-the sintering of a bundle of fibres

Howell [42] gives an example of his analysis, solving the problem of two identical fibres (of initially circular cross-section) sintering together under the action of surface tension as they are stretched out. This is done for the simplest case, $\mathcal{Z}^{*}=0=\alpha$, using the method described in [82]. If instead equation (4.3) is used, with the method outlined in $\S 3.3$ of this thesis (but with the simplification that now $\mathcal{X}(\zeta)$ will be regular on the whole unit disc, so that we only need match singularities in two terms of the equation), the p.d.e.'s governing the parameters of the mapping function are much more quickly obtained. The mapping function used to describe the cross-section is

$$
w(\zeta)=C\left(\frac{1}{1-b \zeta}-\frac{1}{1+b \zeta}\right)=\frac{2 C b \zeta}{1-b^{2} \zeta^{2}}
$$

[^17]

Figure 4.1: Typical cross-sections generated by the map (4.6) when $n=6$. Picture (a) is the cusped configuration, while (b) is the kind of smooth cross-section we might expect to observe in practice.
for $b$ and $C$ both functions of $x$ and $t$. The strictly two-dimensional version of this problem (and the problem of two unequal cylinders coalescing under surface tension) has been solved in [37] and [82], and [42] uses much of the analysis of the latter. Note that the problem is as yet underdetermined, though, since we have said nothing about $u(x, t)$, which appears in the equations governing the parameters of the map.

The "missing link" is an axial stress balance for the fibre, difficult to manipulate analytically, given in [42] as

$$
\begin{equation*}
\frac{\partial}{\partial x}\left(3 S \frac{\partial u}{\partial x}\right)=\operatorname{Re} S\left(\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}\right)-\frac{T}{2 \mu} \frac{\partial \Gamma}{\partial x} \tag{4.5}
\end{equation*}
$$

where $S$ is the cross-sectional area of $\Omega, \Gamma$ is the circumference of the cross-section (i.e. the length of $\partial \Omega$ ), and " $R e$ " is the Reynolds number based on the flow $u$ along the fibre.

An obvious extension of this work is to consider a mapping function of the form

$$
\begin{equation*}
w(\zeta)=\frac{n C b \zeta}{1-b^{n} \zeta^{n}} \tag{4.6}
\end{equation*}
$$

with $b \in(0,1)$ to ensure analyticity of the map, and $C>0$ without loss of generality. In fact, for the map to be univalent we require that

$$
0<b<b_{\text {crit }}=\frac{1}{(n-1)^{1 / n}}
$$

the limit $b \rightarrow 0, C \rightarrow \infty$ giving a circular cross-section, and the limit $b \rightarrow b_{\text {crit }}$ giving a "flowershaped" cross-section, having $n$ inward-pointing cusps (figure 4.1 (a)). Such a map might represent the later stages in the sintering of a bundle of viscous fibres as they are stretched out, provided we could be sure that the interior gaps between the fibres had closed up by this stage, so that the analysis for a simply-connected cross-section is applicable.

We restrict ourselves to working with equation (4.4), for simplicity. We have to match singularities here only at the two points $\zeta=0$ and $\zeta=b$ within the unit disc, appealing to the symmetry of the map. Matching at $\zeta=0$ simply yields the mass conservation result,

$$
\begin{equation*}
\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}(u S)=0 \tag{4.7}
\end{equation*}
$$

where a straightforward integration gives the cross-sectional area $S$ as

$$
S=\frac{\pi n^{2} b^{2} C^{2}\left(1+(n-1) b^{2 n}\right)}{\left(1-b^{2 n}\right)^{2}}
$$

Matching at $\zeta=b$ yields another p.d.e.,

$$
\begin{equation*}
\frac{\partial b}{\partial t}+u \frac{\partial b}{\partial x}=-\frac{T b}{2}\left(2 f_{+}(b)-f_{+}(0)\right) ; \tag{4.8}
\end{equation*}
$$

these two equations (4.7) and (4.8) are equivalent to those given in [42] for the case $n=2$.
The factor $\left(2 f_{+}(b)-f_{+}(0)\right)$ can be explicitly evaluated from the formula (3.8) in terms of elliptic integrals, with the result

$$
2 f_{+}(b)-f_{+}(0)=\frac{2\left(1-b^{2 n}\right)}{n \pi b C} K\left((n-1) b^{n}\right)
$$

where $K(\cdot)$ denotes the complete elliptic integral of the first kind (see appendix B, [8], or [30] for a definition). In deriving this simple form for the right-hand side, use was made of the relation [8]

$$
\begin{equation*}
\frac{1}{1+(n-1) b^{n}} K\left(\frac{2 b^{n / 2}(n-1)^{1 / 2}}{1+(n-1) b^{n}}\right)=K\left((n-1) b^{n}\right) . \tag{4.9}
\end{equation*}
$$

We also need to utilise the axial stress balance (4.5) to close the problem; however, the expression obtained for the circumference $\Gamma(x, t)$ is more intricate,

$$
\Gamma=\frac{4 b n C(n-1)}{1+(n-1) b^{n}}\left\{\frac{n\left(1+(n-1) b^{2 n}\right)}{(n-1)\left(1-b^{n}\right)^{2}} \Pi\left(\frac{-4 b^{n}}{\left(1-b^{n}\right)^{2}}, k\right)-K(k)\right\}
$$

where $k=2 b^{n / 2}(n-1)^{1 / 2} /\left(1+(n-1) b^{n}\right)$, and $\Pi(\cdot, \cdot)$ denotes the complete elliptic integral of the third kind (again, see appendix B, [8], or [30] for a definition). We could use (4.9) on the second term here, but further analytical progress with the first term is difficult, and numerics must be employed to complete the solution. We do not pursue this work further; however we should mention recent work by Richardson [85], which is concerned with the sintering of an almost arbitrary array of circular cylinders of viscous fluid. Many numerical solutions are presented, but in this work the evolution is solely surface tension driven, with no extensional axial velocity.

### 4.3.2 Connectedness considerations

We suggested that the problem considered in §4.3.1 might represent the later stages in the sintering of a bundle of fibres, with the proviso that the "holes" between the fibres, which would necessarily be present at the outset, must have closed up before this analysis (which relies on a simply connected cross-section) can be applicable. An obvious calculation is to check whether a circular hole in an unbounded, two dimensional flow domain, will indeed close up under the action of surface tension only.

The required conformal mapping from the unit disc onto the fluid domain is just

$$
w(\zeta)=\frac{a(t)}{\zeta}
$$

for real, positive $a(t)$. This is different to previous examples, in that the point $\zeta=0$ maps to infinity (rather than $z=0$ ). Rather than reformulating the theory of $\S \S 3.2$ and 3.3 to deal with this, it is easier to use an ad hoc method based on equation (3.10) and the boundary condition (3.13).

The function $f_{+}(\zeta)$ is easily seen to be equal to $1 / a$ everywhere (with $f_{-}(0) \equiv 0$ ). Equation (3.13) then becomes

$$
2 \Re(\zeta \Phi(\zeta))=\dot{a}+\frac{T}{2 \mu}, \quad \text { on }|\zeta|=1
$$

which is trivial to continue analytically, giving

$$
\Phi(\zeta)=\frac{1}{2 \zeta}\left(\dot{a}+\frac{T}{2 \mu}\right)
$$

Substitution into (3.10) gives

$$
\mathcal{X}^{\prime}(\zeta)=\frac{a}{\zeta}\left(\dot{a}+\frac{T}{2 \mu}\right)-\frac{a T}{2 \mu} .
$$

The final condition needed is that the velocity vanish at infinity (as $\zeta \rightarrow 0$ ); the most general conditions allowing this are given in chapter 5, by (5.30) and (5.31). Since the flow is solely surface-tension driven, the pressure at infinity $\left(p_{\infty}\right)$ must be zero in (5.30). Hence we require $\phi(z)$ to be bounded at infinity, that is, $\Phi(\zeta)$ must be bounded at the origin, giving the final result

$$
\begin{aligned}
\dot{a} & =-\frac{T}{2 \mu} \\
\Rightarrow a(t) & =a(0)-\frac{T t}{2 \mu} .
\end{aligned}
$$

The radius of the hole is exactly $a(t)$, so the hole will close in finite time.

### 4.4 Summary

In this chapter, we have seen how the ideas introduced in chapter 3 for the two-dimensional problem, may be extended to deal with the (three-dimensional) problem of viscous fibres undergoing traction (and torsion, although we did not elaborate on this point). This was a consequence of the slender geometry, which meant that asymptotic methods could be employed to make the problem effectively two-dimensional. We considered only fibres; however as was mentioned in $\S 1.3$, similar asymptotic methods may be used for slender bubbles; see for example [7], [43].

As in chapter 3, "moments" of the cross-flow may again be defined by (3.31). In the ZST case (which here is equivalent to the assumption that the effects of traction far outweigh those of surface tension), they are simply convected with the flow along the fibre. In the more realistic NZST case, their evolution is via a much more difficult system of nonlinear partial differential equations, analogous to (3.34).

Finally, in $\S 4.3 .1$, we gave a physically-relevant example of the theory, as applied to the sintering of a bundle of viscous fibres under traction.

## Chapter 5

## Flow in unbounded domains

### 5.1 Introduction

This chapter is concerned with Hele-Shaw flows and Stokes flows on unbounded ${ }^{1}$ fluid domains with a free boundary. The theory and techniques of previous chapters (which assumed a bounded fluid domain) will, on the whole, still carry through for such cases, but the conformal map from the unit disc must now have an isolated singularity within $|\zeta| \leq 1$, corresponding to the single point that maps to infinity.

This chapter crystallises why Stokes flow and Hele-Shaw flow are different. The key idea (which has already been mentioned in §3.6.2) involves the Schwarz function of the free boundary, $g(z, t)$. Driving singularities of the flow are associated with singularities of $g$ for both problems, and so we may regard the flow as being "driven by the Schwarz function". For Hele-Shaw flow, as we saw in $\S 2.3$, the singularities of $g$ must remain fixed within the physical plane, and so may be made to correspond to (fixed) driving singularities. For Stokes flow, they remain fixed in the $\zeta$-plane, at least when we make the technical assumption $\Phi(0)=0$, and so in general we have no hope of keeping the associated driving singularity fixed in the physical plane too.

Problems on unbounded domains are another example of this difficulty, if we try to solve in the time-dependent case for driving singularities at points other than infinity. This is because "infinity" is, by necessity, also a singular point of the flow, in both the fully-infinite and semiinfinite domain cases of such problems. Thus, explicit, unsteady solutions (with $\Phi(0)=0$ ) will, in general, be driven by moving singularities. Even when $\Phi(0) \neq 0$, isolated singularities of the Schwarz function move in a specified manner within the $\zeta$-plane (see (5.43)), which is very restrictive, and in $\S 5.4 .2$ we find that a continuous distribution of singularities is needed if we are to satisfy all the conditions.

No such technical difficulties arise for time-dependent problems with driving singularities at infinity [96], since this is the only singular point of the flow. Likewise, steady problems driven from the origin can be solved, since infinity is not a singular point of such flows. We consider some steady problems in $\S 5.5$.

### 5.2 Literature Review

Before saying more, we first consider the work which has been done on the two problems when the flow domain is unbounded, beginning with the Hele-Shaw case. For the Hele-Shaw problem very many results exist; as mentioned above, most results from the bounded domain case follow through straightforwardly, and there seem to be no surprises. For the case of a finite air bubble, concepts such as Richardson's moments ( $\$ 2.5$ ) can be easily redefined in terms of integrals over

[^18]

Figure 5.1: Schematic diagram showing how a "continuable 5/2-power cusp" solution looks in phase trajectory space within the univalency domain.
large circles containing the bubble (see for example [24]); the evolution equation for $M_{0}(t)$ then reduces to conservation of the bubble area.

The classical Saffman-Taylor fingering solutions [87] (and their time-dependent analogues [88]) are one obvious example of solutions on semi-infinite fluid domains, driven by a uniform pressure gradient at infinity. Solutions exhibiting fingering in a radial geometry have been found by Howison [45]; like Saffman \& Taylor's fingers, these exist for all time, but are driven by a sink at infinity. Further "bubble" solutions have been found by Tanveer [93] and Howison [47] (this latter paper considers the classification of bubble solutions according to the limiting form of the fluid domain). Entov et al. [24] consider bubbles in unbounded domains driven by multipole singularities at infinity. These last solutions are worth remarking on, if only for the fact that they demonstrate steady solutions to the NZST Hele-Shaw problem.

Howison [46] presents bubble solutions in which cusps form in the free boundary within finite time, but where the solution may be continued beyond this time. This extraordinary behaviour is only possible when solutions blow up via the formation of a $(4 n+1) / 2$-power cusp in the free boundary (see the comments of $\S 2.6$ ). In $\S 5.3$ we present a new example which exhibits this behaviour. For a general example of this kind, we may draw the solution trajectories for the parameters of the mapping function within the univalency domain $V$. The particular trajectory which passes through the point on $\partial V$ corresponding to the $5 / 2$-power cusp does so tangentially to the boundary $\partial V$, before re-entering $V$, so that geometrically, the free boundary becomes nonanalytic at a point for an instant, before smoothing again, and the solution continues to exist (see figure 5.1; also figure 5.7 for the local form of the free boundary as this occurs). All known solutions of this kind subsequently blow up via $3 / 2$-power cusp formation however, which is known to be always "terminal" (this may be proved from known results for the related obstacle problem of variational calculus; [90], [64]).

Another Hele-Shaw solution which is of interest, and of which we shall consider the Stokes flow analogue, concerns a rational mapping function,

$$
\begin{equation*}
w(\zeta)=\frac{\alpha \zeta(\zeta-\beta)}{\zeta-\gamma} \tag{5.1}
\end{equation*}
$$

this is considered by Hohlov et al. [33] for the case of real parameters $\alpha, \beta, \gamma$. For $\gamma \in(0,1)$ the map gives unbounded fluid domains; the case $\gamma=1(\beta \neq 1)$ gives a map to a semi-infinite
fluid domain. The authors solve for the problem with a single point sink at the origin driving the flow (the Stokes flow analogue we consider is driven by a dipole singularity at the origin). This situation differs from those mentioned above in that here, the driving singularity is at the origin, whereas the previous cases were driven by prescribed singularities at infinity. The mapping function (5.1) is interesting because the points $(\beta, \gamma)$ on the boundary of the univalency domain in ( $\beta, \gamma$ )-space correspond to fluid domains having slits in them (along arcs of circles).

Finally (for the Hele-Shaw problem) we mention the work of Richardson [79], where a limiting procedure is employed to solve for a problem on a semi-infinite fluid domain. The problem of a point sink, placed off-centre in an initially circular domain, is considered (the geometry of figure 5.2 but with a different driving singularity); the NZST version of this problem has recently been solved numerically in [55]. For an initial circle of radius $r$, centred at $z=\alpha<r$ (with the sink at $z=0$ ), the method outlined in $\S 2.5$ is used to first deduce the correct form of the mapping function, and then to solve the problem. The semi-infinite domain limit is obtained by allowing both $r$ and $\alpha$ to tend to infinity, whilst keeping the quantity $r-\alpha=k$ fixed. This process yields the solution for a point sink placed at the origin in the unbounded initial domain $\{x>-k\}$ (so here again we have a driving singularity at a finite point within the flow domain). In $\S 5.3$ we employ the same methods to solve the problem for a vortex dipole singularity at the origin, in the same geometry.

We now consider what results exist for the Stokes flow case. Most of the work which has been done on the unbounded domain problem involves steady solutions for finite bubbles in (fully) infinite fluid domains, and was mentioned briefly in §1.4.2. One of the first papers of note was Richardson [78] (1968), who solved the problem of a two-dimensional inviscid bubble in the cases of uniform shear, and pure straining, external flow. In a subsequent paper [80] he solved the same problem for a parabolic external velocity profile. In 1972, Buckmaster [7] published results for slender bubbles in three-dimensional axisymmetric slow viscous flow (at small surface tension), finding bubble shapes which appeared to have cusped ends; Antanovskii (though in two dimensions) has also considered the formation of steady-state cusped bubbles [3] and pointed drops [4]. Later work by Youngren \& Acrivos [101] (1976) gave agreement with Buckmaster's results for small values of surface tension, and with other experimental work for larger surface tension values. Slender three-dimensional bubbles have been studied more recently by Howell [42, 43], in extensional flow, and with a time-dependent formulation (the above mentioned work all being for steady flows).

Tanveer \& Vasconcelos [96] have recently published results on the time evolution of twodimensional bubbles, where the motion is driven by a given external flow field at infinity (which may include a source/sink at infinity, so that the bubble area can change). Particular cases considered are when the external flow is simple shear, and pure straining (as in Richardson [78], but time-dependent), and they find a family of exact solutions for a polynomial-type conformal map. Three-dimensional time-dependent axisymmetric bubbles have also been studied by Nie \& Tanveer [70]; in this paper and in [96] the possibility of "pinching" is considered, where, for a shrinking bubble, opposite sides of the bubble touch before the bubble has vanished, and the solution breaks down.

The work of [96] demonstrates that the ideas pioneered by Hopper [37, 38] and Richardson [82] for bounded fluid domains, carry through to the unbounded domain (with finite bubble) case with little modification needed. However, very little work has been published relating to problems on unbounded domains where the driving singularity is at some finite point within the flow domain, and, to our knowledge, all explicit solutions which have been found for such cases are for the much simpler, steady version. In fact, the only notable contribution to this problem of which we are aware is the work of Jeong \& Moffatt [52] (henceforth J \& M), who solve the steady problem for a vortex dipole placed beneath a free surface in a semi-infinite fluid domain. This is intended to model experiments performed in a large tank of fluid, with two counter-rotating cylinders placed beneath the free surface. Antanovskii [2] generalises their work in two ways: firstly, he allows a variable interfacial tension, to model the effect of surfactant, and secondly, he assumes each of the counter-rotating cylinders to be represented by a separate vortex singularity in the flow (although an exact analytical solution is only found in the limit in which these two vortices merge to form


Figure 5.2: The geometry for the problem of a dipole placed off-centre in a circle.
a single vortex dipole, as considered by J \& M).
Our idea was to present a time-dependent version of the work of J \& M (which would hopefully tend to their solution as $t \rightarrow \infty$ ), since no solutions to problems of this kind exist in the current literature. As we shall see though, this is far from trivial, involving complications of the kind hinted at in $\S 5.1$. Before considering such a generalisation, we will briefly review the analysis of J \& M, but before we do this we solve the corresponding ZST Hele-Shaw problem, which turns out to be much more straightforward than the Stokes flow problem.

### 5.3 The Hele-Shaw dipole problem

In this section we show that the ZST Hele-Shaw version of the J \& M dipole problem can be solved without difficulty. It is worth doing this for two reasons; firstly, it represents an interesting new solution for Hele-Shaw, being one which exhibits the "transient $5 / 2$-power cusp" behaviour referred to earlier, and furthermore, it has the driving singularity at a finite point within the fluid domain (as in the Stokes flow literature, most infinite-domain Hele-Shaw solutions have driving singularities at infinity). Secondly, it illustrates the use of the procedure outlined in §2.5, for deducing the form of the mapping function that is needed for a particular geometry.

The analysis is a simple adaptation of that given in Richardson [79], where the same geometry is assumed (i.e. an initially circular domain, with an off-centre singularity; see figure 5.2) but instead a point sink drives the flow. The dipole singularity in our example forces a more complicated mapping function than in that paper; hence the solution we obtain has a different structure-all solutions in [79] blow up in finite time via formation of a single $3 / 2$-power cusp.

Recall the result (2.10) linking the time evolution of the Schwarz function, $g(z, t)$, to the complex potential, $\mathcal{W}(z)$, of the flow. For the case of a vortex dipole singularity of strength $M$ at the origin, in the sense of figures 5.2 and 5.3 , the only singularity of $\mathcal{W}(z)$ is at $z=0$, this being

$$
\mathcal{W}(z)=-\frac{M}{z}+O(1) \quad \text { as } z \rightarrow 0
$$

It follows from (2.10) that, near the origin, the Schwarz function varies according to

$$
\begin{equation*}
\frac{\partial g}{\partial t}=2 \frac{d \mathcal{W}}{d z}=\frac{2 M}{z^{2}}+O(1) \quad \text { as } z \rightarrow 0 \tag{5.2}
\end{equation*}
$$

so that, decomposing the Schwarz function according to (2.25), the singular part must satisfy

$$
g_{e}(z, t)=g_{e}(z, 0)+\frac{2 M t}{z^{2}} .
$$

The Schwarz function for a circular initial domain, with centre at $z=\alpha$ and radius $r>\alpha$, is given in [79] (and in any case is trivial to find) as

$$
\begin{equation*}
g(z, 0)=\alpha+\frac{r^{2}}{z-\alpha} \tag{5.3}
\end{equation*}
$$

hence for $t>0$ the singular part of the Schwarz function is given explicitly by

$$
\begin{equation*}
g_{e}(z, t)=\frac{r^{2}}{z-\alpha}+\frac{2 M t}{z^{2}} \tag{5.4}
\end{equation*}
$$

Let the point $d(t)$ within $\{|\zeta| \leq 1\}$ map to the point $\alpha$ in the physical domain, $w(d)=\alpha$ (we know that the origin maps to the origin, and that the mapping function is analytic on the unit disc). The relation (2.27) then tells us that the complex conjugate mapping function $\bar{w}(1 / \zeta)$ has to have a double pole at $\zeta=0$, a simple pole at $\zeta=d$, and no other singularities. Since $w(0)=0$, $\bar{w}(1 / \zeta)$ must also vanish at infinity, and hence must be of the form

$$
\begin{equation*}
\bar{w}(1 / \zeta)=\frac{a}{\zeta^{2}}+\frac{B}{\zeta}+\frac{C}{\zeta-d}, \tag{5.5}
\end{equation*}
$$

so that assuming $a, B, C, d \in \mathbb{R}$, which amounts to assuming symmetry about the $x$-axis, we have

$$
\begin{equation*}
w(\zeta)=a \zeta^{2}+B \zeta+\frac{C \zeta}{1-d \zeta} \tag{5.6}
\end{equation*}
$$

To determine the parameters $a, B, C$ and $d$ in (5.6), we need to match singularities within the unit disc in equation (2.12). This requires a straightforward local analysis at each of the singularities, and yields the three algebraic equations,

$$
\begin{align*}
a(B+C)^{2} & =2 M t,  \tag{5.7}\\
\frac{B(B+C)^{3}}{a+C d} & =-4 M t  \tag{5.8}\\
C w^{\prime}(d)=C\left(2 a d+B+\frac{C}{\left(1-d^{2}\right)^{2}}\right) & =r^{2} . \tag{5.9}
\end{align*}
$$

The fourth equation needed comes from the requirement $w(d)=\alpha$ (which embodies the fact that the singularities of the Schwarz function remain fixed in the physical plane for Hele-Shaw flow),

$$
\begin{equation*}
a d^{2}+B d+\frac{C d}{1-d^{2}}=\alpha \tag{5.10}
\end{equation*}
$$

These equations are seen to satisfy automatically the correct initial conditions for the geometry $\Omega(0)$, namely

$$
a(0)=0=B(0), \quad C(0)=r\left(1-\frac{\alpha^{2}}{r^{2}}\right), \quad d(0)=\frac{\alpha}{r} .
$$

Note that the system is simpler if we replace (5.8) by $((5.8)+2 \times(5.7))$, i.e.

$$
\begin{equation*}
2 a(a+C d)+B(B+C)=0 \tag{5.11}
\end{equation*}
$$



Figure 5.3: The geometry for the dipole-in-a-half-space problem.

Solving for $a, B, C$ and $d$ gives the evolution of the fluid domain until such time as the solution breaks down, with loss of univalency of the mapping function. We do not solve (5.7)-(5.10) explicitly, however, because we are primarily interested in the infinite domain limit (figure 5.3). In addition, the system (5.7)-(5.10) is specific to an initially-circular geometry (or an initiallyflat geometry in the unbounded domain limit), and it is clear that with the mapping function (5.6), we may consider the evolution of more general initial domains than this. Since we are now proposing a definite form, (5.6), for $w(\zeta, t)$, the governing equations for the more general case are best obtained using the method outlined below equation (2.12) in $\S 2.3$ (and illustrated in $\S 2.4$ ).

We know that (5.2) represents the only singular behaviour of $\partial g / \partial t$ in $\Omega(t)$. We also know that $g(z) \equiv \bar{w}(1 / \zeta)$ (equation (2.12)), which by (5.5) has singularities at points $\zeta=0$ and $\zeta=d$ within the unit disc. Hence, expanding the mapping function to find $\zeta$ as a function of $z$ in the neighbourhood of these points, we may find the local form of $g(z)$ at both points $z=0, z=w(d)$, and use (5.2) to get the o.d.e.'s governing the coefficients. Near $z=0=\zeta$ we have

$$
\zeta=\frac{z}{w^{\prime}(0)}-\frac{w^{\prime \prime}(0) z^{2}}{2 w^{\prime}(0)^{3}}+O\left(z^{3}\right)
$$

so that a local analysis here yields

$$
g(z)=\frac{a w^{\prime}(0)^{2}}{z^{2}}+\frac{a w^{\prime \prime}(0)+B w^{\prime}(0)}{z}+O(1) .
$$

Hence, matching singularities in (5.2),

$$
\begin{align*}
\frac{d}{d t}\left(a w^{\prime}(0)^{2}\right) & =2 M  \tag{5.12}\\
a w^{\prime \prime}(0)+B w^{\prime}(0) & =\text { const. }=k_{1} \tag{5.13}
\end{align*}
$$

Near $\zeta=d$,

$$
\zeta-d=\frac{z-w(d)}{w^{\prime}(d)}+O\left((z-w(d))^{2}\right)
$$

so by (2.12) and (5.5),

$$
g(z)=\frac{C w^{\prime}(d)}{z-w(d)}+O(1)
$$

and (5.2) gives the two equations

$$
\begin{align*}
w(d) & =\text { const. }=\alpha  \tag{5.14}\\
C w^{\prime}(d) & =\text { const. }=r^{2} \tag{5.15}
\end{align*}
$$

Equations (5.7), (5.11), (5.10) and (5.9) are easily seen to be a special case of equations (5.12)(5.15). ${ }^{2}$ The constants in (5.14) and (5.15) are arbitrary; we write them as $\alpha$ and $r^{2}$ to conform with the special case considered first. We are justified in assuming that the constant in (5.15) is positive, since we are assuming that the positive real axis within the unit disc maps onto the positive real axis in the fluid domain.

Taking the infinite domain limit of the problem involves letting $r$ and $\alpha$ tend to infinity whilst keeping $r-\alpha=k$ fixed. Equations (5.12) and (5.13) have trivial limits,

$$
\begin{align*}
a(B+C)^{2} & =2 M t+\text { const. }  \tag{5.16}\\
2 a(a+C)+B(B+C) & =k_{1}, \tag{5.17}
\end{align*}
$$

whilst a careful analysis of equations (5.14) and (5.15) shows that they reduce to the single condition

$$
\begin{equation*}
C-2(a+B)=2 k . \tag{5.18}
\end{equation*}
$$

Geometrically this last equation is what we expect; it says that asymptotically, the free boundary is $|y| \rightarrow \pm \infty, x \sim-k$. It is easily verified that the solution represented by equations (5.16)-(5.18) gives a velocity field which tends to zero at infinity, which we clearly require for a realistic solution.

To see by what possible means this more general solution may blow up, we must consider the subset of $(a, B, C)$-parameter space on which the map with $d=1$ is univalent. For this purpose it is simplest to rewrite the map as

$$
\begin{equation*}
w(\zeta)=a\left(\zeta^{2}+b \zeta+\frac{c \zeta}{1-\zeta}\right) \tag{5.19}
\end{equation*}
$$

so that in fact we only need consider univalency of the map in $(b, c)$-parameter space. The only drawback of writing the map this way is that the limit $a \rightarrow 0$ is degenerate; maps which pass through or start from such configurations must have $c \rightarrow \pm \infty$ ( $b$ can either become unbounded or remain $\mathrm{O}(1))$-the "initially flat" configuration is one such case, as are any solutions which pass through flat configurations.

We may determine the univalency domain $V$ in $(b, c)$-parameter space by looking for boundary curves on which $w^{\prime}\left(e^{i \theta}\right)=0$ for real $\theta$ (the condition that the free boundary has a cusp), and also curves on which $w\left(e^{i \theta}\right)=w\left(e^{-i \theta}\right)$ for real, nonzero $\theta$ (the condition that the free boundary is self-intersecting, using the symmetry of the domain). We omit the analysis, which is not difficult, but rather tedious. The result is that $V$ is split into two parts, in $c>0$ and $c<0$ (see figure 5.4; the line $c=0$ is not relevant to the present discussion, since it represents bounded fluid domains, and our solution trajectories cannot reach it). That part of $V$ in $c>0$ is bounded by the lines

$$
4 b+c=8 \quad \text { and } \quad c=0
$$

whilst the part in $c<0$ is bounded by

$$
b+c+4=0, \quad c=0
$$

and the parabola

$$
c+\left(\frac{b}{2}+1\right)^{2}=0
$$

[^19]

Figure 5.4: The univalency domain in $(b, c)$-space for the mapping function (5.19).


Figure 5.5: Typical free boundaries generated by points $(b, c)$ on the boundary of the univalency domain (figure $5.4)$. The dipole is situated at the origin in each case, and is such that the $x$-axis is a streamline in the positive sense. (1) has $a=1, b=1, c=4$, and has a single cusp in the free boundary; (2) has $a=-1, b=1, c=-5$, and has two cusps in the free boundary, and (3a) has $a=-1, b=4, c=-9$, and shows the free boundary beginning to overlap itself. (3b) is an enlargement of the trapped air bubble in (3a).
which meets the line $b+c+4=0$ at the point $(2 \sqrt{3},-(4+2 \sqrt{3}))$.
With our "normalisation condition" that the positive real axis in the $\zeta$-plane should map to the positive real axis in the $z$-plane, we require $a(0)>0$ for solutions in $c>0$, and $a(0)<0$ for solutions in $c<0$. Geometrically, in the upper part of $V$, the line $4 b+c=8$ corresponds to loss of univalency via formation of a single $3 / 2$-power cusp at the point $z=w(-1)$ on $\partial \Omega$ (figure $5.5(1)$ ). The line $c=0$ is the finite domain limit in which the fluid domain is a cardioid ( $c f$ the limaçon example of $\S 2.4$ ). In the lower part of $V$, the line $b+c+4=0$ gives free boundaries $\partial \Omega$ having two $3 / 2$-power cusps, symmetrically placed about the $x$-axis (figure 5.5 (2)). Along the parabolic portion of $\partial V$ the free boundary is nonanalytic due to self-intersection (figure 5.5 (3)), except for the isolated point $(6,-16)$, where $\partial \Omega$ has a single $5 / 2$-power cusp-at this point the parabola is actually tangent to the line $4 b+c=8$ (which formed part of $\partial V$ in the upper half plane, $c>0$ ). Figures 5.4 and 5.5 should clarify the above explanation.

The analogues of equations (5.16)-(5.18) for the modified mapping function (5.19) are:

$$
\begin{align*}
a^{3}(b+c)^{2} & =2 M t+\text { const. }  \tag{5.20}\\
2(1+c)+b(b+c) & =\frac{k_{1}}{a^{2}}  \tag{5.21}\\
c-2(1+b) & =\frac{2 k}{a} \tag{5.22}
\end{align*}
$$

eliminating $a$ between the last two yields the solution trajectories in $(b, c)$-space as

$$
\frac{2+b^{2}+c(2+b)}{(1+b-c / 2)^{2}}=\kappa
$$

for various constants $\kappa$, determined by the initial conditions (figure 5.6). This equation clearly represents some kind of conic section, depending on the value of $\kappa$ chosen. Changing to axes aligned with the principal axes, it can be rearranged into the form

$$
\left(b-\frac{c}{2}+1+3 \lambda\right)^{2}-4 \lambda\left(b+\frac{c}{4}+1\right)^{2}=9 \lambda(1+\lambda),
$$

where $\lambda=1 /(3 \kappa+1)$. From this it is clear that for $0<\lambda<\infty$ the phase paths are branches of hyperbolae, with asymptotes

$$
\frac{c}{2}(1 \pm \sqrt{\lambda})=b(1 \mp 2 \sqrt{\lambda})+1+3 \lambda \mp 2 \sqrt{\lambda}
$$

(so the gradient of these asymptotes is $2(1 \mp 2 \sqrt{\lambda}) /(1 \pm \sqrt{\lambda})$ ). The two sets of branches are separated by the $\lambda=0$ (straight line) path,

$$
b-\frac{c}{2}+1=0
$$

A positive value of $\lambda$ does not uniquely specify a solution path then, since there are two possible branches. This is not a problem, however, because the value of $\lambda$ is determined by the initial conditions, and obviously the free boundary has to evolve in a continuous manner, so we must stay on the particular path on which we start.

Values of $\lambda$ between $-\infty$ and -1 give elliptical phase paths, $\lambda=-1$ being the value at which the family of ellipses collapses to the single point $(0,-4)$, which actually lies on the boundary $\partial V$. There are no real phase trajectories for $\lambda \in(-1,0)$. The limit $\lambda \rightarrow \pm \infty$ corresponds to the original parameter $\kappa$ approaching the value $-\frac{1}{3}$ from above or below; the limiting phase path is a parabola with equation

$$
\begin{equation*}
Y^{2}=\frac{3}{2}\left(X-\frac{3}{2}\right) \tag{5.23}
\end{equation*}
$$

where $X=1+b-c / 2$ and $Y=1+b+c / 4$ are the co-ordinates along the principal axes of the family of conics.


Figure 5.6: The phase diagram (within the univalency domain) for the Hele-Shaw dipole problem.

Line $b+c+4=0$


Figure 5.7: Enlargement of the transient 5/2-power cusp formation. Pictures (a), (b) and (c) illustrate how the free boundary passes through the $5 / 2$-power cusped configuration (a), to a smooth boundary (b), before ultimately blowing up with two $3 / 2$-power cusps (c).

The phase diagram within the univalency domain is shown in figure 5.6. The arrows on the phase paths are for positive values of the dipole strength $M$ (so the dipole has the sense in figure 5.3 ); if the dipole is reversed, so are these arrows. Points to note about the phase diagram are:

- The existence of the elliptical solution trajectory for $\lambda=-5$, which is tangential to the parabolic part of $\partial V$ at the point $(6,-16)$, and re-enters $V$ immediately (figure 5.7). As discussed in $\S 5.2$, this corresponds to the appearance, and subsequent immediate disappearance, of a $5 / 2$-power cusp in $\partial \Omega$. Note that this trajectory reaches the boundary $b+c+4=0$ of $\partial V$ shortly afterwards, so that the solution does ultimately blow up with the formation of two $3 / 2$-power cusps in $\partial \Omega$.
- The existence of the "degenerate" phase path $b=0, c=-4,(\lambda=-1)$, which lies on $\partial V$. This cannot represent a solution, however, since equations (5.20)-(5.22) cannot be solved for $a(t)$ if $b$ and $c$ are constant.
- The parabolic phase path ( $\lambda \rightarrow \pm \infty$, or $\kappa=-1 / 3$ ), separating the elliptical paths in the lower part of $V$, from the branches of the hyperbolic paths in that part of $V$.

In the lower part of $V$, the paths which "go off to infinity" whilst remaining within $V$ are those hyperbolic paths for $0<\lambda<1$ which lie to the right of the "separatrix" $1+b=c / 2$ (the $\lambda=0$ path), and those for $0<\lambda<1 / 4$ to the left of the separatrix.

For the paths lying to the right of the separatrix, $\lambda=1(\kappa=0)$ gives the hyperbola which has an infinite-gradient asymptote. The corresponding asymptotes of the hyperbolae for $0<\lambda<1$ have gradients lying in the range 2 to $\infty$ ( 2 being the gradient of the separatrix), whilst those for $\lambda>1$ will have asymptotes with negative gradient. These $\lambda>1$ hyperbolae will thus intersect the parabolic part of $\partial V$ before $b$ and $c$ become infinite, and solution breakdown via self-overlapping of the free boundary occurs. The hyperbolae for $0<\lambda<1$, which lie between the $\lambda=0$ separatrix and the $\lambda=1$ path, have corresponding branches in the upper part of $V$. What happens with these solutions is that we reach $c=-\infty$ within finite time, then reappear on the corresponding branch at $c=+\infty$ in the upper part of $V$. We have to be careful, since there are two possible branches for a particular value of $\lambda$, but a little thought tells us that the "corresponding branch" is the branch having the same value of $\lambda$, but now lying to the left of the separatrix in the upper half of $V$, since the free boundary shape must change in a continuous manner.

This transition from $c=-\infty$ to $c=+\infty$ is simultaneous with $a$ passing through the value zero. Only if $b$ remains $O(1)$ as this happens will we have the free boundary passing through the completely flat configuration, which is the special case we considered first. This will be exactly the hyperbolic path whose asymptote has infinite gradient, i.e. the $\lambda=1$ (or $\kappa=0$ ) path, along which $b \rightarrow-2$ as $\kappa \rightarrow-\infty$.

For the paths lying to the left of the separatrix in the lower half of $V, \lambda=1 / 4$ is the value for which one of the asymptotes of the hyperbola has zero gradient. Phase paths for $0<\lambda<\frac{1}{4}$ will thus extend to infinity without leaving the domain $V$; as above, this will occur within finite time, and the solution is continued by reappearing at infinity on the corresponding path in the upper part of $V$, which will here lie to the right of the $\lambda=0$ separatrix. It can be seen, by considering the arrows on the phase paths in the upper half of $V$, that all solutions exhibiting this kind of behaviour will ultimately blow up with formation of a $3 / 2$-power cusp in the free boundary at the point $w(-1)$.

The parabolic phase path is exceptional in that it has no component in the upper part of $V$. The solution represented by this trajectory blows up within finite time when it intersects the parabolic part of $\partial V$, at the point $(b, c)=(21.2328,-134.941)$, with self-intersection of the free boundary.

We have been able to analyse the possible kinds of behaviour, without solving explicitly for $(a, b, c)$. A complete description would involve solving (5.20)-(5.22), but little would be gained by doing this, since the above working captures all the essentials, and certainly all the interesting aspects of the problem.

As an aside, we comment that the analysis may be extended to the case of a general multipole singularity at the origin (as considered by Entov et al. [24]). For a multipole of order n, the
complex potential has the behaviour

$$
\mathcal{W}(z)=-\frac{M}{z^{n}}+O(1) \quad \text { as } z \rightarrow 0
$$

so that the singular part of the Schwarz function is given by

$$
g_{e}(z, t)=\frac{r^{2}}{z-\alpha}+\frac{2 n M t}{z^{n+1}}
$$

instead of (5.4). Thus in this case, by (2.27), the complex conjugate mapping function must have a pole of order $(n+1)$ at $\zeta=0$ and a simple pole at $\zeta=d$ as its only singularities within the unit disc, and must vanish at infinity. The mapping function for this case will therefore be given by

$$
\begin{equation*}
w(\zeta)=\sum_{r=1}^{n+1} a_{r} \zeta^{r}+\frac{C \zeta}{1-d \zeta}, \tag{5.24}
\end{equation*}
$$

(with $d=1$ in the unbounded domain limit). The more complicated form of this function will allow for more varied behaviour of solutions, but for larger values than $n=1$ it is much more difficult to determine the univalency region for the map in parameter space.

### 5.4 The Stokes flow dipole problem

### 5.4.1 Review of Jeong \& Moffatt's steady solution

Before considering time-dependent possibilities, we first review the steady Stokes flow dipole problem as solved by J \& M [52], since this was our original motivation for studying problems on unbounded domains with finite driving singularities. This problem models experiments performed (at low Reynolds number) in a large tank of viscous fluid, with two counter-rotating cylinders placed beneath the free surface. Above a certain critical rate of rotation of the cylinders, a steady state was quickly attained in which the free surface of the fluid above the cylinders had an apparent cusp. It is this steady-state configuration which J \& M's solution describes.

The presentation we give here is slightly different to theirs, since we use the techniques and conventions of this thesis; however in all essentials it is the same. We use the form of the governing equations given in $\S 3.2 .1$; in the simpler time-independent case here, equation (3.26) integrates to

$$
\begin{equation*}
\mathcal{X}(\zeta)+\bar{w}(1 / \zeta) \Phi(\zeta)=0 \tag{5.25}
\end{equation*}
$$

which is just the condition that the free boundary be a streamline, analytically continued to hold globally. $\Phi(\zeta)$ is given (from (3.15)) by

$$
\begin{equation*}
\Phi(\zeta)=\frac{T}{4 \mu}\left[f_{+}(0)-2 f_{+}(\zeta)\right] \zeta w^{\prime}(\zeta, t)+\frac{w^{\prime}(\zeta)}{w^{\prime}(0)} A\left(1-\zeta^{2}\right) \tag{5.26}
\end{equation*}
$$

where we have assumed $\Phi(\zeta)$ to be bounded at the origin (with limit $A$ ), which will be the case if we have a pure vortex dipole at the origin. We have also assumed that $A$ is real, which is equivalent to requiring the geometry to be symmetric about the $x$-axis (if we also have a conformal map with real parameters). Thus our geometry is J \& M's, rotated through $90^{\circ}$, and the free boundary at infinity will be $x=$ constant, with the fluid occupying the region to the right of this boundary (see the Hele-Shaw definition sketch, figure 5.3). The dipole is taken to be situated at the origin, and the conformal map is subject to the usual conditions $w(0, t)=0, w^{\prime}(0, t)>0$.

Some point on the unit circle must map to infinity; with the above restrictions, this can only be the point $\zeta=1$. The geometrical constraint that the free boundary be asymptotically flat forces the behaviour

$$
w(\zeta)=\frac{\beta}{1-\zeta}+O(1) \quad \text { as } \zeta \rightarrow 1
$$

for some real positive $\beta$, and there are no other singularities of $w(\zeta)$ within the unit disc. Consideration of the dipole behaviour at $\zeta=0$ requires (using (5.25))

$$
\begin{equation*}
\bar{w}(1 / \zeta)=-\frac{M}{A w^{\prime}(0) \zeta}+O(1) \quad \text { as } \zeta \rightarrow 0 \tag{5.27}
\end{equation*}
$$

so that the mapping function must be of the form ${ }^{3}$

$$
\begin{equation*}
w(\zeta)=\alpha \zeta+\frac{\beta \zeta}{1-\zeta} \tag{5.29}
\end{equation*}
$$

For this map, the free boundary at infinity is given by

$$
x=\alpha-\frac{\beta}{2}, \quad|y| \rightarrow \infty
$$

and since we must be able to specify the depth of the dipole beneath this asymptotic free surface, we may insist

$$
\alpha=\frac{\beta}{2}-1
$$

without loss of generality (taking this dipole depth to be 1 ). There are now only two unknown quantities, the parameter $\alpha$, and $A \equiv \Phi(0)$. The behaviour at the dipole gives $A$ in terms of $\alpha$, since by (5.27) and (5.29) we must have

$$
\alpha A=-\frac{M}{w^{\prime}(0)} \equiv-\frac{M}{3 \alpha+2} .
$$

We finally have to satisfy the condition that the flow be stagnant at infinity, where the effect of the dipole cannot be felt. The most general behaviour of $\phi(z)$ and $\chi(z)$ allowing this is

$$
\begin{align*}
& \phi(z)=-\frac{p_{\infty}}{4 \mu} z+\bar{c}+O(1 / z)  \tag{5.30}\\
& \chi(z)=c z+O(1) \tag{5.31}
\end{align*}
$$

both holding as $|z| \rightarrow \infty$; this may be seen from (3.2) and (3.3) ( $p_{\infty}$ denotes the pressure field at infinity). For this particular case we take $p_{\infty}=0$; this is necessary if we are to get the correct balance in (5.25) at infinity, because the map (5.29) satisfies

$$
\begin{equation*}
\bar{w}(1 / \zeta) \sim-w(\zeta) \tag{5.32}
\end{equation*}
$$

at leading order, as $\zeta \rightarrow 1$. With our symmetric geometry, the constant $c$ will be real. Thus, if we can ensure that $\Phi(\zeta)$ remains bounded as $\zeta \rightarrow 1$, denoting this limit by $c$, equation (5.25) will automatically ensure that the second of the 'stagnant flow' conditions is satisfied.

Writing $\epsilon=1-\zeta$, (5.26) gives the asymptotic behaviour of $\Phi(\zeta)$ near $\zeta=1$ as

$$
\Phi(\zeta)=\frac{T \beta}{4 \mu \epsilon^{2}}\left(f_{+}(0)-2 f_{+}(1)\right)-\frac{1}{\epsilon}\left[\frac{T \beta}{4 \mu}\left(f_{+}(0)-2 f_{+}(1)-2 f_{+}^{\prime}(1)\right)-\frac{2 A \beta}{w^{\prime}(0)}\right]+O(1) .
$$

[^20]This would necessitate a pole of order $n$ at $\zeta=0$ in $\bar{w}(1 / \zeta)$, giving the general form of the map as

$$
w(\zeta)=\sum_{r=1}^{n} \alpha_{r} \zeta^{r}+\frac{\beta \zeta}{1-\zeta} .
$$

(All quantities appearing in this expression are either given, or else are known functions of $\alpha$ ). It is easy to see from the definition (3.8) that

$$
f_{+}(0)-2 f_{+}(1)=\frac{i}{2 \pi} \int_{0}^{2 \pi} \frac{\sin \theta d \theta}{\left|w^{\prime}\left(e^{i \theta}\right)\right|(1-\cos \theta)}
$$

which is identically zero, by a simple symmetry argument. Thus the singularity of order $1 / \epsilon^{2}$ in $\Phi(\zeta)$ vanishes automatically, and we need only impose the condition

$$
\frac{T}{4 \mu} f_{+}^{\prime}(1)+\frac{A}{w^{\prime}(0)}=0
$$

to remove the order $1 / \epsilon$ singularity. Use of (3.8) reduces this to

$$
\Phi(0)=\frac{T w^{\prime}(0)}{16 \pi \mu} \int_{0}^{2 \pi} \frac{1}{\left|w^{\prime}\left(e^{i \theta}\right)\right|} \frac{d \theta}{(1-\cos \theta)},
$$

which is the single condition needed to complete the solution. For detailed discussion of the results, and comparison of the free boundary shape with experiment, see [52].

### 5.4.2 The time-dependent problem

Having seen how the steady problem works, we consider how we might generalise the analysis to find a time-dependent solution. Recalling the comments of $\S 5.1$, we cannot expect to find such a solution unless we allow the singularity to move relative to the fluid mass, and we find that this is indeed the case, at least when we assume $\Phi(0)=0$.

We begin by observing that, if such a solution exists, it must be realisable as the limit of a dipole-in-circle problem, such as we solved for Hele-Shaw flow in $\S 5.3$. Motivated by this, we attempt the Stokes flow problem, using the method outlined in $\S 3.7$ to find the conformal map. As commented in $\S 3.6 .2$, if a conformal map gives a solution to the ZST time-dependent problem, the same map will also work for the NZST problem, hence we consider the easier ZST case. For a pure dipole singularity at the origin, we require $\Phi(\zeta)$ to be bounded there, but do not yet know whether we may assume it to be zero. (Given that $\Phi(0)$ turned out to be nonzero for the steady problem of $\S 5.4 .1$, we can hardly expect to find a time-dependent solution for which it is zero, but it is instructive to see what happens when we consider both cases.) We consider the simpler situation in which $\Phi(0) \equiv 0$ first.

In this case, and with a dipole singularity at the origin, the function $H_{e}(\zeta, t)$ is given by (3.63), so we need to determine $H_{e}(\zeta, 0)$. The Schwarz function of the initial domain is exactly as for the Hele-Shaw problem, and is given by (5.3); from this the functions $g_{e}(z, 0)$ and $h_{e}(z, 0)$ are immediate as

$$
g_{e}(z, 0)=\frac{r^{2}}{z-\alpha}, \quad h_{e}(z, 0)=r^{2} \log (z-\alpha)
$$

If $d$ is the (unique) point within the unit disc such that $w(d, 0)=\alpha$, then we must have

$$
\begin{equation*}
H_{e}(\zeta, 0)=r^{2} \log (\zeta-d) \tag{5.33}
\end{equation*}
$$

so that for the case of a dipole singularity driving the flow we have

$$
H_{e}(\zeta, t)=r^{2} \log (\zeta-d)-\frac{M \theta(t)}{\zeta}
$$

by (3.63), with $\theta(t)$ as defined there. Since $H^{\prime}(\zeta)=w^{\prime}(\zeta) \bar{w}(1 / \zeta)$, the singular part of this combination within the unit disc is given by

$$
\left[w^{\prime}(\zeta) \bar{w}(1 / \zeta)\right]_{\operatorname{sing}}=\frac{r^{2}}{\zeta-d}+\frac{M \theta(t)}{\zeta^{2}}
$$

These singularities must come from the term $\bar{w}(1 / \zeta)$, the mapping function being analytic on the unit disc. Hence, if a solution to the problem as stated exists, the correct mapping function must be such that $\bar{w}(1 / \zeta)$ has a double pole at the origin, and a simple pole at some (fixed) point $\zeta=d$, as its only singularities within the unit disc. It can have no singularities outside the unit disc since this would entail $w(\zeta)$ being nonanalytic on $|\zeta|<1$. It follows that $w(\zeta)$ must be of the form

$$
\begin{equation*}
w(\zeta)=a \zeta^{2}+b \zeta+\frac{c \zeta}{1-d \zeta} \tag{5.34}
\end{equation*}
$$

for some time-dependent parameters $a, b$ and $c$, which is exactly the map which was used for the Hele-Shaw solution of $\S 5.3 .{ }^{4}$ This will always be the case with the assumption $\Phi(0)=0$; if a Stokes flow solution of this kind exists, it must be such that the mapping function is the same as for the corresponding Hele-Shaw solution.

To find the equations governing the evolution of the parameters in (5.34), we must return to (3.29) and make a quantitative comparison of singularities at $\zeta=0$ and $\zeta=d$. The appropriate behaviour of $\mathcal{X}^{\prime}(\zeta)$ on the right-hand side of $(3.29)$ is $\mathcal{X}^{\prime}(\zeta)=-M /\left(\zeta^{2} w^{\prime}(0)\right)+O(1)$ near $\zeta=0$, whilst $\mathcal{X}^{\prime}(\zeta)$ must be regular at $\zeta=d$. Matching at $\zeta=0$ gives the equations

$$
\begin{align*}
\frac{d}{d t}(a(b+c)) & =\frac{2 M}{b+c}  \tag{5.35}\\
2 a(a+c d)+b(b+c) & =\text { const. }=0 \tag{5.36}
\end{align*}
$$

while eliminating the singularity at $\zeta=d$ yields

$$
\begin{align*}
d & =\text { const. }
\end{align*}=\frac{\alpha}{r}, ~ \begin{gathered}
c  \tag{5.37}\\
c w^{\prime}(d)=c\left(2 a d+b+\frac{c}{\left(1-d^{2}\right)^{2}}\right) \tag{5.38}
\end{gathered}=\text { const. }=r^{2} . ~ \$
$$

Note that equations (5.36) and (5.38) are exactly the same as (5.11) and (5.9) for the Hele-Shaw problem; (5.35) is analogous to (5.7), and (5.37) is analogous to (5.10). The constancy of $d$ here emerged naturally as a consequence of the solution (3.63), but in any case could have been assumed a priori by the comment of $\S 3.6 .2$ that the singularities of the Schwarz function must remain fixed in the $\zeta$-plane. The values of the constants on the right-hand sides of these equations were obtained from the initial conditions on the map necessary to give the circle of radius $r$ centred at $z=\alpha$, namely

$$
c(0)=r\left(1-\frac{\alpha^{2}}{r^{2}}\right), \quad b(0)=0=a(0), \quad d(0)=\frac{\alpha}{r}
$$

the condition on $d(0)$ gives the value of $d$ for all time. We have the three equations (5.35), (5.36) and (5.38) to solve for the functions $a(t), b(t)$ and $c(t)$, which is a well-determined system. At first sight then, it seems that everything is leading to a solution of the physical problem; however when we consider the large domain limit (as we did for the analogous Hele-Shaw problem), we encounter problems, finding that the entire fluid mass translates uniformly relative to the (fixed) dipole (or vice-versa, if we subtract off this translational velocity from the solution).

To see this explicitly, we take the same limit as in $\S 5.3$, letting $r \rightarrow \infty, \alpha \rightarrow \infty$, while keeping $(r-\alpha)$ fixed at $k$ (this being the problem of a dipole placed at $z=0$ in the half-space of fluid $\{x>-k\}$ ). In this limit $d$ approaches the value 1 , and $c(0)$ approaches $2 k$. Equation (5.35) still stands, and (5.36) has the obvious limit obtained by setting $d=1$. A leading-order balance is achieved in (5.38) only if $c(t)$ assumes the constant value, $2 k$. As the limit is approached, it is prudent to look at what is happening far from the dipole, since physically the free surface ought

[^21]to be undisturbed (flat) as $x \rightarrow \pm \infty$, and the velocity field ought to be zero at infinity. For the ZST problem with $\Phi(0)=0$, the velocity field is given by (C.1) with $T=0$, that is,
\[

$$
\begin{equation*}
2(u-i v)=\bar{w}_{t}(1 / \zeta)+\overline{w_{t}(\zeta)}+\frac{w_{t}^{\prime}(\zeta)}{w^{\prime}(\zeta)}(\bar{w}(1 / \zeta)-\overline{w(\zeta)}) \tag{5.39}
\end{equation*}
$$

\]

The map for the infinite domain limit is

$$
w(\zeta, t)=a \zeta^{2}+b \zeta+\frac{c \zeta}{1-\zeta}
$$

with initial conditions as before except that now $c(0)=2 k$. Since $\zeta=1$ maps to infinity, we write $\zeta=1-\epsilon$ for small $\epsilon \in \mathbb{C}$ and examine the velocity field (5.39). The various expressions are found to be

$$
\begin{aligned}
\bar{w}(1 / \zeta) & =-\frac{c}{\epsilon}+(a+b)+\epsilon(2 a+b)+O\left(\epsilon^{2}\right) \\
w(\zeta) & =\frac{c}{\epsilon}+(a+b-c)-\epsilon(2 a+b)+O\left(\epsilon^{2}\right) \\
\bar{w}_{t}(1 / \zeta) & =-\frac{\dot{c}}{\epsilon}+(\dot{a}+\dot{b})+\epsilon(2 \dot{a}+\dot{b})+O\left(\epsilon^{2}\right) \\
w_{t}(\zeta) & =\frac{\dot{c}}{\epsilon}+(\dot{a}+\dot{b}-\dot{c})-\epsilon(2 \dot{a}+\dot{b})+O\left(\epsilon^{2}\right) \\
\frac{w_{t}^{\prime}(\zeta)}{w^{\prime}(\zeta)} & =\frac{\dot{c}}{c}+\frac{\epsilon^{2}}{c^{2}}[c(2 \dot{a}+\dot{b})-\dot{c}(2 a+b)]+\frac{2 \epsilon^{3}}{c^{2}}(a \dot{c}-\dot{a} c)+O\left(\epsilon^{4}\right)
\end{aligned}
$$

combining these gives the velocity at large distances from the dipole as

$$
u-i v=-\frac{\dot{c}}{\epsilon}+(\dot{a}+\dot{b})+O(\epsilon)
$$

The leading-order term here vanishes, since we already know $c$ is a constant (equal to $2 k$ ) from the balance of terms in (5.38). We also require

$$
a(t)+b(t)=\text { const. }=0
$$

combined, these two conditions give $c-2(a+b)=2 k, c f(5.18)$, which is exactly the requirement that the asymptotic free boundary at infinity be fixed relative to the dipole. These conditions are clearly incompatible with equations (5.35) and (5.36) (with $d=1$ in the latter). Insisting that (5.35) and (5.36) hold, as they must certainly do to fit the conditions at the dipole, the best we can then do is to take $c(t)$ to be constant, giving bounded, but nonzero, velocity at infinity. Physically, this corresponds to the entire fluid mass translating with speed $a(t)+b(t)$ relative to the fixed dipole, which is a highly artificial geometry. The unrealistic nature of this large domain limit implies that the finite domain solution represented by the system (5.34)-(5.38) must also be artificial, otherwise it would give a sensible limit. ${ }^{5}$

The unacceptability of this particular solution is a consequence of the assumption $\Phi(0)=0$ which was made, since if a solution exists to the problem of a pure dipole in this geometry, $\Phi(0)$ is necessarily regular at the origin, but need not vanish there (see the comments following (3.14) in $\S 3.1)$. We therefore consider what form the mapping function must take in this case. The relevant governing equations for the Schwarz function are now (3.57) and (3.54); by the remarks following (3.57) the solution for $H_{e}(\zeta, \tau)$ may be written down from (3.58) as

$$
\begin{aligned}
H_{e}(\zeta, \tau)=\frac{C_{0}}{\pi} \log \zeta-\frac{1}{\pi} \int_{0}^{\tau} \frac{\left(C_{0}+M \pi / A\left(\tau^{\prime}\right)\right) d \tau^{\prime}}{\tanh \left(\tau^{\prime}-\tau+\tanh ^{-1} \zeta\right)} & -\frac{1}{\pi} f\left(\tanh ^{-1} \zeta-\tau\right) \\
& -\frac{1}{\pi} \int_{0}^{\tau} C_{1}\left(\tau^{\prime}\right) d \tau^{\prime}
\end{aligned}
$$

[^22]where $C_{0}$ and $C_{1}$ are, respectively, the zeroth and first "moments", and $f$ is a function to be determined. Note that we are now working with the scaled time variable $\tau$, defined in (3.51), and that since we have no point sink at the origin, the "moment" $C_{0}$ is a constant equal to the area of the fluid domain ( $\pi r^{2}$ here). We have initial conditions (5.33) as before, from which the function $f$ is seen to be
$$
f(\nu)=\pi r^{2} \log (\tanh \nu)-\pi r^{2} \log (\tanh \nu-d)
$$

After simplifying and rearranging, the solution for $H_{e}$ becomes

$$
\begin{aligned}
H_{e}(\zeta, \tau) & =r^{2} \log \zeta-r^{2} \log (\zeta-\tanh \tau)+r^{2} \log \left(\zeta-\tanh \left(\tau+\tanh ^{-1} d\right)\right) \\
& -r^{2} \int_{0}^{\tanh \tau} \frac{(1-\zeta x) d x}{(\zeta-x)\left(1-x^{2}\right)}-M \int_{0}^{\tanh \tau} \frac{(1-\zeta x) d x}{A\left(\tau-\tanh ^{-1} x\right)(\zeta-x)\left(1-x^{2}\right)}
\end{aligned}
$$

(plus some function of $\tau$, irrelevant to the discussion). Hence,

$$
\begin{align*}
& H_{e}^{\prime}(\zeta, \tau)=\frac{r^{2}}{\zeta}-\frac{r^{2}}{\zeta-\tanh \tau}+\frac{r^{2}}{\zeta-\tanh \left(\tau+\tanh ^{-1} d\right)} \\
& \quad+r^{2} \int_{0}^{\tanh \tau} \frac{d x}{(\zeta-x)^{2}}+M \int_{0}^{\tanh \tau} \frac{d x}{A\left(\tau-\tanh ^{-1} x\right)(\zeta-x)^{2}} \tag{5.40}
\end{align*}
$$

As discussed in $\S 3.7$, this must give us the singularities of $\bar{w}(1 / \zeta)$ within the unit disc. The above expression for $H_{e}^{\prime}(\zeta, \tau)$ reveals these to be simple poles at $\zeta=0, \zeta=\tanh \tau$, and $\zeta=$ $\tanh \left(\tau+\tanh ^{-1} d\right)$, as well as a line singularity along the line segment $(0, \tanh \tau)$ on the positive real axis. Hence, the form of the mapping function needed to give a solution for this particular problem must be such that

$$
\begin{gathered}
\bar{w}(1 / \zeta)=\frac{a}{\zeta}+\frac{b}{\zeta-\tanh \tau}+\frac{c}{\zeta-\tanh \left(\tau+\tanh ^{-1} d\right)}+\int_{0}^{\tanh \tau} \frac{R(x, \tau)}{(\zeta-x)^{2}} d x \\
+\int_{0}^{\tanh \tau} \frac{Q(x, \tau)}{\zeta-x} d x
\end{gathered}
$$

so that

$$
\begin{gather*}
w(\zeta)=a \zeta+\frac{b \zeta}{1-\zeta \tanh \tau}+\frac{c \zeta}{1-\zeta \tanh \left(\tau+\tanh ^{-1} d\right)}+\int_{0}^{\tanh \tau} \frac{\zeta^{2} R(x, \tau)}{(1-x \zeta)^{2}} d x \\
+\int_{0}^{\tanh \tau} \frac{\zeta Q(x, \tau)}{1-x \zeta} d x \tag{5.41}
\end{gather*}
$$

for some unknown functions $a(\tau), b(\tau), c(\tau), R(x, \tau), Q(x, \tau)$. Note that this analysis is all in terms of the scaled time variable, hence if we wish to solve for such a mapping function, we must use the governing equations in the form (3.60), (3.61). This form of $w(\zeta)$ is clearly very complicated, and the algebra involved in substituting into (3.60) will be very messy; however, the arguments leading to (5.41) were deductive, and in principle the problem can be solved. The parameter $A(\tau)$ provides the extra degree of freedom, which is necessary to enable the "momentum condition" to be imposed. For the unbounded domain limit of the problem, this will be automatic if we insist that the conditions both at the dipole and at infinity hold. For a bounded domain, the momentum $P$ of the flow is given by (3.17) with $T=0$; with $\Phi(\zeta)$ given by (3.61) this becomes

$$
P=\frac{A(\tau)}{i w^{\prime}(0, \tau)} \int_{|\zeta|=1} w^{\prime}(\zeta) \bar{w}(1 / \zeta)\left(w_{\tau}(\zeta)+w^{\prime}(\zeta)\left(1-\zeta^{2}\right)\right) d \zeta
$$

which must vanish.

As an aside, note that we may also find the "moment generating function" $\mathcal{F}_{1}$ from (3.68) and the remark following it, as

$$
\begin{gather*}
\mathcal{F}_{1}(\zeta, \tau)=\int_{0}^{\tau}\left(C_{0}+\frac{M \pi}{A\left(\tau^{\prime}\right)}\right) \tanh \left(\tau^{\prime}-\tau+\tanh ^{-1} \zeta\right) d \tau^{\prime}+f_{1}\left(\tanh ^{-1} \zeta-\tau\right) \\
+\int_{0}^{\tau} C_{1}\left(\tau^{\prime}\right) d \tau^{\prime} \tag{5.42}
\end{gather*}
$$

The arbitrary function $f_{1}$ is determined by the data (5.33), using the relations (3.54) and (3.67), and is

$$
f_{1}(\nu)=-C_{0} \log (1-d \tanh \nu)
$$

Equating the constant term on the right-hand side of (5.42) to zero then yields a consistency condition analogous to (3.70), namely

$$
\begin{aligned}
C_{1}(\tau) & =\int_{0}^{\tau}\left(C_{0}+\frac{M \pi}{A\left(\tau^{\prime}\right)}\right) \operatorname{sech}^{2}\left(\tau^{\prime}-\tau\right) d \tau^{\prime}+\frac{C_{0} d \operatorname{sech}^{2} \tau}{1+d \tanh \tau} \\
\Rightarrow C_{1}(\tau) & =\int_{0}^{\tau} \frac{M \pi}{A\left(\tau^{\prime}\right)} \operatorname{sech}^{2}\left(\tau^{\prime}-\tau\right) d \tau^{\prime}+C_{0} \tanh \left(\tau+\tanh ^{-1} d\right)
\end{aligned}
$$

this gives a relation between $C_{1}(\tau)$ and $A(\tau)$ (remember $C_{0} \equiv \pi r^{2}$, a constant). The other "moments" also follow from (5.42), should we want them.

We have assumed a specific geometry and driving mechanism throughout this section, but it is clear that the same deductive method of finding the form of $w(\zeta, \tau)$ will apply quite generally, provided we can solve (3.52) for $H_{e}(\zeta, \tau)$. Given the complexity of (5.41) it is unlikely that will be able to determine solutions of this kind analytically; however we have shown how, in principle, they may be constructed.

Recalling our comments of $\S 5.1$, the root of the difficulty lies in finding a mapping function such that the distribution of singularities of the Schwarz function within the unit disc in $\zeta$-space allows all conditions in the physical domain to be fulfilled. We have already seen (from our discussion of the case $\Phi(0)=A=0$ ) that this is certainly not possible for the case where these singularities remain fixed in $\zeta$-space. The above shows that, even in the case $A \neq 0$ when the singularities are allowed to move, a Schwarz function which has only isolated singularities within the unit disc is not good enough; a continuous line distribution of singularities is needed, and moreover, this line itself must vary in time.

We remark that if we were to attempt a general problem (with $A \neq 0$ ) using a Schwarz function with a distribution of isolated singularities at points $\zeta_{r}=\sigma_{r}(\tau) \neq 0$, say, within the unit disc, then it is easily seen from the governing equation (3.60) that these singularities must vary according to

$$
\begin{equation*}
\frac{d \sigma_{r}}{d \tau}=1-\sigma_{r}^{2} \tag{5.43}
\end{equation*}
$$

(in contrast to the case $A=0$, where the $\sigma_{r}$ remain fixed). With the scaled time variable $\tau$, they are thus moving in a specified manner within the unit disc (except for the singularity at $\zeta=0$ corresponding to the driving singularity, which remains fixed). Behaviour of this kind is every bit as restrictive as fixed singularities, hence we cannot expect to do any better for problems with multiple, fixed driving singularities in this case. It seems that the continuous distribution of singularities is essential if such problems are to be treated by conformal mapping methods.

### 5.5 Steady Stokes flow reconsidered

Since the time-dependent problem with driving singularity at a finite point proves so intractable, we reconsider steady problems of this kind (little literature exists even for this simpler problem).

Although we were able to consider the ZST equations for the time-evolving problem with singularities, surface tension is essential in the steady problem. One can consider the small surface tension limit of steady, NZST solutions ( $c f$ chapter 6 , where we do this for the unsteady, finite domain problem), but this requires careful asymptotics, and is definitely not the same as setting $T=0$ in the steady equations (which gives either trivial solutions, or no solution).

The most convenient form of the equations to use is (5.25) and (5.26). The solution technique is best illustrated by example, and shows how the ideas introduced by $\mathrm{J} \& \mathrm{M}$ can be generalised. For the sake of definiteness, we again consider a dipole singularity situated at the origin in an unbounded fluid domain (though we could easily consider more general singularities), but in contrast to their solution, we assume the fluid to contain a finite air bubble.

Assuming $\Phi(0)=A \neq 0$ then, (5.25) implies that $\bar{w}(1 / \zeta)$ must have a simple pole at the origin, to balance with the dipole singularity. There will also be some point $\zeta=\gamma$ within the unit disc which maps to infinity; as usual, we consider a map with real coefficients, so that without loss of generality $\gamma \in(0,1)$. Thus we consider a map of the form

$$
\begin{equation*}
w(\zeta)=\frac{\alpha \zeta(\zeta-\beta)}{\zeta-\gamma} \tag{5.44}
\end{equation*}
$$

and this should give a solution. Note that the map (5.29) considered by J \& M is the special case $\gamma=1$, so we are in effect generalising their solution. Note also that this is exactly the map (5.1) referred to in $\S 5.2$, for which a Hele-Shaw solution has been found with a point sink singularity driving the flow [33].

The usual conditions apply at the dipole, namely

$$
\mathcal{X}(\zeta)=\frac{M}{\zeta w^{\prime}(0)}+O(1), \quad \Phi(\zeta)=A+O(\zeta), \quad \text { as } \zeta \rightarrow 0
$$

so matching the order $1 / \zeta$ singularity at $\zeta=0$ in (5.25) gives

$$
\begin{equation*}
\frac{\gamma M}{\alpha \beta}+\alpha A=0 \tag{5.45}
\end{equation*}
$$

We also have to deal with the conditions at $\zeta=\gamma$ (infinity, in the physical plane). The most general conditions allowing stagnant flow at infinity are (5.30) and (5.31), and for the mapping function (5.44) we can no longer take $p_{\infty}=0$. After a local analysis of (5.26) at the point $\zeta=\gamma$, condition (5.30) yields the three equations

$$
\begin{array}{r}
\frac{T}{4 \mu}\left(f_{+}(0)-2 f_{+}(\gamma)\right)+\frac{A\left(1-\gamma^{2}\right)}{\alpha \beta}=0 \\
\frac{2 \gamma^{2} A}{\alpha \beta}-\frac{T}{4 \mu}\left(f_{+}(0)-2 f_{+}(\gamma)-2 \gamma f_{+}^{\prime}(\gamma)\right)=-\frac{p_{\infty}}{4 \mu} \\
\frac{A}{\beta} \gamma(1-\beta \gamma)+\frac{T \alpha \gamma}{4 \mu}\left\{f_{+}(0)-2 f_{+}(\gamma)+(\gamma-\beta)\left[2 f_{+}^{\prime}(\gamma)+\gamma f_{+}^{\prime \prime}(\gamma)\right]\right\} \\
=c-\frac{p_{\infty}}{4 \mu} \alpha(2 \gamma-\beta) \tag{5.48}
\end{array}
$$

Condition (5.31), in conjunction with equation (5.25), yields another condition,

$$
\begin{equation*}
c=\frac{p_{\infty}}{4 \mu} \bar{w}(1 / \gamma) . \tag{5.49}
\end{equation*}
$$

We have five equations, then, for the six unknowns $\alpha, \beta, \gamma, A, p_{\infty}$ and $c$. The system is not underdetermined however, since we have an arbitrary lengthscale, represented by the parameter $\alpha$. We may simplify the system (5.45)-(5.49) by introducing the scalings

$$
\begin{equation*}
\hat{A}=\frac{\mu}{T} A, \quad P_{\infty}=\frac{\alpha p_{\infty}}{T}, \quad C a=\frac{M \mu}{T \alpha^{2}} \tag{5.50}
\end{equation*}
$$

here, $C a$ is a Capillary number for the flow (measuring the relative effects of viscosity and surface tension), and the quantity $P_{\infty}$ is a dimensionless line pressure. If we also redefine the functions $f_{+}(\cdot)$ by writing

$$
\begin{equation*}
F_{+}(\zeta)=\alpha f_{+}(\zeta) \tag{5.51}
\end{equation*}
$$

then they are independent of the scaling parameter $\alpha$. With these scalings, and eliminating $c$ between (5.48) and (5.49), we obtain the system:

$$
\begin{array}{r}
\frac{\beta \hat{A}}{\gamma}+C a=0, \\
F_{+}(0)-2 F_{+}(\gamma)+\frac{4 \hat{A}}{\beta}\left(1-\gamma^{2}\right)=0, \\
\frac{4 \hat{A}}{\beta}\left(1+\gamma^{2}\right)+2 \gamma F_{+}^{\prime}(\gamma)=-P_{\infty}, \\
P_{\infty} \frac{\gamma^{4}-\gamma^{2}-\beta \gamma+1}{\gamma\left(1-\gamma^{2}\right)(\gamma-\beta)}=-\frac{4 \hat{A}}{\beta}+\gamma^{2} F_{+}^{\prime \prime}(\gamma) . \tag{5.55}
\end{array}
$$

We now have four equations for the four unknowns $\beta, \gamma, \hat{A}$ and $P_{\infty}$. The Capillary number will be specified as part of the problem. It is a simple matter to eliminate $P_{\infty}$ and $\hat{A}$, giving just two equations to be solved for the mapping parameters $\beta$ and $\gamma$,

$$
\begin{align*}
& \beta^{2}\left(F_{+}(0)-2 F_{+}(\gamma)\right)=4 C a \gamma\left(1-\gamma^{2}\right)  \tag{5.56}\\
& \frac{\gamma^{4}-\gamma^{2}-\beta \gamma+1}{\gamma\left(1-\gamma^{2}\right)(\gamma-\beta)}\left(4\left(1+\gamma^{2}\right) C a-2 \beta^{2} F_{+}^{\prime}(\gamma)\right)=4 C a+\beta^{2} \gamma F_{+}^{\prime \prime}(\gamma) \tag{5.57}
\end{align*}
$$

These equations are still very difficult though, since the functions $F_{+}(\cdot)$ depend nonlinearly on $\beta$ and $\gamma$, being defined through the expressions (3.8). A better notation emphasising this dependence would perhaps be $F_{+}(\cdot ; \beta, \gamma)$, but this is rather unwieldy.

Clearly, in the limit $\gamma \rightarrow 1$ we require $P_{\infty}=0$ for a balance of terms in (5.55). In this limit, equation (5.53) reduces to an identity, and what remains is equivalent to the problem solved by $\mathrm{J} \& \mathrm{M}$ for the dipole in a half space.

The set of solution pairs $(\beta, \gamma)$ to (5.56) and (5.57) describe a set of equilibrium bubble shapes which are solutions to the dipole-at-the-origin problem. Since we expect that there are very many such solution pairs, a sensible scheme for finding solutions to these equations (which will need to be done numerically) is to specify the value of $\gamma$ we wish to consider, and, eliminating $C a$ between equations (5.56) and (5.57), find the corresponding value(s) of $\beta$ which satisfies the equations (the relevant Capillary number will then follow from either of the above equations). Note that the mapping function (5.44) can also describe bounded fluid domains if we allow $\gamma>1$; the governing equations in this case will be different however, since the singularities of equation (5.25) within the unit disc then occur at $\zeta=0, \zeta=1 / \gamma$ and have to be matched appropriately.

Any solution pairs $(\beta, \gamma)$ which are found must be checked for univalency, since only univalent mapping functions give acceptable solutions. For unbounded fluid regions, the univalency domain in $(\beta, \gamma)$ parameter space is the union of the two regions

$$
\frac{1}{\beta}<\gamma<1 \quad(\beta>0, \gamma \in(0,1))
$$

and

$$
\beta<-\left(2+\frac{1}{\gamma}\right) \quad(\beta<0, \gamma \in(0,1))
$$

recall that we assumed $\gamma \in(0,1)$ at the outset. In $\beta>0,<0$ we require $\alpha>0,<0$ (respectively) to satisfy the normalisation condition $w^{\prime}(0)>0$. Typical free boundary shapes described by this
map are shown in figure 5.8; note that the extremal conformal maps corresponding to points $(\beta, \gamma)$ on the boundary of the univalency domain in $\beta>0$ describe fluid regions in which the bubble has collapsed to a slit, which is a circular arc (the limiting case of figure 5.8 (b))-such domains will be obtained as solutions as we allow the Capillary number to become unbounded (the ZST limit).

As another example, we may consider the solution of $\mathrm{J} \& \mathrm{M}$ as being the limit of yet another family of maps. In their solution the parameter $p_{\infty}$ was identically zero, because the map satisfied (5.32) near the point $\zeta=1$ (the preimage of infinity). We may consider more general maps having $p_{\infty}$ identically zero, provided we choose an appropriate form for them. We know that the conditions (5.30), (5.31) are the most general giving stagnant flow at infinity; we also know that (5.25) must hold globally. Hence if $\Phi(\zeta)$ is to remain $O(1)$ as $\zeta \rightarrow \gamma$, with $\mathcal{X}(\zeta)$ behaving as specified by (5.31), the function $\bar{w}(1 / \zeta)$ must have a pole at $\zeta=\gamma$; that is to say, if $w(\zeta)$ has a pole at $\zeta=\gamma$ within the unit disc, it must also have a corresponding pole at the inverse point, ( $\zeta=1 / \gamma$, in the case $\gamma \in \mathbb{R}$ ), outside the unit disc. We again consider a dipole singularity at the origin, so the function $\bar{w}(1 / \zeta)$ must have another simple pole at $\zeta=0$. Hence a possible mapping function giving a (steady) solution has the form

$$
\begin{equation*}
w(\zeta)=\alpha\left[\zeta+\beta \zeta\left(\frac{1}{\zeta-\gamma}-\frac{\gamma}{1-\gamma \zeta}\right)\right] \tag{5.58}
\end{equation*}
$$

We could consider more complicated options, but restrict ourselves to this simplest possibility satisfying all the requirements, since the solution of J \& M again emerges as a special limiting case. Note that unlike the last example, this map cannot describe bounded fluid domains.

The solution procedure is essentially the same as for the previous example, but the algebra is a little easier. Matching the singularity at the dipole in (5.25) gives (with the scalings of (5.50))

$$
\hat{A}\left(1-\beta\left(\gamma+\frac{1}{\gamma}\right)\right)+C a=0
$$

For the condition at infinity, we need only ensure that $\Phi(\zeta)$ has a finite limit as $\zeta \rightarrow \gamma$, since for this particular choice of mapping function the condition on $\mathcal{X}(\zeta)$ at infinity will then follow automatically from (5.25). With the functions $f_{+}(\cdot)$ redefined as in (5.51) this is easily seen to require

$$
\frac{\hat{A}\left(1-\gamma^{2}\right)}{1-\beta(\gamma+1 / \gamma)}+\frac{\gamma}{4}\left(F_{+}(0)-2 F_{+}(\gamma)\right)=0
$$

and

$$
\frac{\hat{A}\left(1+\gamma^{2}\right)}{1-\beta(\gamma+1 / \gamma)}+\frac{\gamma^{2}}{2} F_{+}^{\prime}(\gamma)=0
$$

Eliminating $\hat{A}$ between these three equations gives two equations for $\beta$ and $\gamma$, which also contain the Capillary number,

$$
\begin{align*}
\frac{C a\left(1-\gamma^{2}\right)}{(1-\beta(\gamma+1 / \gamma))^{2}} & =\frac{\gamma}{4}\left(F_{+}(0)-2 F_{+}(\gamma)\right)  \tag{5.59}\\
\frac{C a\left(1+\gamma^{2}\right)}{(1-\beta(\gamma+1 / \gamma))^{2}} & =\frac{\gamma^{2}}{2} F_{+}^{\prime}(\gamma) \tag{5.60}
\end{align*}
$$

Again, one may seek solution pairs $(\beta, \gamma)$ to these equations by postulating a particular value for $\gamma \in(0,1)$, eliminating $C a$ between equations (5.59) and (5.60), and finding the corresponding value(s) of $\beta$ which gives a solution (the corresponding Capillary number then follows from either equation). Thus, the equation to be solved for $\beta$, with $\gamma$ specified, is

$$
\frac{1-\gamma^{2}}{1+\gamma^{2}}=\frac{F_{+}(0)-2 F_{+}(\gamma)}{2 \gamma F_{+}^{\prime}(\gamma)}
$$



Figure 5.8: Typical free boundary shapes described by the mapping (5.44). Case (a) has $\alpha=-0.4, \beta=-5, \gamma=0.5$, and corresponds to a dipole such that the $x$-axis is a streamline from negative to positive. Case (b) has $\alpha=1, \beta=1.4, \gamma=0.8$, and has the $x$-axis as a streamline from positive to negative.


Figure 5.9: Typical free boundary shape described by the mapping (5.58). The parameter values used here are $\alpha=-1, \beta=3.5, \gamma=0.65$. The dipole at the origin is such that the $x$-axis is a streamline from negative to positive.
the $\beta$-dependence in this equation is entirely implicit in the functions $F_{+}(\cdot)$. Maps for solution pairs $(\beta, \gamma)$ describe a family of equilibrium bubble shapes in a flow with a dipole singularity at the origin. Only solution pairs $(\beta, \gamma)$ giving univalent maps are relevant; a typical configuration is shown in figure 5.9. For the map (5.58), the univalency domain in $(\beta, \gamma)$-space is bounded by the lines

$$
\beta=\frac{(1+\gamma)^{2}}{2 \gamma}, \quad \gamma=1
$$

so the allowed parameter regime is

$$
\beta>\frac{(1+\gamma)^{2}}{2 \gamma}, \quad \text { for } \gamma \in(0,1)
$$

There is no subset of the region $\beta<0, \gamma \in(0,1)$ which corresponds to a univalent map (we can find univalent maps with $\beta<0, \gamma \in(-1,0)$, but these are equivalent to those already considered). For points on the univalency boundary, the bubble shape described has a single cusp in the free boundary on the $x$-axis (at the point closest to the dipole). The situation shown in figure 5.9 corresponds to a dipole such that the $x$-axis is a streamline in the positive sense; for a dipole of the opposite strength we have the "mirror image" situation (with $\beta<0, \gamma \in(-1,0)$ ); there is no analogue of figure 5.8 (b).

### 5.6 Summary

This work of this chapter falls into two parts; the (ZST) Hele-Shaw results, and the Stokes flow results. We began by reviewing the relevant literature for each, which is substantial, for both the steady and time-dependent versions of the problems. Nevertheless, for the Stokes flow problem, there is a gap in the literature: no solutions (obtainable by the complex variable methods used in this thesis) exist for time-dependent problems on unbounded flow domains, having a free boundary, and driven by a singularity at some finite point within the flow. Our task, which proved to be far from trivial, was to find such a solution by generalising the well-known (dipole driven) steady solution of [52] to the time-dependent case.

The Hele-Shaw version of this problem was solved in $\S 5.3$ and found to be a reasonably straightforward adaptation of a problem solved in [79], but with more interesting solution behaviour, since the presence of the dipole singularity (rather than a point sink) necessitated a more complicated mapping function. In particular, the existence of a "transient $5 / 2$-power cusp" solution was found (see also [46], [50]), where the free boundary formed a cusp near the dipole, then immediately smoothed, with a little air entering through the cusp. The free boundary soon afterwards became nonanalytic again, with the formation of two $3 / 2$-power cusps in the free boundary (figure 5.7).

For the slow viscous flow analogue, we saw that no physically-relevant solution to the problem exists in the simplest case with $\Phi(0)=0$; since the Goursat function $\Phi(\zeta)$ must be bounded at a dipole singularity we were forced to conclude that $\Phi(0)$ be finite and nonzero. In this situation we were able to solve for the singular part of the primitive of the Schwarz function, and hence deduce the distribution of singularities of $\bar{w}(1 / \zeta)$ (the Schwarz function itself) within the unit disc. The main result was that a solution is only possible if the Schwarz function has a time-dependent line singularity within the unit disc. Due to the complexity of the conformal map needed, the solution was not completed.

We concluded the chapter by considering two (dipole driven) steady problems on unbounded fluid domains, both of which may be considered as generalising the solution of [52].

## Chapter 6

## Stokes flow with small surface tension


#### Abstract

We now abandon the simplification we have hitherto assumed in most of our solutions, and move on to consider the time-dependent problem with positive surface tension, or the NZST problem. We have mentioned already that remarkable analytical progress has been made with this problem, notably by Hopper [37, 38, 39], Richardson [82] and Howison \& Richardson [49] (and see also Tanveer \& Vasconcelos [96] in this context); Hopper uses somewhat more convoluted methods than ours to obtain his solutions. Two-dimensional problems that have been fully solved by the methods of $\S 3.3$ or by Hopper's method include:


- the coalescence under surface tension of two (equal or unequal) circular cylinders of fluid [37, 82];
- the coalescence under surface tension of a cylinder and a half-space of fluid [39];
- a limaçon-shaped fluid domain evolving under the action of surface tension only [82];
- the evolution of domains described by polynomial mapping functions of the form $w(\zeta)=$ $a\left(\zeta-b \zeta^{n} / n\right)$ for any integer $n>2$, evolving under the action of both surface tension and a point sink at the origin [49];
- the evolution of bubbles in shear flow with surface tension included [96];
- the evolution of expanding/contracting bubbles in quiescent flow with surface tension included [96].

The work that will most interest us here is that of Howison \& Richardson [49], which we shall henceforth refer to as HR'95, since they include the effects of both surface tension and a driving mechanism. In addition, they introduce a new concept, which we shall call a weak solution to the problem, and which we will be exploiting to solve a new problem.

### 6.1 Review of "weak" solutions

We begin by giving a short review of some of the work of HR'95, using their example to illustrate the "weak solution" concept. They consider fluid domains $\Omega(t)$ having a single point sink of strength $Q$ at the origin, which are described by the family of mapping functions

$$
\begin{equation*}
z=w(\zeta, t)=a\left(\zeta-\frac{b}{n} \zeta^{n}\right), \quad|\zeta| \leq 1 \tag{6.1}
\end{equation*}
$$

for $a, b$ real and positive functions of time, and integers $n \geq 2$. The maps (6.1) are univalent on the unit disc only if $b<1$, with $(n-1)$ inward-pointing $3 / 2$-power cusps forming simultaneously on $\partial \Omega$ if $b=1$.


Figure 6.1: The different regions in the small surface tension "limaçon" problem, using matched asymptotics.

We can circumvent some of the analysis of HR'95 if we use the results of $\S 3.6$, since we there gave the "moment" evolution equations (3.34), and we have a polynomial map, for which the "moments" were evaluated explicitly in equation (3.36). The form of the NZST equations (3.34) makes it clear why a polynomial of the form above is so much easier than a general polynomial; it means that we only need find $f_{+}(0)$, without worrying about any of the higher derivatives $f_{+}^{(r)}(0)$. The only tricky bit then is calculating $f_{+}(0)$ using (3.8); once this is done the evolution equations

$$
\begin{gather*}
\frac{d S}{d t}=\frac{d}{d t}\left[\pi a^{2}\left(1+\frac{b^{2}}{n}\right)\right]=-Q  \tag{6.2}\\
\frac{d}{d t}\left(a^{2} b\right)=-(n-1) \frac{T}{\pi \mu} a b K(b) \tag{6.3}
\end{gather*}
$$

are immediate. Here, $S(t)$ denotes the cross-sectional area of $\Omega(t)$ and $K(\cdot)$ denotes the complete elliptic integral of the first kind (see [8], [30], or appendix B, for example).

The authors considered an $(a, b)$ phase plane within the univalency domain $0 \leq b \leq 1, a \geq 0$, a solution trajectory $(a(t), b(t))$ reaching the boundary $b=1$ being associated with formation of $3 / 2$-power cusps. Solution breakdown is inevitable when $T=0$, with $b\left(t^{*}\right)=1, a\left(t^{*}\right)>0$ for
some positive "blow up" time $t^{*}$ (as expected, by the time-reversal argument of §1.4.2). However, when $T>0$ they found that one always has complete extraction of the fluid from the domain, with extraction time $t_{E}=S(0) / Q$ such that $a\left(t_{E}\right)=0, b\left(t_{E}\right)<1$. This naturally led them to consider the limiting case $T \rightarrow 0$ where, combining the previous two observations, cusps form in $\partial \Omega$ at time $t^{*}<t_{E}$, and persist until time $t_{E}$ in a kind of "weak solution" scenario, where a nonanalytic free boundary is permissible. This corresponds to a degenerate case of equation (6.3), where $K(b)$ on the right-hand side is singular as $b \uparrow 1$ (in fact the elliptic integral has the asymptotic behaviour $K(1-\epsilon) \sim-(1 / 2) \log (\epsilon / 8)$ as $\epsilon \rightarrow 0)$, but $T \rightarrow 0$ to counteract this effect. The net result is that $b(t)$ is 'pinned' at 1 for $t>t^{*}$, whilst from (6.2), $a(t)$ evolves according to

$$
\begin{equation*}
\frac{d}{d t}\left[a^{2}\left(1+\frac{1}{n}\right)\right]=-\frac{Q}{\pi} \tag{6.4}
\end{equation*}
$$

until $t=t_{E}$, ZST theory holding for $0<t<t^{*}$, of course. These weak ${ }^{1}$ solutions are all of similarity type for $t^{*}<t<t_{E}$, since only the scaling parameter $a$ is changing; the shape of the free boundary remains the same throughout. So for instance, the solution for $n=2$ (which is just the "limaçon" example of $\S 3.4$ ) will evolve as a shrinking cardioid for times $t>t^{*}$ in the limit $T \rightarrow 0$. The authors investigate the velocity field of the fluid in the neighbourhood of the "cusp", and find it to be finite, being exactly the velocity of the free boundary itself (so there is no entrainment of air into the cusp). This can be easily checked using the expression (C.1) in the $T \rightarrow 0$ limit.

We note that if $T$ is small and positive, the logarithmic nature of the singularity in $K(b)$ as $b \uparrow 1$ in (6.3) means that $b(t)$ must be exponentially close to 1 before surface tension effects become important, a fact borne out in experiments, where 'almost-cusps', having radii of curvature which are exponentially small in the Capillary number, can be observed; see for example [52]. ${ }^{2}$

In this small surface tension case, $0<T \ll 1$, matched asymptotics may also be used to solve the problem, with three distinct régimes (figure 6.1 illustrates this for the case $n=2$ ). The outer solution will be the $T \rightarrow 0$ shrinking cusped shape of HR'95. In the inner region, the free boundary will be locally parabolic, a configuration which was solved for in the NZST case by Hopper [40] (the "Stokes flow Ivantsov" solution). The invariance of Stokes flow under rigid-body motions means that this solution can be a travelling wave of arbitrary speed. The apex curvature of the particular parabola observed is a function of the far-field flow imposed (which here is due to the point sink). There will also be an intermediate region in which the inner and outer solutions are matched. In this region, the geometry is such that the free boundary may be linearised onto a slit, and the Stokes equations solved on an unbounded domain with appropriate matching conditions. Similar remarks apply to our solution of $\S 6.2$. We shall not consider matched asymptotics in this thesis-see [67] for full details of the method applied to a similar Stokes flow problem (the coalescence of two identical circular cylinders).

### 6.2 The cubic polynomial map

We now consider the ideas of HR'95 described above, applied to a general cubic polynomial mapping in the limit $T \rightarrow 0$. The ZST case of the analogous Hele-Shaw problem was solved by Huntingford in [50]. As explained, we expect the evolution of $\Omega(t)$ to follow ZST theory until the "blow up" time $t^{*}$, at which point we relax the restriction on $\partial \Omega$ to permit solutions with persistent cusps in the free boundary. For ease of manipulation we change notation slightly from that above, writing

$$
\begin{equation*}
w(\zeta, t)=a(t)\left(\zeta+\frac{b(t)}{2} \zeta^{2}+\frac{c(t)}{3} \zeta^{3}\right) \tag{6.5}
\end{equation*}
$$

[^23]The scaling factor $a$ may clearly be taken to be real and positive for all time. By suitably rotating the co-ordinates in the initial domain $\Omega(0)$, the general case with both $b(0)$ and $c(0)$ complex may be reduced to an initial map with just one complex coefficient. For simplicity we shall assume $b(0), c(0) \in \mathbb{R}$, which will then ensure $b(t)$ and $c(t)$ are real for $t>0$; this is equivalent to the assumption that $\Omega(t)$ is symmetric about the $x$-axis. We return to the limitations of our assumption in $\S 6.2 .1$. For this case, (3.34) and (3.36) yield

$$
\begin{align*}
\frac{d}{d t}\left[a^{2}\left(1+\frac{b^{2}}{2}+\frac{c^{2}}{3}\right)\right] & =-\frac{Q}{\pi}  \tag{6.6}\\
\frac{d}{d t}\left[a^{2} b\left(1+\frac{2}{3} c\right)\right] & =-\frac{T}{2 \mu} f_{+}(0) a^{2} b\left(1+\frac{2}{3} c\right)-\frac{T}{\mu} f_{+}^{\prime}(0) \frac{2}{3} a^{2} c  \tag{6.7}\\
\frac{d}{d t}\left[a^{2} c\right] & =-\frac{T}{\mu} f_{+}(0) a^{2} c \tag{6.8}
\end{align*}
$$

With $T=0$ these equations are valid until the time $t^{*}$ at which the map ceases to be univalent. As in [50] we must consider the domain $V$ in $(b, c)$-space for which (6.5) is univalent ${ }^{3}$ on $|\zeta| \leq 1$, and find the phase trajectories of the system (6.7), (6.8) within $V$. The determination of this univalency domain (in the more general case of complex coefficients) is the subject of [15]. For real coefficients, $V$ is symmetric about the $c$-axis (so we lose nothing by restricting attention to the right-half plane), and is bounded in $b>0$ by the lines

$$
\begin{equation*}
c=1, \quad b=1+c, \quad \text { and } \quad \frac{b^{2}}{4}+4\left(\frac{c}{3}-\frac{1}{2}\right)^{2}=1 \tag{6.9}
\end{equation*}
$$

The line $b=1+c$ corresponds to formation of a single $3 / 2$-power cusp on $\partial \Omega$, except for the isolated points $(0,-1)$ (where we have two $3 / 2$-power cusps, symmetrically placed about both axes), and ( $8 / 5,3 / 5$ ) (where we have a single $5 / 2$-power cusp). The line $c=1$ corresponds to two $3 / 2$-power cusps on $\partial \Omega$ (symmetrically placed about both axes when $b=0$ ), and the elliptical segment of $\partial V$ (which extends from $b=8 / 5$ to $b=4 \sqrt{2} / 3$ ) corresponds to loss of univalency by the free boundary beginning to overlap itself. The domain $V$, together with the phase paths for the $T \rightarrow 0$ solution, is shown in figure 6.4. Figure 6.2 shows free boundary "blow-up" shapes for various parameter values on $\partial V$.

Equations (6.7) and (6.8) give the ZST phase paths within $V$ as the curves

$$
\begin{equation*}
\frac{b}{c}\left(1+\frac{2 c}{3}\right)=\text { const }=k \tag{6.10}
\end{equation*}
$$

for various $k \in \mathbb{R}$. In contrast to the Hele-Shaw result of [50], we find no phase paths which meet $\partial V$ tangentially and then re-enter $V$; all ZST solutions blow up with the phase path hitting $\partial V$ obliquely. A tangent phase path is associated with the instantaneous formation of a cusp, which immediately smooths (when the phase path re-enters $V$ ), and the free boundary becomes analytic again (the only known examples of such behaviour for the Stokes flow [85] and Hele-Shaw problems involve transient 5/2-power cusps).

The ZST evolution is then fully determined, and we now consider the effect of small positive surface tension, as we approach $\partial V$ along a phase path. Using the definition (3.8), we are able to find exact expressions for $f_{+}(0)$ and $f_{+}^{\prime}(0)$ in terms of elliptic integrals. These exact expressions are necessary if we wish to consider the problem with $O(1)$ surface tension, but not very illuminating for the present discussion of the limit $T \rightarrow 0$; hence we relegate the details to appendix B. The main point to note is that they are singular only on the straight-line portions of $\partial V$ (i.e. those portions corresponding to blow-up via cusp formation rather than by overlapping) and so only in the neighbourhood of these lines will surface tension effects be significant, justifying our earlier assumption that ZST theory is adequate for $t<t^{*}$. To find the phase paths near $\partial V$, we combine

[^24]

Figure 6.2: Free boundary shapes described by the map (6.5) for various points $(b, c)$ on the boundary $\partial V$ of the univalency domain. The values used are: $\left(b_{1}, c_{1}\right)=(0,1),\left(b_{2}, c_{2}\right)=(1,1),\left(b_{3}, c_{3}\right)=(4 \sqrt{2} / 3,1)$, $\left(b_{4}, c_{4}\right)=(1.8,0.8461),\left(b_{5}, c_{5}\right)=(8 / 5,3 / 5),\left(b_{6}, c_{6}\right)=(1,0)$, and $\left(b_{7}, c_{7}\right)=(1 / 5,-4 / 5)$. Pictures (3b) and (4b) are magnifications of the nonunivalent region, showing how the free boundary begins to overlap itself; the former case is cusped and self-overlapping, while the latter is smooth. The value $a=1$ was used to generate each picture, hence the shapes do not have equal areas.
(6.7) and (6.8), writing $a^{2} c=A$, and $a^{2} b(1+2 c / 3)=B$, to give ${ }^{4}$

$$
\begin{equation*}
\frac{d B}{d A}=\frac{B}{2 A}+\frac{2}{3} \frac{f_{+}^{\prime}(0)}{f_{+}(0)} \tag{6.11}
\end{equation*}
$$

and hence it is only the ratio of $f_{+}^{\prime}(0)$ to $f_{+}(0)$ which is important.
We need to consider two separate cases, according as to whether the ZST solution breaks down by reaching $c=1$, or by reaching $b=1+c$ (refer forward to figure 6.4). Consider first the class of solutions for which $c \uparrow 1$ within $\partial V$, along a ZST phase path. Once $c$ has reached a value close to 1 , it is 'trapped' near $c=1$ until either the solution blows up (with $c=1$ and attendant cusp formation, or with $c \simeq 1, b \simeq 4 \sqrt{2} / 3$ on the elliptical portion of $\partial V$, and self-overlapping of the free boundary), or all fluid is extracted, since if $c$ decreased much below 1, ZST theory would again take over, forcing it back up towards $c=1$ on a ZST phase path. It follows that only $a$ and $b$ will be varying appreciably with time, and so $A \approx a^{2}, B \approx 5 a^{2} b / 3$. The results of appendix B show that near $c=1$,

$$
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \approx-\frac{b}{2} \approx-\frac{3 B}{10 A},
$$

hence ( 6.11 ) becomes

$$
\frac{d B}{d A} \approx \frac{3 B}{10 A}
$$

giving $B=$ (const.) $\times A^{3 / 10}$, or, in terms of the mapping function parameters,

$$
\begin{equation*}
b \approx(\text { const. }) \times a^{-7 / 5}, \quad c \approx 1 \tag{6.12}
\end{equation*}
$$

Knowing that ZST theory will hold until $c \approx 1$, we may take $t^{*}$ (the ZST "blow up" time) to be zero without loss of generality and proceed from there, so that, in the limit $T \rightarrow 0, c(t) \equiv 1$ throughout the motion. Thus, from (6.6) and (6.12), the equations to be solved are

$$
\begin{align*}
a^{2}\left(\frac{4}{3}+\frac{b^{2}}{2}\right)=\frac{S(0)-Q t}{\pi} & =a_{*}^{2}\left(\frac{4}{3}+\frac{b_{*}^{2}}{2}\right)-\frac{Q t}{\pi}  \tag{6.13}\\
\text { and } \quad b & =b_{*}\left(\frac{a_{*}}{a}\right)^{7 / 5} \tag{6.14}
\end{align*}
$$

where we use $S(t)$ to denote the area of $\Omega(t)$, and $a_{*}, b_{*}$ denote the starting values of $a$ and $b$ ( $c_{*}=1$, remember). The right-hand side of (6.13) is simply a linearly decreasing function of time, reaching zero at "extraction time" $t_{E}=S(0) / Q$. Substituting from (6.14) in (6.13) gives

$$
\begin{equation*}
G(b)-G\left(b_{*}\right)=-\frac{6 Q t}{\pi a_{*}^{2} b_{*}^{10 / 7}}, \quad \text { where } \quad G(b):=b^{-10 / 7}\left(8+3 b^{2}\right) \tag{6.15}
\end{equation*}
$$

Now, $G(b)$ is positive and monotone decreasing in $b$ on the range of interest (namely $0 \leq b \leq$ $4 \sqrt{2} / 3$ ), so (6.15) tells us that $b$ must be monotone increasing in $t$, from the starting value $b_{*}$. Hence the phase path must follow the line $c=1$ in this direction, ending either at time $t_{E}$, or when it reaches $b=4 \sqrt{2} / 3$. Complete extraction cannot occur in this régime, since (6.13) and (6.14) give the area of the fluid domain as

$$
S(t)=\pi\left(\frac{4}{3} a^{2}+\frac{b_{*}^{2} a_{*}^{14 / 5}}{2 a^{4 / 5}}\right),
$$

which is always positive. Hence we deduce that the phase path reaches $b=4 \sqrt{2} / 3$ before all the fluid has been extracted, and the solution breaks down with $\partial \Omega$ beginning to overlap itself (figure 6.2 (3a) and (3b)).

[^25]We now consider the case of solutions approaching the straight-line portion $b=1+c$ of $\partial V$ along a ZST phase path, observing, by the same argument as above, that a phase path will be 'trapped' near this line once it is sufficiently close to it (figure 6.4). We may thus eliminate either $b$ or $c$ in the ZST limit, and we choose to work with $b$ (so $c=b-1$ ). In this case, $A \approx a^{2}(b-1)$ and $B \approx a^{2} b(2 b+1) / 3$. The asymptotic evaluation of the ratio $f_{+}^{\prime}(0) / f_{+}(0)$ as the line $b=1+c$ is approached is performed in appendix B . This limit is found to be nonuniform on the range $b \in(0,8 / 5)$, being equal to -1 everywhere except at the single point $b=0$. Thus for $b>0$ (6.11) becomes

$$
\frac{d B}{d A} \approx \frac{B}{2 A}-\frac{2}{3}
$$

which has solution

$$
B=-\frac{4 A}{3}+\lambda \sqrt{|A|},
$$

for some constant $\lambda$. We again take $t^{*}=0$ without loss of generality, and our initial conditions must satisfy $b_{*}=1+c_{*}$ (where now $0 \leq b_{*} \leq 8 / 5$ ). In terms of $a$ and $b$ then, we have

$$
\frac{a}{|b-1|^{1 / 2}}\left(2 b^{2}+5 b-4\right)=\frac{a_{*}}{\left|b_{*}-1\right|^{1 / 2}}\left(2 b_{*}^{2}+5 b_{*}-4\right) \equiv 3 \lambda,
$$

holding together with the mass conservation equation (6.6) which, after some rearrangement, and putting $c=b-1$ (since we remain on this part of the univalency boundary), becomes

$$
\left(\frac{a}{a_{*}}\right)^{2} h(b)-h\left(b_{*}\right)=-\frac{6 Q t}{\pi a_{*}^{2}},
$$

for $h(b)$ defined by

$$
h(b):=5 b^{2}-4 b+8 \equiv 6\left(1+\frac{b^{2}}{2}+\frac{(b-1)^{2}}{3}\right) .
$$

Combining the previous two equations, eliminating the ratio $a / a_{*}$ between them, we finally arrive at an analogue of (6.15),

$$
\begin{equation*}
|b-1| \frac{h(b)}{g(b)^{2}}-\left|b_{*}-1\right| \frac{h\left(b_{*}\right)}{g\left(b_{*}\right)^{2}}=-\frac{6 Q t}{\pi a_{*}^{2}} \frac{\left|b_{*}-1\right|}{g\left(b_{*}\right)^{2}}, \tag{6.16}
\end{equation*}
$$

where $g(b):=2 b^{2}+5 b-4$. Ignoring the exceptional cases $b_{*}=1, \lambda=0$ for the moment (on our range of interest, $\lambda=0$ occurs if and only if $b_{*}=b_{c}=(-5+\sqrt{57}) / 4$ ), we see that the right-hand side of (6.16) is a monotone decreasing function of time, and so the left-hand side must be also, i.e. $F(b):=|b-1| h(b) / g(b)^{2}$ decreases with time. The area of the fluid domain is given by

$$
S(t)=\frac{\pi a_{*}^{2} h\left(b_{*}\right)}{6} \frac{F(b)}{F\left(b_{*}\right)},
$$

so complete extraction occurs if and only if $F(b)$ falls to zero; this corresponds to extraction time $t_{E}=\pi a_{*}^{2} h\left(b_{*}\right) /(6 Q)$. A plot of $F(b)$ on $(0,8 / 5)$ is given in figure $6.3(b=8 / 5$ is the point at which the form of $\partial V$ changes, the small elliptical portion of $\partial V$ for $8 / 5<b<4 \sqrt{2} / 3$ corresponding to blow up of solutions by overlapping of the free boundary). Important features to note are that:

- $F(b)$ vanishes only at $b=1$;
- $F(b)$ has a singularity at $b=b_{c}$, corresponding to a critical point in the phase diagram;
- $F(b)$ is monotone increasing (to infinity) on $\left(0, b_{c}\right)$, monotone decreasing (to zero) on ( $b_{c}, 1$ ), and monotone increasing on $(1,8 / 5)$;



Figure 6.3: The function $F(b)$ governing evolution on the part $b=1+c$ of $\partial V$. (Note the difference in scales between the two plots.)

- $F^{\prime}(b)=0$ at $b=8 / 5$, and only there, corresponding to the formation of the 5/2-power cusp.

Hence for $b_{*} \in\left(b_{c}, 1\right)$ and $b_{*} \in(1,8 / 5)$, the phase path will approach the point $b=1, c=0$, with complete extraction occurring when we reach this point, since we must have $F(b)$ decreasing with $t$. By contrast, if $b_{*} \in\left(0, b_{c}\right)$ we must have the phase path approaching $b=0, c=-1$. Since $F(0)>0$, this point is reached before all the fluid has been extracted, but due to the symmetry of the phase diagram about the $c$-axis, we are forced to stay at this point. For the moment we ignore the complications hinted at by the nonuniformity of the limit $f_{+}^{\prime}(0) / f_{+}(0)$ at this point.

Recall now the comment in footnote (3), that we have actually been considering the projection of a univalency cylinder by suppressing the parameter $a$. We are thus in one of the special cases considered in HR'95; the subsequent evolution will be of the 'similarity' type discussed there, with $b \equiv 0, c \equiv-1$, and the parameter $a$ changing in accordance with the corresponding mass conservation equation. The full phase diagram in the $(b, c)$-plane is given in figure 6.4 , with phase paths that are in some way 'special' represented by dashed lines. The bold arrows indicate the sense in which the phase paths 'turn around' as they hit $\partial V$.

It is now apparent that the 'exceptional cases' $b_{*}=1, b_{*}=b_{c}$ mentioned earlier are stable and unstable (respectively) critical points in the phase diagram, and thus also represent possible 'similarity' solutions of the kind studied in HR'95, the dotted phase path $c=0$ being exactly one of those solutions. Note that for this special solution, reaching $b=1$ no longer need be synonymous with total extraction, since the right-hand side of (6.16) is now identically zero; indeed, by the analysis of HR' 95 we do remain a finite time at $(1,0)$ before extraction is complete. The points $(0,-1)$ and $(0,1)$ are also critical points, stable (but see $\S 6.2 .1$ ) and unstable respectively, and again, are members of the family of similarity solutions of HR'95. We may summarise our results as follows:

- Phase paths which hit $\partial V$ at $(1,0)$ or $(0,-1)$ terminate there and represent stable similarity solutions, since adjacent phase paths are also entering these points.
- Paths which hit $\partial V$ at $(0,1)$ and $\left(b_{c}, 1-b_{c}\right)$ terminate there and represent similarity solutions which are unstable, since neighbouring paths are diverging.
- Paths for which $c_{*}=1, b_{*} \in(0,4 \sqrt{2} / 3)$ turn to the right and follow $\partial V$ along $c=1$, reaching $(4 \sqrt{2} / 3,1)$ before extraction is complete, at which point the free boundary begins to overlap itself. The present analysis then no longer applies, and the solution cannot be continued.


Figure 6.4: The univalency diagram (restricted to the right-half ( $b, c$ )-plane) for the cubic polynomial mapping function. The shaded region corresponds to a nonunivalent map.

- Paths hitting the curved portion of $\partial V$ likewise represent self-overlapping fluid domains, and cannot be continued.
- Paths for which $c_{*}=b_{*}-1,1<b_{*}<8 / 5$ turn around and enter the point $(1,0)$; reaching this point is simultaneous with complete extraction.
- Ditto for $b_{c}<b_{*}<1$.
- Paths for which $c_{*}=b_{*}-1,0<b_{*}<b_{c}$ turn around and enter $(0,-1)$, reaching this point before extraction is complete; subsequent evolution is of 'similarity' type and is discussed in HR'95.

In addition, an analysis of the velocity field in the neighbourhood of the cusp has been carried out, using the expression (C.1) with $T \rightarrow 0$. The same result as in HR'95 was found (although we do not give the analysis): the velocity at the cusp is exactly the (finite) velocity of the free boundary at that point, so there is no entrainment of air into the cusp.

The 5/2-power cusp is an interesting borderline case, being the point of transition between ZST solutions which break down via formation of a $3 / 2$-power cusp, and those which break down via overlapping of the free boundary. This path must still turn around and enter $(1,0)$; however, the fact that $F^{\prime}(8 / 5)=0$ (figure 6.3) implies that this path 'only just makes it'. Geometrically, the $5 / 2$-power cusp immediately becomes a $3 / 2$-power cusp, which then persists. The point $(1,0)$ on $\partial V$ corresponds to a cardioid, but since reaching this point is simultaneous with total extraction, this configuration is not actually attained.

The existence of the point $b_{c}$ is also interesting. As we move along $\partial V$ from $(1,0)$ towards $(0,-1), \partial \Omega(t)$ evolves continuously from a cardioid (with a cusp on the left-hand side), to a fully symmetric shape having cusps on both sides. As is does so, a 'dimple' develops on the right-hand side (see (6) and (7) in figure 6.2), which becomes more pronounced, eventually turning into the second cusp at $(0,-1)$. It is perhaps not surprising then that there is some critical point beyond which the 'dimple' is too large to disappear, and the ultimate shape has to have two cusps. If the dimple is small enough (i.e. $b_{*}>b_{c}$ ), then the ultimate shape will have just one cusp. For the solutions with $c_{*}=1$, however, the possible geometries are such that the two-cusp state is always unstable, and ultimate overlapping of the free boundary has to occur.

### 6.2.1 Complex coefficients

Recall that, near the start of $\S 6.2$, we stated that the assumption of real coefficients in the mapping function (6.5) was equivalent to assuming symmetry of $\Omega(t)$ about the $x$-axis. The results obtained seem to have a remarkably rich structure nonetheless; however they are somewhat deceptive, as consideration of the case with complex coefficients reveals.

A little thought about the conclusions of $\S 6.2$ throws up an apparent contradiction: the point $(0,-1)$ in $(b, c)$-space is stated to be a stable equilibrium point, whilst the point $(0,1)$ is an unstable equilibrium point. But the two configurations are actually identical, one being a rotation through angle $\pi / 2$ of the other. In fact, the conclusions regarding the point $(0,-1)$ were a little suspect anyway, since we knew the limit $f_{+}^{\prime}(0) / f_{+}(0)$ to be nonuniform at this point, but the analysis away from this point did indicate that it should be a stable equilibrium.

In the preceding analysis, we have been considering a single, two-dimensional cross-section of what is actually a four-dimensional univalency domain $V_{4}$ in complex $(b, c)$-space. In fact, bearing in mind the comments of footnote (3), the full univalency domain for the map (6.5) will be a cylinder in five-dimensional space, but the dependence on the scaling parameter $a$ is of no consequence. Determination of this domain is the subject of Cowling \& Royster's (henceforth C $\& R)$ paper [15]. There, the authors note that the cross-section $\Im\{b\}=0$ of $V_{4}$ is symmetric about the planes $\Re\{b\}=0$ and $\Im\{c\}=0$, and so it may be assumed without loss of generality that $\Im\{b\}=0$ and $\Re\{b\} \geq 0$ (since this will still generate all possible free boundary shapes, up to rotations and reflections). Writing $c=\gamma+i \mu$ and taking $b \in \mathbb{R}^{+}$, their paper then determines this three-dimensional cross-section $V_{*}$ of $V_{4}$ for which the map (6.5) is univalent on the unit disc; however this domain is not simple, and is given in an implicit form which is difficult to use.

Before exploring the structure of this domain further, we consider the changes wrought in the evolution equations for the coefficients by allowing them to be complex. We assume still that $a \in \mathbb{R}^{+}$, but now write $b=\beta+i \lambda$ and $c=\gamma+i \mu$, for real $\beta, \lambda, \gamma, \mu$. The equations governing the ZST problem are found from (3.32) and (3.36), and are

$$
\begin{align*}
\frac{d}{d t}\left[a^{2}\left(1+\frac{|b|^{2}}{2}+\frac{|c|^{2}}{3}\right)\right] & =-\frac{Q}{\pi}  \tag{6.17}\\
a^{2}\left(b+\frac{2 c \bar{b}}{3}\right) & =\text { constant }  \tag{6.18}\\
a^{2} c & =\text { constant. } \tag{6.19}
\end{align*}
$$

To obtain the phase paths in the four-dimensional $(\beta, \lambda, \gamma, \mu)$-space we must equate real and imaginary parts in these equations. The first is wholly real already, and in any case (as we have observed for the real coefficients case) is unnecessary for determination of the phase paths. The four real equations resulting from (6.19) and (6.18) are

$$
\begin{align*}
a^{2} \gamma & =k_{1},  \tag{6.20}\\
a^{2} \mu & =k_{2},  \tag{6.21}\\
a^{2}\left(\beta+\frac{2}{3}(\beta \gamma+\lambda \mu)\right) & =k_{3},  \tag{6.22}\\
a^{2}\left(\lambda+\frac{2}{3}(\beta \mu-\lambda \gamma)\right) & =k_{4}, \tag{6.23}
\end{align*}
$$

which, for various values of these four arbitrary constants $k_{1}$ to $k_{4}$, will give paths in $(\beta, \lambda, \gamma, \mu)$ space (after elimination of $a$ ). We now recall the statement of $\mathrm{C} \& \mathrm{R}$ that it is sufficient to consider the situation $\lambda=0, \beta>0$. Suppose we seek such solutions to the above equations (6.20)-(6.23). The first two are unchanged, whilst the second two become

$$
\begin{align*}
a^{2} \beta\left(1+\frac{2 \gamma}{3}\right) & =k_{3}  \tag{6.24}\\
a^{2} \beta \mu & =k_{4} . \tag{6.25}
\end{align*}
$$

Equations (6.20) and (6.21) give

$$
\frac{\gamma}{\mu}=\text { constant }
$$

whilst (6.24) and (6.25) give

$$
\frac{1}{\mu}+\frac{2 \gamma}{3 \mu}=\text { constant }
$$

which together imply that either both $\mu$ and $\gamma$ must be constant, or else $\mu=0$. Supposing the first case, with $\mu \neq 0$, then to satisfy the equations we need both $a$ and $\beta$ to be constant also, in which case the mass conservation equation cannot hold (except in the trivial case $Q=0$ ). Hence we must have $\mu=0$, showing that the only family of solutions for which $b \in \mathbb{R}$ throughout the evolution are those already found for which $c \in \mathbb{R}$ also.

The result of $\mathrm{C} \& \mathrm{R}$ essentially says that restricting attention to $V_{*}$ yields all possible free boundary shapes, the remainder of $V_{4}$ consisting of rotations and reflections of shapes which are contained within $V_{*}$. For a map with constant coefficients it is then sufficient to consider $V_{*}$, since any free boundary configuration can be generated by some point within $V_{*}$ provided the axes are suitably chosen. With time-dependent coefficients, we may choose axes such that the initial configuration $\Omega(0)$ is generated by a point of $V_{*}$; however the above shows that only if $\mu=0$ will the configuration for $t>0$ also be generated by a point of $V_{*}$. Solution trajectories for $\mu \neq 0$ will migrate to regions of $V_{4}$ outside $V_{*}$.

C \& R's observation is therefore of limited use, since the only family of solution trajectories lying wholly within the three-dimensional cross-section $V_{*} \subset V_{4}$ is the family of real solutions already studied-all other solution trajectories will simply intersect $V_{*}$ at a single point. The full four-dimensional space $V_{4}$ will be horribly difficult (if not impossible) to determine and study. We consider instead whether we might find a three-dimensional solution family for the case in which $c$ is real, but $b$ is complex. Setting $\mu=0$ in equations (6.20)-(6.23) gives

$$
\begin{gathered}
a^{2} \gamma=k_{1}, \quad k_{2}=0 \\
a^{2} \beta\left(1+\frac{2 \gamma}{3}\right)=k_{3}, \quad a^{2} \lambda\left(1-\frac{2 \gamma}{3}\right)=k_{4}
\end{gathered}
$$

since (6.21) has reduced to an identity, we are able to eliminate $a$ from these equations to find phase trajectories in $(\beta, \lambda, \gamma)$-space: these will be determined by the two equations

$$
\frac{\beta}{\gamma}\left(1+\frac{2 \gamma}{3}\right)=\text { constant }, \quad \frac{\lambda}{\gamma}\left(1-\frac{2 \gamma}{3}\right)=\text { constant. }
$$

To get an idea of this three-dimensional cross-section of $V_{4}$, call it $V_{\ddagger}$, we consider simple twodimensional cross-sections. The cross-section $\lambda=0$ is the case already studied (the domain $V$ given by (6.9)). The cross-section $\beta=0$ corresponds to maps of the form

$$
\frac{w(\zeta)}{a}=\zeta+\frac{i \lambda}{2} \zeta^{2}+\frac{\gamma}{3} \zeta^{3}
$$

Making the substitutions $\zeta=-i \hat{\zeta}, \gamma=-\hat{\gamma}$ and $w(\zeta)=-i \hat{w}(\hat{\zeta})$ we find that

$$
\frac{\hat{w}(\hat{\zeta})}{a}=\hat{\zeta}+\frac{\hat{\lambda}}{2} \hat{\zeta}^{2}+\frac{\hat{\gamma}}{3} \hat{\zeta}^{3}
$$

so the intersection of $V_{\ddagger}$ with this cross-section is exactly the domain $V$, but inverted with respect to $\gamma$; call it $V_{\dagger}$. Likewise, we will have a ZST solution family lying entirely within $V_{\dagger}$, with phase paths exactly as for the real coefficients case, but inverted with respect to $\gamma$. The $T \rightarrow 0$ limit is also inferred from the earlier analysis.

The other two-dimensional cross-section of $V_{\ddagger}$ we can look at is $\gamma=0$. This is particularly easy, the map now being

$$
\frac{w(\zeta)}{a}=\zeta+\frac{(\beta+i \lambda)}{2} \zeta^{2}
$$

so that $w^{\prime}(\zeta)=0$ only if $\zeta=-1 /(\beta+i \lambda)$, and the map is univalent on the disc

$$
\beta^{2}+\lambda^{2} \leq 1
$$

A solution family again lies in this cross-section (which we call $V_{o}$ ), with solution trajectories which are straight lines

$$
\frac{\beta}{\lambda}=\text { constant }
$$

as can be seen from (6.22) and (6.23) with $\gamma=0=\mu$. All points on the univalency boundary are equivalent, in the sense that the free boundary shapes represented by the maps are just rotations of the same cardioid. The $T \rightarrow 0$ limit of this solution family will be of the "similarity solution" type, with initial limaçons becoming cardioids (before all the fluid has been extracted) which then persist in a self-similar fashion until extraction is complete.

The schematic diagram 6.5 indicates how the three-dimensional domain $V_{\ddagger}$ fits together. Given the equivalence of the cross-sections $V$ and $V_{\dagger}$, we now see plainly the equivalence of the points


Figure 6.5: The three-dimensional univalency domain $V_{\ddagger} \subset V_{4}$, and its two-dimensional cross-sections $V, V_{\dagger}$ and $V_{o}$. The arrows on $V_{\dagger}$ indicate how the point $\{b=0, c=-1\}$ destabilises ( $c f$. figure 6.4).
$\{b=0, c=1\}$ and $\{b=0, c=-1\}$, and the arrows in figure 6.5 show how this configuration destabilises.

This is obviously not the full story, since we have considered only a limited subset $V_{\ddagger}$ of $V_{4}$, which happens to contain a family of solutions of which the real coefficients case is a subfamily. In fact this sub-family appears twice within $V_{\ddagger}$, as we have seen (in $V$ and in $V_{\dagger}$ ) so there is considerable repetition even within this limited subset. $V_{\ddagger}$ does not contain all possible free boundary configurations. On the other hand, the cross-section $V_{*}(\lambda=0)$ of $V_{4}$ is a minimal set of possible free boundary shapes if axes are chosen appropriately, but complex-parameter solutions do not lie wholly within this space. Evolution in time cannot be determined by studying $V_{*}$ then, unless the analysis is somehow modified to allow the coordinates within the fluid domain to rotate suitably in time-we do not consider this possibility.

In conclusion, it seems that a comprehensive study requires the determination of the full domain $V_{4}$, since we can say little more by considering three-dimensional cross-sections. We have at least resolved the apparent paradox of the real coefficients case, which appeared to show that identical configurations were at once stable and unstable.

### 6.3 Summary

In this chapter our principal concern has been with the (time-dependent) NZST Stokes flow problem, in the limiting case that the small positive surface tension tends to zero. We began by listing the more notable contributions to the NZST problem, most of which assume an $O(1)$ surface tension parameter. Only HR'95 have previously considered the $T \rightarrow 0$ limit in such problems (the "weak solution" concept), and their work is described in §6.1. Clearly, it only makes sense to do this when considering problems for which the ZST version undergoes finite time blow-up (the unstable suction problem, recall the comments in §1.4.2). For such problems the $T \rightarrow 0$ limit is not the same as the ZST problem: in the latter, the cusp formation in the free boundary is terminal, with solution breakdown occurring, but for the former this is not so.

Following HR' 95 we found a new weak solution for the suction problem, with a cubic polynomial mapping function with real coefficients, which we discussed at some length. Some of these solutions were found to permit the extraction of all the fluid from the fluid domain, while some underwent a topological change (self-overlapping), beyond which the solution could not be continued. In contrast to the ZST Hele-Shaw result of [50] (using the same mapping function), there is no continuable $5 / 2$-power cusp solution in the ZST case. ${ }^{5}$

The quadratic and cubic "similarity solution" families of HR'95 were seen to be a subset of our new family. Our solutions were seen to be unsatisfactory, however, in that two identical configurations are apparently both stable and unstable, a paradox which is a consequence of the "real coefficients" simplification. Hence, in $\S 6.2 .1$, we investigated the "complex coefficients" case. While a complete solution was found to be too complicated analytically, we were able to resolve the apparent contradiction, and arrive at an understanding of how our solution family fits into the much larger family of complex cubic solutions. In particular, it would appear that the generic "limiting configuration" (as extraction time is neared) for those solutions which do not undergo a change of topology, is the cardioid solution of HR'95, this being the only stable point of the univalency boundary.

It was also found (although we did not give the analysis) that the velocity field at the cusp is finite, and equal to the velocity of the free boundary at that point, so that there is no entrainment of air into the cusp (see HR'95 for the analysis in the simpler case considered there). Viewed as a regularisation of the Stokes flow suction problem, this may be contrasted with the analogous slit regularisation for the Hele-Shaw problem, which was mentioned in §1.4.1, and is discussed in detail in chapter 7 .

[^26]
## Chapter 7

## Crack and Anti-crack solutions to the Hele-Shaw model


#### Abstract

We recall the comments of $\S 1.4 .1$ where we mentioned the theory of "cracks" and their ZST limit, "slits", in the Hele-Shaw problem. In this chapter we shall give a brief overview of the theory of cracks and slits, and introduce the new concept of what we call anticracks, which although very different in behaviour to cracks, are in some sense complementary. We shall be considering the small, positive surface tension problem throughout, so it might be expected that the ZST model applies; however, we will see that the the nonzero surface tension is a crucial part of the theory.


### 7.1 Overview of cracks and slits

The crack/slit theories were developed by several authors in a series of papers [36, 62, 33] between 1988 and 1994. The models are an attempt to regularise the ill-posed ZST Hele-Shaw suction problem, which is known to exhibit finite-time blow-up via cusp formation in all but very special cases. The ZST model has to be invalid as such cusped configurations are approached, because the curvature $\kappa$ of the free boundary becomes very large at the forming cusp. When $\kappa=O(1 / T)$, where $T$ is the (small) surface tension parameter, the boundary condition $p=0$ on $\partial \Omega$ is no longer a valid approximation to the actual boundary condition $p=\kappa T$ on $\partial \Omega$. Thus, surface tension effects become important as we near cusp formation (the ZST blow-up time), but only in the neighbourhood of the cusp. This statement forms the basic premise of the models: they are, in effect, a local regularisation, acting only at isolated points of the boundary, which on physical grounds we expect to be those points at which blow-up of the ZST problem occurs. For definiteness, we assume a point sink-driven flow.

In the case of the crack model [62], the proposal is that the subsequent morphology (for times close to the ZST blow up time) is that of a thin finger of air (a crack) which penetrates the fluid domain and propagates rapidly towards the sink. By "rapidly", we mean that the motion of the rest of the free boundary is negligible compared to the motion of the crack, as long as this continues. Asymptotic methods are employed to study the evolution, and the crack geometry for later times is determined as an analytic continuation of some assumed initial geometry. Slits are cracks of zero thickness, but the slit model can be derived independently of the crack model, as we shall see.

Before presenting the theories, we mention some of the experimental and numerical evidence supporting them. Kopf-Sill \& Homsy [59], and Couder et al. [14] have observed narrow fingers (of thickness approximately $1 / 10$ the channel width) propagating in Saffman-Taylor experiments. These can be observed only under very carefully monitored conditions; in [59] for instance, the plates of the Hele-Shaw cell had to be scrupulously clean. The fingers destabilise via dendritic instabilities along the sides of the finger, which are on a much shorter lengthscale than the finger length itself, and are neglected in the crack theory to be described.


Figure 7.1: The geometry of (a) a finite crack; (b) a semi-infinite crack, along the $x$-axis (driven by a sink at infinity).

Kelly \& Hinch [55] and Nie \& Tian [69] have computed crack-type morphologies numerically. In [55], the problem of off-centre suction from an initially circular disc is solved using a boundary integral algorithm (the ZST version of this problem was solved analytically by Richardson in [79]). For small values of the surface tension parameter, the free boundary is observed to follow ZST theory approximately until the curvature is relatively high, then a thin finger of air advances towards the sink. In the computations of [69], similar geometries are found; these authors actually find that the solution breaks down via the "finger" reaching the sink before all the fluid has been sucked out.

Consider first the crack model. We begin with the assumption that a crack (either finite, occupying $-c(t)<x<c(t)$, or semi-infinite, occupying $-\infty<x<c(t)$; see figure 7.1) has already formed along the $x$-axis, and is described by

$$
y= \pm \epsilon h(x, t) ; \quad h(x, 0) \equiv h_{0}(x) \text { known }
$$

for some small parameter $\epsilon$, which is also used to rescale time, $t=\epsilon \tau$. Three regions are considered: the first is the outer region, in which the crack may be linearised onto the $x$-axis as a slit. In this region, with the above scalings, the evolution equation for the crack profile (in the case of the finite crack $-c(t)<x<c(t))$ is readily derived as

$$
\begin{equation*}
\frac{\partial h}{\partial \tau}=\frac{1}{\sqrt{c(\tau)^{2}-x^{2}}} \tag{7.1}
\end{equation*}
$$

It is straightforward to integrate this to find the crack profile for times $\tau>0$ if $x$ lies in the range $|x|<c(0)$, but for $|x|>c(0)$ the analytic continuation mentioned above is necessary, which relies
on the assumption that

$$
h(c(\tau), \tau)=0, \quad \tau \geq 0
$$

(Here $c(\tau)$ is assumed monotone increasing, which will be the case if $p_{\infty}<0$, by the maximum principle for Laplace's equation.) With this assumption the inverse function $\omega(x)$, such that $c(\omega(x)) \equiv x$, may be defined, and explicit expressions for $h(x, \tau)$ in both regions may be written down,

$$
\begin{align*}
& h(x, \tau)=h_{0}(x)+\int_{0}^{\tau} \frac{d \tau^{\prime}}{\sqrt{c\left(\tau^{\prime}\right)^{2}-x^{2}}}, \quad|x|<c(0)  \tag{7.2}\\
& h(x, \tau)=\int_{\omega(x)}^{\tau} \frac{d \tau^{\prime}}{\sqrt{c\left(\tau^{\prime}\right)^{2}-x^{2}}}, \quad|x|>c(0) \tag{7.3}
\end{align*}
$$

The latter equation here is the analytic continuation of the former. For the particular example of a finite crack with an initially elliptical profile, the shape of the crack for later times turns out to be a narrow ellipse [62].

There is also an inner region for the problem, in which the crack is considered to have an $O(1)$ thickness (by rescaling $y$ with $\epsilon$ ), and a tip region, in which both $(x-c(t))$ and $y$ are scaled with $\epsilon^{2}$, and surface tension effects are important. In [36], conjectures are made which suggest that the matching at the tip is only possible if $\epsilon=T^{1 / 3}$. (In this paper, a balance of terms in the tip region is achieved by scaling distances with $T^{2 / 3}$, time with $T^{-1 / 3}$, and pressure variations with $T^{1 / 3}$; with these scalings the tip speed is $O\left(T^{-1 / 3}\right)$.)

The procedure of analytic continuation of the initial geometry $h_{0}(x)$ is ill-posed, often leading to the formation of singularities in the free boundary within finite time, with subsequent blow-up of the model. Examples of how this can occur are given in [62], and speculation is made about how the crack might evolve through certain types of singularities via tip-splitting.

The slit model (Hohlov et al. [33]) on the other hand is well-posed, relying on conformal mapping ideas which by now should be familiar to the reader. A slit is essentially a crack of zero thickness, which again is postulated to propagate rapidly into the fluid region. In this limiting case, "rapidly" will mean on a timescale such that the rest of the free boundary is actually stationary whilst the slit is in motion. On physical grounds, as for the crack model, the expectation is that slits will grow from the ZST blow-up points of the free boundary; however, the slit model suffers from being seriously under-determined, permitting slits to propagate from arbitrary points of the free boundary along arbitrary paths, and to branch in any chosen way.

If a well-posed slit model exists, it ought to be realisable as the (ZST) limit of some crack model, and we will see that this is indeed the case. The theoretical framework for the model relies on first rewriting the $\mathrm{P}-\mathrm{G}$ equation (2.4) in the form

$$
\begin{equation*}
\Re\left(\frac{w_{t}(\zeta)}{\zeta w^{\prime}(\zeta)}\right)=\frac{-Q}{2 \pi\left|w^{\prime}(\zeta)\right|^{2}} \quad \text { on }|\zeta|=1 \tag{7.4}
\end{equation*}
$$

(where the prime denotes $\partial / \partial \zeta$ ); as usual we take $Q>0$ to represent a sink, since we are only interested in the suction problem. We have already mentioned that the model is under-determined; however, physical considerations suggest that we are interested in slit evolution for times $t>t^{*}$ ( $t^{*}$ being the blow-up time for the ZST model), and from the point(s) on $\partial \Omega\left(t^{*}\right)$ at which the cusp forms. Time is thus rescaled for $t>t^{*}$ with some small parameter $\epsilon$,

$$
\begin{equation*}
t-t^{*}=\epsilon \tau \tag{7.5}
\end{equation*}
$$

and the "slit mapping" is denoted by

$$
\begin{equation*}
W(\zeta, \tau)=w\left(\zeta, t^{*}+\epsilon \tau\right) \tag{7.6}
\end{equation*}
$$

For simplicity, suppose a cusp has formed at $w\left(-1, t^{*}\right)$ on the real negative axis, with the sink lying at $z=0$ (as in figure 2.3, for example). To lowest order in $\epsilon,(7.4)$ becomes

$$
\Re\left(\frac{W_{\tau}(\zeta)}{\zeta W^{\prime}(\zeta)}\right)=0
$$

holding on $|\zeta|=1$, except at $\zeta=-1$, where the right-hand side is in fact unbounded (so we expect something like a delta-function there). In the absence of other singularities on $|\zeta|=1$ Hohlov et al. derive the appropriate form of (7.4) as

$$
\Re\left(\frac{W_{\tau}(\zeta)}{\zeta W^{\prime}(\zeta)}\right)=\Re\left(-a(\tau)\left(\frac{1-\zeta}{1+\zeta}\right)\right)
$$

for an arbitrary real, positive function $a(\tau)$. Rescaling time again, and analytically continuing this equation, they obtain a version of Löwner's differential equation for the mapping function $W$,

$$
\begin{equation*}
\frac{W_{\tau}(\zeta)}{\zeta W^{\prime}(\zeta)}=-\left(\frac{1-\zeta}{1+\zeta}\right) \tag{7.7}
\end{equation*}
$$

holding globally. This is a linear, hyperbolic p.d.e., with solution

$$
W(\zeta, \tau)=F\left(e^{-\tau} K(\zeta)\right)
$$

for arbitrary functions $F(\cdot)$. Here $K(\cdot)$ is the Koebe map of univalent function theory (see [20]), defined by

$$
K(\zeta)=\frac{\zeta}{(1-\zeta)^{2}}
$$

which maps the unit disc onto the whole complex plane, minus the slit $(-\infty,-1 / 4]$ along the real axis. The function $F(\cdot)$ is determined from the initial data,

$$
F(K(\zeta))=W(\zeta, 0) \equiv w\left(\zeta, t^{*}\right)
$$

Hence the "slit mapping" $W(\zeta, \tau)$ maps the unit disc onto the region $\Omega\left(t^{*}\right)$, minus the image (under $F$ ) of the slit $\left[-1 / 4,-e^{-\tau} / 4\right]$, so that the smooth portion of the free boundary remains fixed at $\partial \Omega\left(t^{*}\right)$, whilst the point which was at $w\left(-1, t^{*}\right)$ has travelled into $\Omega\left(t^{*}\right)$ as far as the point $F\left(-e^{-\tau} / 4\right)$. This is illustrated schematically in figure 7.2. By choosing a suitable distribution of delta-functions on the right-hand side of the P-G equation, more general versions of Löwner's equation are obtained, and the above can be generalised to a model which allows slits to propagate from arbitrary points of the free boundary, and in a specified manner (this is described in [33]), hence the indeterminacy of the model referred to earlier. However, the justification for writing down (7.7) had a physical basis, in that the singularity on the right-hand side was placed at $\zeta=-1$ because this is where the zero of $w^{\prime}\left(\zeta, t^{*}\right)$ occurred.

The link with the crack model of $[36,62]$ can now be demonstrated. With time scaled as in (7.5), we assume that the "slit" map $W(\cdot)$ of (7.6) represents only the first term in a simple perturbation of the actual mapping function, in powers of $\epsilon$ (the crack thickness), so that the free boundary will be an $O(\epsilon)$ distance away from the slit. Thus, (7.6) is replaced by

$$
w(\zeta, t)=W(\zeta, \tau)+\epsilon W_{1}(\zeta, \tau)+O\left(\epsilon^{2}\right)
$$

We again substitute into (7.4); at lowest order we retrieve the slit problem, but at order $\epsilon$ we get

$$
\begin{equation*}
\Re\left(\zeta \frac{\partial W}{\partial \zeta} \frac{\overline{\partial W_{1}}}{\partial \tau}+\zeta \frac{\partial W_{1}}{\partial \zeta} \frac{\overline{\partial W}}{\partial \tau}\right)=-\frac{Q}{2 \pi} \quad \text { on }|\zeta|=1 \tag{7.8}
\end{equation*}
$$

For the simple "paradigm" problem of a finite crack $|x|<c(\tau)$ driven by a sink at infinity, with sides $y= \pm \epsilon h(x, \tau)$, the slit mapping is exactly

$$
W(\zeta, \tau)=\frac{c}{2}(\zeta+1 / \zeta)
$$

Substitution of this form for $W$ into (7.8) with $\zeta=e^{i \theta}$ and the normalisation $Q=2 \pi$, and noting that at leading-order $h(x, \tau)=\Im\left(W_{1}\left(e^{i \theta}\right)\right)$, and that $x=c(\tau) \cos \theta$, leads eventually to

$$
\frac{\partial h}{\partial \tau}=\frac{1}{\sqrt{c(\tau)^{2}-x^{2}}}
$$

Time $\tau=0$


Time $\tau>0$



Figure 7.2: Schematic diagram showing how a general slit solution works.
which is exactly equation (7.1) of the crack model.
We can also study the crack model in terms of the Schwarz function of the free boundary, and its singularities. We know that the free boundary, assuming it is analytic, may be written in the form

$$
\bar{z}=g(z, t)
$$

where the Schwarz function $g$ is analytic in some neighbourhood of the free boundary. It was shown in $\S 2.3$ (equation (2.10) and the comments below it) that although $g(z, t)$ may have singularities both inside and outside the fluid domain, those within the fluid remain constant in both position and time, whilst those outside the fluid may move around, and vary in strength. In the example of $\S 2.4$, blow-up of the ZST problem was associated with an external square-root singularity of $g$ reaching the free boundary.

For the symmetric crack $y= \pm \epsilon h(x, \tau)$, either finite or semi-infinite, the Schwarz function representation takes the form

$$
\bar{z}=z \pm 2 i \epsilon h(z, \tau)+O\left(\epsilon^{2}\right)
$$

(the + , - referring to the lower, upper sides of the crack respectively) and hence the Schwarz function of the crack boundary has singularities wherever $h$ does. Consider the singularity near the crack tip for the finite crack lying along $-c(\tau)<x<c(\tau)$. The leading-order behaviour of $h$ may be found from (7.3) as

$$
h(x, \tau) \sim \frac{1}{c(\tau)}\left[\frac{2(c(\tau)-x)}{c(\tau)}\right]^{1 / 2}
$$

an expression which is easily verified by checking in (7.1), with $x=c-\delta$ for some small $\delta$. A similar result holds for the case of the symmetric semi-infinite crack (which is a parabolic, or "Ivantsov" crack) [62]. Hence in general, the assumption is that the function $h$ will have a squareroot singularity at the crack tip, which may be interpreted as a square-root singularity of the Schwarz function, just inside, and $O(\epsilon)$ distant from, the tip (note that "inside the tip" here is actually outside the fluid domain, since the crack is the narrow finger of air).

Since we have now seen how the two models are linked, we shall henceforth use the term "crack" to denote either a crack or a slit solution, on the understanding that the slit is the ZST limit of the crack. We reserve the term "slit" for emphasis, when we are considering only the slit model. It is helpful at this stage to summarise by listing the more important points of the theory, for later comparison with the anticrack results and conjectures.

- Cracks are born at singularities of the ZST problem, and can propagate only in the case $T>0$.
- Crack tip speeds and crack widths are determined by surface tension effects near the tip.
- As $T \rightarrow 0$, the Schwarz function $g(z)$ has a square-root singularity at the crack tip.

We conclude this section by mentioning the work of King et al. [56] on Hele-Shaw flows where the initial geometry has a corner of internal angle $\phi$ in the boundary (that is, $\phi$ is the angle measured within the fluid domain; see the definition sketch of figure 7.9), since we shall cite their results later as evidence for the anticrack theory. They prove that, for the ZST suction problem, if the angle $\phi$ lies strictly between $\pi$ and $2 \pi$ (the case $2 \pi$ being some kind of inward-pointing cusp in the free boundary), there is no solution to the model for later times. In the case that the corner angle is $2 \pi$ there is still no solution for $t>0$, save in the special case that the boundary has a $(4 n+1) / 2$-power cusp (for integer $n$ ); this result is deduced from a similar result for the related obstacle problem of variational calculus [90, 64]. This is the kind of "blow-up" geometry we have in mind for regularisation by slit propagation; this result shows that the surface tension effects at the slit tip are an essential part of the model.

### 7.2 Introduction to Anticracks

Having reviewed the fundamental ideas behind cracks and slits, we turn now to what we view as the complementary phenomenon of anticracks. These too will only arise in the ill-posed "suction" (and not in the "blowing") problem.

As the name suggests, in this case, instead of having thin fingers of air penetrating the fluid domain, we now have thin fingers of fluid which are "left behind" as the free boundary progresses (refer forward to figure 7.5). The two situations are very different, despite the superficial similarities in the geometry. We know (by the maximum principle for harmonic functions) that, at least for the ZST problem, the free boundary must always be moving down the pressure gradient (for example, towards the sink, in a problem with a single point sink). Hence, although fingers of air are able to propagate into the fluid in a general suction problem, fingers of fluid cannot propagate out but must, as we said, be "left behind". Thus the anticracks themselves are very stagnant (the motion of the tip will be seen to be exponentially small for large time), although the rest of the free boundary is able to propagate smoothly.

Such structures are observable in Hele-Shaw experiments (Paterson [73] and Chen [11]), and large families of exact solutions to the ZST model exist (Howison [45], Mineev-Weinstein \& PonceDawson [66]) which demonstrate anticrack formation. In the experiments of [73] and [11] a less viscous fluid is injected at a constant rate into an expanse of more viscous fluid in a large HeleShaw cell, and time-lapse photographs of the evolving interface are presented. The photographs of both papers are remarkably similar, showing clearly that long thin fingers (the anticracks) of the more viscous fluid are left behind as the free boundary advances. Typical results of the experiments of [73] are reproduced in figure 7.3. It is difficult to see from these pictures, but there is a tendency for the tips of the anticracks to be slightly bulbous for large times.

Paterson performs a linear stability analysis for the problem of an expanding circular "bubble" of the less viscous fluid, which provides a surface tension dependent "selection mechanism" for the fingers, predicting firstly at what bubble radius they will start to form, and secondly, which of the unstable finger wavelengths is fastest-growing when this radius is reached. We reconsider this analysis in $\S 7.6$.

The radial fingering solutions of [45], which are briefly considered in $\S 7.3 .3$, can be made to mimic closely the experimental results of [73] and [11] if the parameters in the conformal map are chosen appropriately (figure 7.3). Essentially, "choosing appropriately" means that the mapping function is chosen to give a judicious distribution of singularities of the Schwarz function within the fluid domain, which by the results of $\S 2.3$, must remain fixed within the flow domain, and which the free boundary cannot cross. In [45], a large class of fingering solutions in a channel geometry is also found, which are generalisations of the time-dependent Saffman-Taylor finger [88]. When wide finger solutions of this type are considered as a periodic array, the anticracks become apparent as the strips of fluid separating these air fingers, which were adjacent to the wall and asymptotically stagnant in the channel geometry. The solutions of [66], although non-periodic, are very similar to these solutions.

Remarkably similar geometries to the radial fingering solutions (and experiments) have recently been computed by Elliott \& Gardiner [23], for the closely-related problem of the growth of a seed of solid into a supercooled liquid (figure 7.4). The authors use the phase field equations (with suitably chosen parameters) to approximate the isotropic Stefan problem with Gibbs-Thomson undercooling. The phase field equations are known to exhibit a wide range of possible behaviours in different parameter-group limits; in particular, various types of Stefan problem can be represented in this way, as well as the Hele-Shaw problem itself (see [9] for more discussion of these matters).

In this chapter, we regard cracks and anticracks as two different possible regularisations of the ill-posed "suction" problem. A natural question to ask, then, is: What will determine which instability is observed in a particular physical situation? This is a key question, to which we return in $\S 7.9$; although we are unable to give a categorical answer, we make a conjecture (backed up by analytical, numerical and experimental evidence). Another question of interest concerns the effect of surface tension on anticracks. We saw in $\S 7.1$ that the crack/slit theory is crucially dependent

Figure 7.3: Examples of the radial fingering solutions of [45], together with a photograph of one of Paterson's experiments [73].


Figure 7.4: Phase-field computations of the "free boundary" (actually a level set of the phase parameter $\phi$ occurring in the phase field model) for the growth of a seed of solid into a supercooled liquid. This picture was kindly supplied by Dr A. R. Gardiner [23].
on surface tension, since there is no solution to the ZST problem beyond the time at which it blows up with formation of a $3 / 2$-power cusp [56] (which is the kind of situation we expect to be regularised by crack formation). For the case of anticracks, however, there exist large classes of solutions to the ZST problem which exist for all time, with analytic free boundaries. These boundary curves should not be very different to those for the corresponding small positive surface tension problem, then-although surface tension must be the selection mechanism determining which solution is observed. We also remark that the tendency of the anticrack tips to become slightly bulbous for large times observed in [73] and [11] must be attributable to surface tension effects, since the tip is the most highly curved part of the free boundary in a narrow anticrack.

### 7.3 Exact ZST anticrack solutions

Before making any more general comments, we present some exact ZST solutions to the HeleShaw problem which exhibit anticrack formation. It is useful to do this, firstly because it makes clear what the mathematical description of an anticrack is, and secondly, because by the above comments regarding the small surface tension problem, we expect such solutions to be able to provide a good description of the actual behaviour observed. We will see, however, that the ZST problem is underdetermined, having solutions which allow a more or less arbitrary array of anticracks to be generated by suitable choice of the parameters in the conformal maps.

### 7.3.1 The "generic" anticrack

We first present what we consider to be the "generic" (and certainly the simplest) Hele-Shaw anticrack solution, since it captures all the essentials of the behaviour. It is a solution on a semiinfinite fluid domain, driven by a constant negative pressure gradient at infinity, and is most easily obtained by use of a conformal map from the right-half plane onto the fluid domain. The map we consider is very simple,

$$
\begin{equation*}
w(\zeta, t)=\zeta+U t+\alpha \log (a(t)+\zeta) \quad \Re(\zeta) \geq 0 \tag{7.9}
\end{equation*}
$$

where $U$ and $\alpha$ are positive constant parameters, and $a(t)$ is a function of time (assumed real, without loss of generality). This map is easily seen to be a solution of the problem, for arbitrary $\alpha$, provided $a$ satisfies the condition

$$
\begin{equation*}
a+U t+\alpha \log a=\text { const. }=a_{0}+\alpha \log a_{0} \tag{7.10}
\end{equation*}
$$

where $a_{0}=a(0)$. This may be seen either by using the P-G equation (2.6), or by the results of $\S 2.3$ (that those singularities of the Schwarz function within the physical domain must remain fixed), noting that for a map from the right-half plane onto the fluid domain, the Schwarz function is given by

$$
\begin{equation*}
g(z(\zeta)) \equiv \bar{w}(-\zeta) \tag{7.11}
\end{equation*}
$$

Equation (7.10) is not explicitly solvable for $a(t)$, but the large-time behaviour, when the anticrack is well-developed, is easy to obtain. The only possible large-time balance is $a \rightarrow 0$ as $t \rightarrow \infty$, with the constant $\alpha$ necessarily positive. (If $\alpha<0$ a solution for $a(t)$ must cease to exist within finite time.) Since here we are only interested in the large-time behaviour, we assume initial conditions such that the right-hand side of equation (7.10) vanishes; the asymptotic behaviour of $a(t)$ is then

$$
\begin{equation*}
a(t) \sim \exp (-U t / \alpha) \tag{7.12}
\end{equation*}
$$

The anticrack tip is at

$$
\begin{align*}
z_{t i p} & =w(0) \\
& =U t+\alpha \log a \\
& =-a, \tag{7.13}
\end{align*}
$$

the last equality following from (7.10) with zero right-hand side. Hence with the asymptotic behaviour of (7.12), the anticrack tip approaches exponentially close to the origin, but never actually reaches it. This asymptotic stagnation of the tip is associated with a logarithmic singularity of the Schwarz function (given by (7.11)) at the point

$$
\begin{aligned}
z=w(a) & =U t+a+\alpha \log (2 a) \\
& =\alpha \log 2
\end{aligned}
$$

which the free boundary cannot cross while the solution exists. The free boundary itself is given by the image of the imaginary $\zeta$-axis, $\zeta=i \eta$. Taking real and imaginary parts in $x+i y=w(i \eta)$ gives

$$
\begin{aligned}
x & =U t+\frac{\alpha}{2} \log \left(a^{2}+\eta^{2}\right) \sim U t+\frac{\alpha}{2} \log \left(\eta^{2}+\exp (-2 U t / \alpha)\right) \\
y & =\eta+\alpha \tan ^{-1}(\eta / a) \sim \eta+\alpha \tan ^{-1}(\eta \exp (U t / \alpha))
\end{aligned}
$$

There are three distinct parts to the free boundary, corresponding to different régimes for $\eta$. The anticrack tip is described by $|\eta| \ll \exp (-U t / \alpha) \ll 1$, or equivalently, $|\eta| \ll a$, so that according to the above expressions,

$$
\begin{aligned}
x \sim 0, \quad y & \sim \alpha \tan ^{-1}(\eta \exp (U t / \alpha)) \\
& \sim \alpha \eta \exp (U t / \alpha)
\end{aligned}
$$

(although a better approximation for $x$ is given by (7.13) and (7.12)). For a narrow anticrack, $\alpha \ll 1$, and if we rescale $x=\alpha X, y=\alpha Y$, then in the tip region,

$$
2 X \sim \log \left(1+\eta^{2} / a^{2}\right), \quad Y \sim \tan ^{-1}(\eta / a)
$$

and the general equation of the anticrack tip is

$$
Y=\cos ^{-1}\left(e^{-X}\right)
$$

The anticrack sides are given by $\exp (-U t / \alpha) \ll|\eta| \ll \alpha \pi / 2$, where

$$
x \sim U t+\alpha \log |\eta|, \quad y \sim \pm \frac{\alpha \pi}{2} .
$$

The advancing free boundary is given by $|\eta| \gg \alpha \pi / 2$, and for $\eta$ in this range,

$$
x \sim U t+\alpha \log |y|
$$

Intermediate values of $\eta$ represent the smooth transition between these three regions; in particular, we refer to the transition between the anticrack sides and the advancing free boundary as the anticrack root. A typical large-time free boundary is shown in figure 7.5. All such free boundaries are of self-similar form in any case, there being only one free parameter in this solution $(\alpha / U$, the ratio of the anticrack width to the uniform pressure gradient). Away from the tip, the rest of the anticrack cannot "feel" its presence, and the solution is essentially a travelling wave,

$$
\begin{equation*}
z=w(\zeta)=\zeta+U t+\alpha \log \zeta \tag{7.14}
\end{equation*}
$$

with an infinitely long anticrack of width $\alpha \pi$.
The pressure field within the anticrack is given by (2.1) and (2.5) as

$$
p=-\Re(\zeta)
$$

Near the tip,

$$
z \sim U t+\alpha \log (a+\zeta)
$$

so with $a$ given approximately by (7.12), the pressure field in the tip region is approximately

$$
p \sim \exp (-U t / \alpha)\left(e^{z / \alpha}-1\right)
$$



Figure 7.5: A typical "generic anticrack" solution. The free boundary is shown for times $t=t_{1}, t_{2}, t_{3}$, with $t_{3}>t_{2}>t_{1}$.

### 7.3.2 Solutions with many anticracks

Having seen how the solution for a single anticrack works, it is a straightforward matter to generalise to an arbitrary array of parallel anticracks, by suitable choice of the logarithmic singularities in the mapping function $w(\zeta)$. The solutions of this section were also given in [66] in the context of general Laplacian pattern formation (which of course includes the ZST Hele-Shaw problem), and are closely related to the "channel geometry" solutions of [45] and [88], although the derivation as given here was independent of this work.

The obvious form of mapping function to try is

$$
\begin{equation*}
w(\zeta)=\zeta+U t+\sum_{r=1}^{N} \alpha_{r} \log \left(a_{r}(t)+\zeta\right) \tag{7.15}
\end{equation*}
$$

where all the $\alpha_{r}$ are assumed to be real positive constants, and the $a_{r}(t)$ functions of time with positive real parts. This will give a solution of the P-G equation (2.6) provided the singularities of the Schwarz function (7.11) remain fixed in the physical region, that is, provided the following conditions hold:

$$
\begin{equation*}
a_{n}(t)+U t+\sum_{r=1}^{N} \alpha_{r} \log \left(\overline{a_{r}(t)}+a_{n}(t)\right)=k_{n}, \quad n=1, \ldots, N, \tag{7.16}
\end{equation*}
$$

for $N$ complex constants $k_{n}$. The parameters $\alpha_{n}$ represent the anticrack widths (for large times, when the anticracks are well-developed), and $(-U)$ is the pressure gradient at infinity. As with the generic solution, $U$ may be scaled out of the problem by dividing each of the $\alpha_{n}$ by $U$. The large-time behaviour of these solutions requires a little more care than the previous example, since we are now dealing with complex quantities. Writing $a_{n}=p_{n}+i q_{n}$ and $k_{n}=\lambda_{n}+i \mu_{n}$ the equations (7.16) become:

$$
\begin{align*}
p_{n}+\sum_{r=1}^{N} \frac{\alpha_{r}}{2} \log \left[\left(p_{n}+p_{r}\right)^{2}+\left(q_{n}-q_{r}\right)^{2}\right] & =\lambda_{n}-U t,  \tag{7.17}\\
q_{n}+\sum_{r=1}^{N} \alpha_{r} \tan ^{-1}\left(\frac{q_{n}-q_{r}}{p_{n}+p_{r}}\right) & =\mu_{n} . \tag{7.18}
\end{align*}
$$

A large-time balance in (7.17) is only possible if each of the functions $p_{n}(t)$ is decaying exponentially with time; the $q_{n}(t)$ will remain $O(1)$. If we assume that $q_{n} \neq q_{m}$ for $n \neq m$ then the asymptotic behaviour of (7.17) will be

$$
p_{n}(t) \sim \frac{1}{2} \exp \left(\frac{\lambda_{n}-U t}{\alpha_{n}}\right)
$$

whilst that of (7.18) will be

$$
q_{n}(\infty)=\mu_{n}+\frac{\pi}{2} \sum_{r \neq n} \alpha_{r} \operatorname{sgn}\left(q_{r}(\infty)-q_{n}(\infty)\right)
$$

If we assume the ordering is chosen such that $q_{1}(0)<q_{2}(0)<\ldots<q_{N}(0)$ (which will then persist for later times), we have

$$
\begin{equation*}
q_{n}(\infty)=\mu_{n}+\frac{\pi}{2} \sum_{r>n} \alpha_{r}-\frac{\pi}{2} \sum_{r<n} \alpha_{r}, \tag{7.19}
\end{equation*}
$$

hence the $q_{n}$ approach known, constant values as $t \rightarrow \infty$.
This solution exists globally in time, with the parameter restrictions we have imposed. It describes an array of $N$ parallel anticracks, the $n$ 'th one having its tip at the point

$$
z_{n}=w\left(-i q_{n}\right)
$$



Figure 7.6: A typical solution generated by (7.15), showing 4 well-developed anticracks. Here, $\pi \alpha_{i}=$ $0.5,1,0.8,0.4$, for $i=1,2,3,4$, respectively; $\mu_{i}=-3,-2,0.5,1.5$, and $\lambda_{i}=0.5$ for each $i$.
such that the corresponding logarithmic singularity of the Schwarz function lies (fixed) within the finger tip, preventing the free boundary progressing beyond this point. Asymptotically then, the anticrack tips are fixed at the points

$$
\begin{equation*}
x_{n}(\infty) \sim \lambda_{n}-\alpha_{n} \log 2+\sum_{r \neq n} \alpha_{r} \log \left|q_{r}(\infty)-q_{n}(\infty)\right| \tag{7.20}
\end{equation*}
$$

with the $q_{k}(\infty)$ given by (7.19), and

$$
\begin{align*}
y_{n}(\infty) & \sim-q_{n}(\infty)+\frac{\pi}{2} \sum_{r>n} \alpha_{r}-\frac{\pi}{2} \sum_{r<n} \alpha_{r} \\
& =-\mu_{n} . \tag{7.21}
\end{align*}
$$

The width of the $n$ 'th anticrack is $\alpha_{n} \pi$. This implicitly assumes that the distance between adjacent anticrack tips, $\left|\mu_{n+1}-\mu_{n}\right|$, is greater than the sum of the two anticrack half-thicknesses, $\pi\left(\alpha_{n}+\alpha_{n+1}\right) / 2$, so that the anticracks are all distinct, although this is not necessary for the validity of the solution. All that we require is that the $\alpha_{n}$ be real and positive, and that the


Figure 7.7: Sketch showing the geometry when we have an array of "fat anticracks" with narrow spacing generated by (7.15). The gaps between the "anticracks" may be viewed as cracks.
$\mu_{n}$ be real and distinct. Note that since we have explicit expressions (7.20), (7.21) for the largetime tip co-ordinates $\left(x_{n}(\infty), y_{n}(\infty)\right)$, we may choose these, together with the $\left(\alpha_{n}\right)$, as the free parameters in the solution. Hence with this family of solutions, we can generate a more or less arbitrary array of anticracks, having specified widths and tip co-ordinates. A typical solution is shown in figure 7.6. The family of solutions given in [66] was even more general, in that they allowed the parameters $\alpha$ to be complex (provided they have positive real parts); this can give solutions with non-parallel anticracks.

Several comments may be made about this class of solutions. Firstly, given the above interpretation of the arbitrary constants $\alpha_{n}$ and $\mu_{n}$ in the solution, it is possible to choose these parameters so as to give an array of very fat "anticracks", which are very close together (we use the inverted commas here because usually we envisage anticracks as being narrow structures, hence the name). The strips of air separating them may then be viewed as cracks, since the motion of the "stagnant" anticrack bases is negligible relative to these, which was the case in the crack theory. This is essentially a system of narrow Saffman-Taylor fingers, see figure 7.7, and illustrates the complementarity of cracks and anticracks.


Figure 7.8: Solution of the form (7.15) exhibiting what we interpret as crack and anticrack formation. The values $\alpha_{1}=0.1$ (the anticrack), and $\alpha_{2}=-3$ were used.

Secondly, if we allow any of the real constants $\alpha_{n}$ to be negative, then the free boundary inevitably develops cusps within finite time (as we would expect from the interpretation of the $\alpha_{n}$ as the anticrack widths), since there is no possible large-time balance in equations (7.17). Hence by suitable choice of the parameters in the map, we may generate solutions which first form well-developed anticracks, then break down via cusp formation (figure 7.8). Such solutions can be interpreted as exhibiting anticrack and crack formation, since as the cusped configuration is neared, the crack theory outlined in $\S 7.1$ will take over from the ZST theory, operating on a much faster timescale.

The solutions represented by (7.15) may, for suitably "small" initial data, be regarded as a perturbation to a travelling wave planar front. One can study the effect of positive surface tension on such a travelling wave solution, by superimposing a small sinusoidal perturbation on the interface and performing a straightforward linear stability analysis. The basic travelling wave we perturb is (in dimensional variables, with $b$ the gap width in the Hele-Shaw cell, and $\mu$ the viscosity of the fluid),

$$
p=\frac{12 \mu}{b^{2}} U(U t-x) \quad(x>U t)
$$

where $x=U t$ is the free boundary. Perturbing the free boundary to

$$
x=U t+\epsilon e^{\beta t} \sin \left(\frac{n \pi y}{L}\right)+O\left(\epsilon^{2}\right)
$$

we find the dispersion relation

$$
\beta=\frac{|n| \pi}{L}\left(U-\frac{T n^{2} \pi^{2} b^{2}}{12 \mu L^{2}}\right) .
$$

Here, $T$ is the usual surface tension parameter, which when positive is seen to have the effect of stabilising shorter wavelengths. Assuming walls at $y= \pm L / 2$, there is a critical (minimum) speed for a perturbation to be maintained,

$$
U^{*}=\frac{T \pi^{2} b^{2}}{12 \mu L^{2}}
$$

which will typically be much less than unity, and hence usually exceeded. The fastest-growing wavenumber is the $n=1$ mode, so according to this analysis, if we have a channel of width $L$ (not small), and channel length much greater than $L$, an initially flat interface might be expected to evolve to a single dominant finger (although this would rapidly take us beyond the realm of the linear theory) - the Saffman-Taylor experiment.

### 7.3.3 Howison's radial "anticrack" solutions

For completeness, and also because of the remarkable similarity with the experimental pictures of [73] and [11], and the numerical simulations of [23], we present the family of solutions found by Howison [45]. These solutions differ from those above, being in a radial geometry, although the essential anticrack behaviour is the same.

We map from the unit disc onto the fluid domain, which is here the exterior of some finite bubble. The mapping function used is

$$
\begin{align*}
w(\zeta)=\frac{a(t)}{\zeta}+\beta_{1} \sum_{k=1}^{N} & \varpi^{-k} \log \left(c_{1}(t) \varpi^{k}-\zeta\right) \\
& +\beta_{2} \sum_{k=1}^{N} \varpi^{-k-1 / 2} \log \left(c_{2}(t) \varpi^{k+1 / 2}-\zeta\right) \tag{7.22}
\end{align*}
$$

for $a, c_{1}, c_{2}$ real functions of time, with $a>0, c_{1}>1, c_{2}>1$; the $\beta_{i}$ are positive constants to be chosen (analogous to the $\alpha_{i}$ in the solution (7.15), in that $\pi \beta_{i}$ turns out to be the anticrack width), and $\varpi$ is an $N^{\prime}$ 'th root of unity (so $\varpi^{N}=1$ ). More complicated maps, allowing asymmetric arrays of fingers to be generated, can be dealt with, but (7.22) is the case considered in detail in [45].

The motion is assumed to be driven by a $\operatorname{sink} Q$ at infinity. The fact that the singularities of the Schwarz function must remain fixed in the physical plane yields two invariants of the motion,

$$
a c_{i}+\beta_{i} \sum_{k=1}^{N} \varpi^{-k} \log \left(c_{i}^{2} \varpi^{k}-1\right)+\beta_{j} \sum_{k=1}^{N} \varpi^{-k-1 / 2} \log \left(c_{i} c_{j} \varpi^{k+1 / 2}-1\right)=K_{i}
$$

holding for $i=1, j=2$, and also for $j=1, i=2$, for some real constants $K_{1}, K_{2}$. The third and final equation governing the evolution is the condition that the rate of change of the bubble area should be equal to $Q$,

$$
a^{2}-N a\left(\beta_{1} c_{1}+\beta_{2} c_{2}\right)=\frac{Q t}{\pi}+K_{0}
$$

for some real constant $K_{0}$. These equations could also have been derived by direct substitution of (7.22) into the P-G equation (2.4), although this is much more tedious.

If the initial values $c_{1}(0)$ and $c_{2}(0)$ are large, then the initial domain will be nearly circular (as may be seen from $w(\zeta, 0)$ ). In any event, provided their initial values are greater than one, both $c_{1}$ and $c_{2}$ will decrease monotonically to unity as $t \rightarrow \infty$, and the bubble will leave behind $2 N$ fingers of fluid, alternate fingers having widths $\pi \beta_{1}$ and $\pi \beta_{2}$. A typical solution is shown in figure 7.3 ; we refer to [45] for the remaining details of the solution.

### 7.4 The Schwarz function of an anticrack

The solutions obtained so far indicate that a general ZST anticrack has a logarithmic singularity of the Schwarz function just inside the tip, which is fixed, and which the free boundary cannot reach in finite time (contrast this with the square-root singularity just outside the tip, for a crack). We have already seen how this works for the logarithmic singularity, but it is also easy to demonstrate that the free boundary cannot reach a more general internal singularity, such as $g(z) \sim \lambda(\beta-z)^{m}$ for some constants $\lambda, \beta$, and $m>0 .{ }^{1}$ We note, however, that although we conjecture this logarithmic singularity to be generic for the ZST anticrack, it cannot be a singularity in the NZST problem (even if we allow it to move), as may be seen from (2.11). Internal singularities of the Schwarz function will not now remain fixed in space (save those which coincide with the driving singularity) and must move in such a way as to cancel the singularity on the right-hand side of (2.11). Motivated by a link between (2.11) and the well-known "Harry-Dym" equation, it was conjectured in [48] that the canonical moving singularity in the NZST problem is of the form

$$
g(z) \sim \frac{\alpha(t)}{\left(z-z_{0}(t)\right)^{1 / 3}} \quad \text { as } z \rightarrow z_{0}(t)
$$

for some $\alpha(t), z_{0}(t)$. It is easy to show that the coefficients in this leading-order singularity, which is of course much larger than $\log \left(z-z_{0}\right)$ as $z \rightarrow z_{0}$, satisfy

$$
\dot{z}_{0}=\frac{4 T}{\sqrt{3 \alpha^{3}}}
$$

We know that in the ZST limit we need $\dot{z}_{0}=0$, and we also expect $\alpha=0$ for the generic anticrack, so dependence like $\alpha \sim T^{k}$ for $0<k<2 / 3$ would do, for instance. If our generic ZST anticrack is a sensible $T \rightarrow 0$ limit of some NZST anticrack solution, and if the conjecture of [48] about the canonical NZST singularity is correct, then there must be a secondary logarithmic singularity, which can persist as $T \rightarrow 0$, and which will dominate in this limit.

It is still the case that the free boundary cannot cross internal singularities of $g(z)$ while the solution exists. Given the photographic evidence of [73, 11], and the excellent agreement with the ZST solutions of [45], one therefore expects very little variation in $z_{0}(t)$ for large time. Further progress with the NZST problem is difficult, and we do not pursue it.

### 7.5 Results from formal asymptotics

Having set the scene by presenting some exact ZST anticrack solutions, we now digress a little to consider some results due to King et al. [56], which rely on formal asymptotics in the neighbourhood of singular points of the flow. We mentioned this work in $\S 7.1$, citing the result that in the ill-posed ZST suction problem, there exists no solution to the problem for $t>0$ if the initial free boundary $\partial \Omega(0)$ contains a corner of internal angle $\phi \in(\pi, 2 \pi)$. In fact, much more general results than this are obtained, which we summarise below.

Consider the ZST suction problem, where the initial domain $\Omega(0)$ is nonanalytic, having a corner of internal angle $\phi$ in $\partial \Omega(0)$ at $\mathbf{x}=\mathbf{0}$ (figure 7.9). Then:

[^27]

Figure 7.9: The local geometry with a corner of internal angle $\phi$ in the fluid.

- If $0<\phi \leq \pi / 2$, a solution can exist for $t>0$; if it does, the boundary $\partial \Omega(t)$ continues to have a corner of angle $\phi$ at $\mathbf{x}=\mathbf{0}$.
- For $\pi / 2<\phi<\pi$, a solution can exist for $t>0$; if it does, then for $t>0$ the corner angle switches instantaneously to $(\pi-\phi)$.
- For $\phi=\pi$, if $\partial \Omega(0)$ is analytic at $\mathbf{x}=\mathbf{0}$ then the free boundary $\partial \Omega(t)$ may move such that $\mathbf{0} \notin \Omega(t)$ for $t>0$, but if $\partial \Omega(0)$ is nonanalytic at $\mathbf{x}=\mathbf{0}$ then either $\phi=\pi+$, and there is no solution for $t>0$, or else $\phi=\pi-$, and as long as the solution exists there is an outward-pointing cusp at $\mathbf{x}=\mathbf{0}$.
- For $\pi<\phi \leq 2 \pi$ (excluding the case of $(4 n+1) / 2$-power cusps) there is never a solution for $t>0$.

Further to these results, King [57] considered the "borderline" case $\phi=\pi$ (with the free boundary nonanalytic at $\mathbf{x}=\mathbf{0}$ ) in more detail. In the simplest case considered, an initial free boundary

$$
x=\alpha|y|^{k} \quad k \in(1,2), \quad \text { as } y \rightarrow 0
$$

is assumed (for the usual ZST suction problem, with the fluid occupying the region to the right of this curve). A necessary condition for a solution to exist for $t>0$ is found to be that $\alpha$ must be positive, and if a solution does exist for $t>0$, the subsequent free boundary must have an outward-pointing cusp fixed at the origin, described locally by

$$
|y|=f(x, t) \sim \alpha \sin (k \pi / 2) x^{k}, \quad \text { as } x \rightarrow 0+
$$

The $\alpha<0$ case corresponds to $\phi=\pi+$ in the possibilities itemised above; the free boundary has a continuous tangent vector but is nonanalytic, and we expect this situation to be regularised by crack formation if $T>0$ (or slit propagation in the limit $T \rightarrow 0$ ), as discussed in $\S 7.1$. When $\alpha>0$, $\phi=\pi-$; again the free boundary is nonanalytic but with a continuous tangent vector. In this case, however, a solution to the ZST problem can exist for $t>0$ (the third of the list of possibilities). In terms of our anticrack theory, we interpret the stagnant outward-pointing cusp as a limiting case of an anticrack, somehow analogous to a slit, in that the free boundary is nonanalytic. Unlike slits, though, we only expect such solutions to occur when we have nonanalytic initial data, and also, surface tension is not necessary (even in a limiting sense) for such solutions to exist. In fact, the dependence (or otherwise) on surface tension is one of the main differences between the crack and anticrack theories; for cracks (and slits) surface tension is vital if solutions to the models are to exist at all, but for anticracks, scores of exact ZST solutions exist, which can give good agreement with observable free boundary shapes if we choose the parameters correctly. However,
surface tension is the crucial mechanism which selects the "right" solution; in particular, such non-classical solutions are only to be expected in the limit $T \rightarrow 0$. In the next section we consider how this works for simple "test cases".

### 7.6 Paterson's analysis

The ZST solutions presented in $\S 7.3 .3$ suffer from the same lack of determinacy as those of $\S 7.3 .2$, since we have an infinite family of possible solutions, and we are able to specify the number of anticracks which form, the position of their tips, and their widths. It is hoped that the addition of surface tension to the model will resolve this indeterminacy. A linear perturbation analysis was carried out by Paterson [73] for the problem of a sinusoidally-perturbed expanding circular bubble, which revealed that there is indeed a surface tension dependent selection mechanism at work. We present a slightly modified (and extended) version of this analysis.

Consider an almost circular bubble in an unbounded expanse of fluid, expanding under the influence of a sink of strength $Q>0$ at infinity. We nondimensionalise the problem using the following scalings (where the subscript "dim" denotes the dimensional quantity):

$$
\begin{array}{rll}
\text { length } & : \quad r_{d i m}=a r, \\
\text { time } & : & t_{d i m}=\frac{\pi a^{2}}{Q} t, \\
\text { pressure } & : & P_{d i m}=\frac{Q}{\mathcal{M} \pi} P, \\
\text { surface tension } & : & T_{d i m}=\frac{a Q}{\mathcal{M} \pi} T . \tag{7.26}
\end{array}
$$

In the above, the lengthscale $a$ is taken to be the bubble radius at the time the perturbation is assumed to be imposed, and the quantity $\mathcal{M}$ is the mobility, $\mathcal{M}:=b^{2} /(12 \mu)$. Velocities must be scaled with $Q /(\pi a)$ to make them dimensionless, and the curvature of the interface scales with $1 / a$.

In the dimensionless variables, using the notation $r=R(\theta, t)$ to denote the free boundary in plane polar co-ordinates (and with the rest of the notation as usual), the problem is

$$
\begin{align*}
\nabla^{2} P & =0 \quad \text { in } r>R(\theta, t)  \tag{7.27}\\
P & =-T \kappa \quad \text { on } r=R(\theta, t)  \tag{7.28}\\
\frac{\partial P}{\partial n} & =-V_{n} \quad \text { on } r=R(\theta, t) \tag{7.29}
\end{align*}
$$

Note that the sign of $\kappa$ is reversed compared with the model of $\S 1.2 .1$ because here we take it to be the dimensionless curvature measured relative to the bubble, rather than the fluid domain (so the curvature of the undisturbed free boundary is 1 ). The boundary condition at infinity is

$$
P \sim-\frac{1}{2} \log r, \quad \text { as } r \rightarrow \infty .
$$

The "base state" solution $\left(P_{0}, R_{0}\right)$ about which we perturb is easily seen to be

$$
\begin{align*}
P_{0}(r, t) & =-\frac{1}{2} \log \left(\frac{r}{R_{0}(t)}\right)-\frac{T}{R_{0}(t)}  \tag{7.30}\\
R_{0}(t) & =\sqrt{t} \tag{7.31}
\end{align*}
$$

in the dimensionless co-ordinates. Suppose that at time $t=1$ (which is the dimensionless start time), some small perturbation is imposed on the free boundary,

$$
R(\theta, 1)=1+\epsilon \sin n \theta, \quad 0<\epsilon \ll 1 .
$$

We assume that the solution to the perturbed problem can be expressed as a simple perturbation expansion in the small parameter $\epsilon$,

$$
\begin{aligned}
& P=P_{0}+\epsilon P_{1}+\cdots \\
& R=R_{0}+\epsilon R_{1}+\cdots
\end{aligned}
$$

where each $P_{i}(r, \theta, t)$ must be harmonic in $r>R(\theta, t)$. For the boundary conditions, we need the expression for the curvature $\kappa$ of a general polar curve $r=R(\theta, t)$; this is given by

$$
\kappa=\frac{1}{R}\left[\left(\left(\frac{R^{\prime}}{R}\right)^{\prime}-1\right)^{2}+\left(\frac{R^{\prime}}{R}\right)^{2}\right]^{1 / 2}
$$

where the prime is understood to denote $d(\cdot) / d \theta$. In terms of the asymptotic expansions, then, the boundary condition (7.28) becomes

$$
\begin{align*}
& P_{0}+\epsilon\left(P_{1}+R_{1} P_{0 r}\right)+\epsilon^{2}\left(P_{2}+R_{1} P_{1 r}+\frac{R_{1}^{2}}{2} P_{0 r r}+R_{2} P_{0 r}\right)+\cdots  \tag{7.32}\\
& =-\frac{T}{R_{0}}\left[1-\frac{\epsilon}{R_{0}}\left(R_{1}^{\prime \prime}+R_{1}\right)+\frac{\epsilon^{2}}{R_{0}^{2}}\left(2 R_{1}^{\prime \prime} R_{1}+\frac{3}{2}\left(R_{1}^{\prime}\right)^{2}+R_{1}^{2}-R_{0}\left(R_{2}^{\prime \prime}+R_{2}\right)\right)+\cdots\right],
\end{align*}
$$

while (7.29) is

$$
\begin{aligned}
R_{0 t}+P_{0 r}+\epsilon\left[R_{1 t}+P_{1 r}+R_{1} P_{0 r r}\right]+\epsilon^{2}\left[R_{2 t}+P_{2 r}+R_{1} P_{1 r r}+R_{2} P_{0 r r}\right. & \left.+\frac{R_{1}^{2}}{2} P_{0 r r r}\right]+\cdots \\
& =\epsilon^{2} \frac{R_{1 \theta} P_{1 \theta}}{R_{0}^{2}}+\cdots,
\end{aligned}
$$

both holding on $r=R_{0}(t)$, having been linearised down onto this curve. We already have the basic solution, so we can immediately write down the $O(\epsilon)$ problem as

$$
\begin{aligned}
\nabla^{2} P_{1} & =0 \quad \text { in } r>\sqrt{t}, \\
R_{1 t}+P_{1 r}+\frac{R_{1}}{2 t} & =0 \quad \text { on } r=\sqrt{t}, \\
P_{1}-\frac{R_{1}}{2 \sqrt{t}} & =\frac{T}{t}\left(R_{1}^{\prime \prime}+R_{1}\right) \quad \text { on } r=\sqrt{t} \\
R_{1}(\theta, 1) & =\sin n \theta
\end{aligned}
$$

we are interested in the solution for times $t>1$. The solutions to this system which have bounded pressure at infinity are of the form

$$
\begin{align*}
P_{1} & =A(t)\left(\frac{R_{0}}{r}\right)^{n} \sin n \theta  \tag{7.33}\\
R_{1} & =C(t) \sin n \theta \tag{7.34}
\end{align*}
$$

Substitution into the boundary conditions reveals $A$ and $C$ to satisfy the coupled equations and boundary conditions,

$$
\begin{align*}
\frac{d C}{d t}-\frac{n A}{\sqrt{t}}+\frac{C}{2 t} & =0  \tag{7.35}\\
A-\frac{C}{2 \sqrt{t}}+\frac{T C}{t}\left(n^{2}-1\right) & =0  \tag{7.36}\\
C(1) & =1
\end{align*}
$$

Eliminating $A(t)$ gives the single equation for $C(t)$,

$$
\begin{equation*}
\frac{d C}{d t}=C \frac{(n-1)}{2 t}\left\{1-\frac{2 \operatorname{Tn}(n+1)}{\sqrt{t}}\right\} \tag{7.37}
\end{equation*}
$$

with solution

$$
\begin{equation*}
C(t)=t^{(n-1) / 2} \exp \left\{-\alpha\left(1-\frac{1}{\sqrt{t}}\right)\right\} \tag{7.38}
\end{equation*}
$$

where $\alpha:=2 T n\left(n^{2}-1\right)$. $A(t)$ may then be found from (7.35) or (7.36), but since we are interested in the free boundary rather than the pressure field, we only need find $A(t)$ if we wish to solve for orders $\epsilon^{2}$ and higher. We now consider what information can be extracted from equation (7.37) about wavenumber "selection".

The critical condition for a perturbation to be maintained is $d C / d t=0$, which gives a quadratic equation for the critical wavenumber $n_{c}$, from which we find the critical wavelength $\lambda_{c}$. The solutions are

$$
\begin{align*}
& n_{c}=\left(\frac{R_{0}}{2 T}+\frac{1}{4}\right)^{1 / 2}-\frac{1}{2}  \tag{7.39}\\
& \lambda_{c}=2 \pi R_{0} /\left\{\left(\frac{R_{0}}{2 T}+\frac{1}{4}\right)^{1 / 2}-\frac{1}{2}\right\} \tag{7.40}
\end{align*}
$$

where we substituted back in for $R_{0}$ instead of $\sqrt{t}$, to emphasise the simple interpretation in terms of the (approximate) bubble radius. In the dimensional variables we have

$$
\lambda_{c}=2 \pi a /\left\{\left(\frac{Q a}{2 \pi \mathcal{M} T}+\frac{1}{4}\right)^{1 / 2}-\frac{1}{2}\right\} .
$$

When the circumference $2 \pi a$ of the bubble is less than this critical wavelength, the free boundary is stable to small perturbations, and the interface remains approximately circular. Once the circumference has exceeded $\lambda_{c}$, the "growth factor" $d C / d t$ is positive, and perturbations are able to grow.

Further to this, Paterson considered which, in an unstable situation, would be the fastestgrowing wavelength, since this is the one we would expect to be observed in practice. The condition for maximum growth is

$$
\frac{\partial}{\partial n}\left(\frac{d C}{d t}\right)=0
$$

and applying this to equation (7.37) gives the maximal-growth wavenumber $n_{m}$ and wavelength $\lambda_{m}$ as

$$
\begin{align*}
& n_{m}=\frac{1}{\sqrt{ } 3}\left(\frac{R_{0}}{2 T}+1\right)^{1 / 2}  \tag{7.41}\\
& \lambda_{m}=2 \sqrt{ } 3 \pi R_{0} /\left(\frac{R_{0}}{2 T}+1\right)^{1 / 2} \tag{7.42}
\end{align*}
$$

Obviously the instabilities observed in the experiments of [73] (and [11]) quickly develop to amplitudes beyond the scope of this linear stability; however, Paterson does find that the number of fingers which develop (before secondary bifurcations) is $n_{m}$, and that when the fingers first form, they have the wavelength predicted by this theory. Note the dependence of these maximal growth parameters on the radius (and hence on time). Although the free boundary will not remain approximately circular for long once the instability has set in, these expressions indicate that we do expect further bifurcations to occur. For example, if we interpret $R_{0}$ as the mean bubble radius, we might expect secondary fingering to occur when $R_{0}$ has grown such that the original $n_{m}$ has increased to $n_{m}+1$.

This analysis was carried out with a view to explaining the observations of [73], but consider first a general hypothetical experimental situation in which $n$ fingers develop (before any secondary
bifurcations). This value of $n$ must, by the preceding analysis, satisfy the condition (7.41) when the instability first becomes apparent; at this stage, the dimensionless bubble radius $R_{0}$ is unity, so the value of the dimensionless surface tension parameter giving rise to this instability must satisfy

$$
\begin{equation*}
T=\frac{1}{2\left(3 n^{2}+1\right)} \tag{7.43}
\end{equation*}
$$

We then know the corresponding value of $\alpha$ (as defined in this section), and hence if a suitable value for $\epsilon$ is chosen, we can track the evolution of the interface as predicted by this stability analysis, until the asymptotic expansion is no longer valid. From the boundary condition (7.32), we see that the perturbation expansion is only valid if the parameter $\epsilon$ is much smaller than the dimensionless surface tension, $T$. If $T \ll 1$, as turns out to be the case in the experiments of [73], then once the disturbance to the interface is of the same order of magnitude as $T$, a new analysis is needed.

For Paterson's experiment, the value of $a$ (the bubble radius) at which the instability visibly sets in, can be observed from the time-lapse photographs. With this value of $a$ the dimensionless surface tension $T$ can be found from (7.26), and checked against the value (7.43) predicted by the theory. We then have all the information we need in order to consider the different stages of this early anticrack evolution.

### 7.6.1 The case $\epsilon \ll T$

We refer back to figure 7.3. In our notation, the parameter values of [73] are

$$
Q=9.3 \mathrm{~cm}^{2} \mathrm{~s}^{-1}, \quad \mathcal{M}=3.6 \times 10^{-4} \mathrm{~cm}^{4} \mathrm{dyne}^{-1} \mathrm{~s}^{-1}, \quad a \sim 3 \mathrm{~cm}
$$

The dimensional surface tension parameter $T_{\text {dim }}$ has a value of 63 dyne $\mathrm{cm}^{-1}$, and the number of fingers $n$ which develops is 8 . The value of the dimensionless surface tension $T$ (and hence of the parameter $\alpha$ ) as predicted by (7.43) with $n=8$ is

$$
\begin{align*}
T & =1 / 386=2.59067 \times 10^{-3}  \tag{7.44}\\
\alpha & =2.611 \tag{7.45}
\end{align*}
$$

which by (7.26) corresponds to a value $a=2.957$ for the radius at which the instability sets in. Thus the observed values for $a$ and $n$, with the other measured parameter values, do satisfy approximately the expected maximal growth conditions.

The solutions to the $O(1)$ and $O(\epsilon)$ problems are as given in (7.30), (7.31), and (7.33)-(7.38). We can in fact solve the problem to $O\left(\epsilon^{2}\right)$ in this phase of the evolution, to see what effect (if any) the higher harmonics are having on the solution. The equations may be written down immediately from the earlier systematic linearisations of the boundary conditions. We omit the details, which are rather gruesome since we first have to fully solve the order $\epsilon$ problem and substitute from this (and from the $\mathrm{O}(1)$ problem) into the boundary conditions. The solutions are of the form

$$
\begin{aligned}
P_{2}(r, \theta, t) & =A(t)\left(\frac{R_{0}}{r}\right)^{2 n} \cos 2 n \theta+B(t) \\
R_{2}(\theta, t) & =D(t) \cos 2 n \theta+E(t)
\end{aligned}
$$

and substitution of this form into the boundary conditions gives coupled equations for $A$ and $D$, and for $B$ and $E$. The only quantity of interest is $D(t)$, since we are not interested in the pressure field, and $E(t)$ is only some additive function which will not affect the free boundary shape. Hence we eliminate $A(t)$ from the first of our pairs of equations. If we write

$$
\hat{D}(t)=\exp \left\{2 \alpha\left(1-\frac{1}{\sqrt{t}}\right)\right\} D(t)
$$



Figure 7.10: Graph showing the relative sizes of the coefficients of the terms $\sin n \theta$ (in the $\epsilon R_{1}$ term of $R$ ) and $\cos 2 n \theta$ (in the $\epsilon^{2} R_{2}$ term of $R$ ) as functions of time, in the $\epsilon \ll T$ régime. The $\sin n \theta$ cofficient is the upper curve.
and define the parameter $\beta:=12 T n^{3}$, we find the following ordinary differential equation for $\hat{D}(t)$,

$$
\frac{d}{d t}\left\{\frac{\hat{D}}{t^{n-1 / 2}} e^{-\beta / \sqrt{t}}\right\}=\frac{e^{-\beta / \sqrt{t}}}{4}\left\{\frac{2 n-1}{t^{2}}-\frac{2 n T\left(8 n^{2}-3\right)}{t^{5 / 2}}\right\} .
$$

The solution satisfying $\hat{D}(1)=0$ (corresponding to a perfectly sinusoidal initial perturbation) is found, by direct integration, to be

$$
\begin{aligned}
\hat{D}= & \frac{t^{n-1 / 2}}{2}\left\{\frac{(2 n-1)}{\beta^{2}}\left[1+\frac{\beta}{\sqrt{t}}-(1+\beta) \exp \left(-\beta\left(1-\frac{1}{\sqrt{t}}\right)\right)\right]\right. \\
& \left.-\frac{\left(8 n^{2}-3\right)}{3 n^{2} \beta^{2}}\left[1+\frac{\beta}{\sqrt{t}}+\frac{\beta^{2}}{2 t}-\left(1+\beta+\frac{\beta^{2}}{2}\right) \exp \left(-\beta\left(1-\frac{1}{\sqrt{t}}\right)\right)\right]\right\} .
\end{aligned}
$$

This expression does have a uniform limit as $T \rightarrow 0$, despite the presence of the $1 / \beta^{2}, 1 / \beta^{3}$ terms on the right-hand side. If we expand the exponentials for small $\beta$ we find

$$
\hat{D}=\frac{(2 n-1)}{4}\left(1-\frac{1}{t}\right) t^{n-1 / 2}+O(\beta)
$$

as $\beta \rightarrow 0$ (which, as we shall see, agrees with the perturbation analysis for the ZST problem, providing a check on the analysis). With $T$ given by (7.44), $\beta=15.917$.

For the parameter values given here, it is easy to show (by comparison of the coefficients; figure 7.10) that the term $\epsilon^{2} D(t) \cos 2 n \theta$ (from $\epsilon^{2} R_{2}$ ) in the expression for the free boundary is always negligible for times of interest, while the term $\epsilon R_{1}$ grows steadily according to (7.38), until its magnitude becomes comparable to $T$. This analysis is then no longer valid, and we pass to the régime $\epsilon \sim T$. Since the perturbation retains its initial shape, only becoming more pronounced, this régime is not terribly interesting, but as we have seen, it is crucial in determining the number of fingers which develop, and when they develop, in a given situation.

### 7.6.2 The case $1 \gg \epsilon \sim T$

By the above observation that the term $\epsilon^{2} R_{2}$ remains negligible throughout the régime $\epsilon \ll T$, it is reasonable to assume an initial perturbation $\epsilon \sin n \theta$ here. Writing $T=\epsilon \chi$ for some $O(1)$
quantity $\chi$, the solution is found to be

$$
\begin{equation*}
R=R_{0}+\epsilon R_{1}+\epsilon^{2} R_{2}+O\left(\epsilon^{3}\right) \tag{7.46}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{0}=\sqrt{t}, \quad R_{1}=t^{(n-1) / 2} \sin n \theta, \tag{7.47}
\end{equation*}
$$

and

$$
\begin{align*}
R_{2}=\frac{(2 n-1)}{4}\left(1-\frac{1}{t}\right) t^{n-1 / 2} \cos 2 n \theta-2 n \chi\left(n^{2}-1\right) & \left(1-\frac{1}{\sqrt{t}}\right) t^{(n-1) / 2} \sin n \theta \\
& +\frac{1}{4 \sqrt{t}}\left(1-t^{n-1}\right) \tag{7.48}
\end{align*}
$$

Setting $\chi=0$ in here gives the ZST case. Note that the (dimensional) bubble radius $a$ will have increased during the first stage of the instability (during which the perturbations grow relatively slowly, by (7.38)), hence by the time this second stage is reached, the value of the dimensionless surface tension $T$ will be smaller, by (7.26). Perturbations now grow quite rapidly, and will quickly reach an amplitude larger than $T$. We thus consider the final stage of the small perturbation theory, when the perturbations are large compared with the surface tension parameter (now somewhat smaller than (7.44), as commented above).

### 7.6.3 The case $T \ll \epsilon \ll 1$

The free boundary will now evolve according to the solution (7.46), (7.47), (7.48), but with $\chi=0$, as long as this expansion remains valid. An evolution sequence based on the experimental parameters of [73], is shown in figure 7.11 (for dimensionless times $t=1,1.15,1.3$ ). The value of $\epsilon$ used was 0.05 , which is much greater than the value of $T$ in (7.44), but still much less than 1 . The initial stages of the anticrack formation can be clearly observed, as the effect of the $\cos 2 n \theta$ term from (7.48) grows. The crucial factor here leading to anticrack, rather than crack formation, is that the sign of the coefficient of $\cos 2 n \theta$ is the same as that of $\sin n \theta$ (figure 7.12); had it been of opposite sign, we would have seen cracks beginning to form.

### 7.6.4 Conclusions for Paterson's anticracks

In the preceding sections we have reviewed the experiments of [73], and extended the analysis of that paper. We saw that, for small values of the dimensionless surface tension $T$, the initial perturbation analysis is only valid when the perturbations are of very small amplitude, and that when the amplitude becomes comparable to $T$, a different analysis is needed. Nevertheless, it is during this initial stage that the wavenumber "selection" occurs. Higher-order terms (than the first) in the perturbation expansion are irrelevant during this initial stage. With the $\epsilon \gg T$ theory, solving the problem to order $\epsilon^{2}$, the early stages of anticrack formation can be observed (figure 7.11). This analysis itself then breaks down, and nonlinear theory is needed.

It is clear then that anticrack "selection" is determined at a very early stage of the instability, when the perturbations are small compared with the dimensionless surface tension, which is itself small. For later times, surface tension effects are negligible, and ZST theory gives a good approximation to the free boundary shape. Given the degree of similarity between the exact ZST solutions of [45] and the photographs of [73], it seems reasonable to conjecture that the nonlinear régime referred to above is described by solutions of the kind in [45], at least until secondary bifurcations occur, with the birth of new anticracks (and even such bifurcation can be described by the exact ZST theory of [45], if we can predict when it will occur). Note that the selection of the finger widths, which appears as an arbitrary parameter in the solutions of [45], is not determined by this argument. It seems reasonable to assume, though, that the anticrack widths will be approximately $1 /(2 n)$ times the bubble circumference at the time the instability sets in, that is, $a \pi / n$. With the value of $a$ found in $\S 7.6 .1$ and $n=8$, this does give a value for the


Figure 7.11: Evolution of the free boundary of Paterson's expanding bubble for dimensionless times $t=1,1.15,1.3$. This plot applies to the régime in which the amplitude of the perturbations is much greater than $T$, so that the ZST perturbation theory is applicable.


Figure 7.12: Graph showing the cofficients of the terms $\sin n \theta$ (in the $\epsilon R_{1}$ term of $R$ ) and $\cos 2 n \theta$ (in the $\epsilon^{2} R_{2}$ term of $R$ ) as functions of time, for the $\epsilon \gg T$ régime. The $\sin n \theta$ coefficient is the one with initial value 1 .
finger widths which agrees well with the experimental photos of [73]. In general, for very small values of $T$ we expect to get many fingers, by (7.43). Provided the dimensional surface tension $T_{d i m}$ is small too, then by (7.26) the radius $a$ should still be $O(1)$, so that we get a very small anticrack thickness $\epsilon=a \pi / n$, as we expect in the small surface tension problem. Since for small $T, n \sim 1 / \sqrt{T}$ by (7.43), we expect that $\epsilon \sim \sqrt{T}$ (c.f. the crack theory, where $\epsilon \sim T^{1 / 3}$ ).

### 7.7 Fractal Hele-Shaw

As an interesting aside to the last section, we present a "thought experiment" due to Howison [44] which relies on the ideas introduced by the stability analysis. Recall that we found a minimum wavelength (7.40) for a perturbation to the expanding bubble to be sustained, and also the expected number of fingers $n_{m}$ which then form (7.41). Suppose we have a situation where the maximal growth rate corresponds to the value $n_{m}=2$. In this case, two anticracks will start to develop when the circumference of the expanding bubble reaches the value $R=2 \lambda_{m}$; call the bubble radius $R_{0}$ at this stage (which we think of as the 0 'th stage).

If we assume narrow anticracks, and a symmetric configuration, then when the circumference of the bubble has doubled again, we expect to be able to "fit in" another two anticracks; the bubble radius will be $4 R_{0}^{+}$at this stage (where the " + " denotes the fact that we only expect these new anticracks to form when we exceed this radius). Since anticrack tips remain stagnant, the two original anticracks are now of length $3 R_{0}$. Likewise, when the bubble radius has grown to $16 R_{0}^{+}$, we can fit in four new anticracks.

Under ideal conditions (whence "thought experiment") we may consider this process repeating indefinitely, with the bubble radius quadrupling between successive bifurcations. At the $n$ 'th stage the bubble radius will be $R_{n}:=2^{2 n} R_{0}$, with $2^{n}$ anticracks already developed, and another $2^{n}$ about to form. The longest of these will be those two which formed at the $n=0$ stage, now of length

$$
R_{0}\left(2^{2 n}-1\right)
$$



Figure 7.13: The evolving anti-slit structure at the stage $n=2$, with 2 anti-slits of length $L_{0}, 2$ of length $L_{1}$, and 4 new anti-slits about to form.
and the shortest will be those $2^{n-1}$ which formed at the $(n-1)$ 'th stage, now of length

$$
R_{0}\left(2^{2 n}-2^{2(n-1)}\right)=\frac{3}{4} 2^{2 n} R_{0}=\frac{3}{4} R_{n}
$$

Analogous to slits (as the ZST limit of cracks) we may consider anti-slits as possible ZST limits of anticracks, these being vanishingly thin "spikes" of fluid left behind as the free boundary evolves (figure 7.13). In this limit we can evaluate the fractal dimension of the evolving structure quite easily, by scaling the expanding bubble (which is now circular, penetrated by $2^{n}$ spikes of varying lengths) down onto the unit disc. Thus at the $n$ 'th stage we have a structure with

| 2 | spikes of length | $L_{0}:=1-2^{-2 n}$, |
| :---: | :--- | :--- |
| $2^{1}$ | spikes of length | $L_{1}:=1-2^{-2(n-1)}$, |
| $2^{2} \quad$ | spikes of length | $L_{2}:=1-2^{-2(n-2)}$, |
|  | $\vdots$ |  |
| $2^{k} \quad$ | spikes of length | $L_{k}:=1-2^{-2(n-k)}$, |
| $\quad \vdots$ |  |  |
| $2^{n}$ | spikes of length | $L_{n}:=1-2^{-2} \equiv 3 / 4$, |

with $2^{n}$ new anti-slits about to form. Note that when we scale the problem in this way, $2^{-2 n}$ is the smallest lengthscale we can consider (although we take the large $n$ limit), and when new anti-slits form at each stage, they immediately have length $3 / 4$. The formula for the fractal dimension $D$ of such a structure is given by Turcotte [97] as

$$
L=C h^{-(D-1)},
$$

where $L$ is the total length of the fractal curve, $C$ is some constant of proportionality, and $h$ is what Turcotte refers to as the "measuring rod" length, which is essentially a measure of the resolution at which we are considering the fractal. This formula holds in a limiting sense, as $h \rightarrow 0$. Here, $h$ is just the lengthscale $2^{-2 n}$, and $L$ is twice the sum of the lengths of all the spikes of fluid (since each spike has two sides) plus the circumference of the circle,

$$
\begin{aligned}
L & =2 \sum_{k=0}^{n} 2^{k} L_{k}+2 \pi \\
& =\frac{12}{7} 2^{n}-2+\frac{2}{7} 2^{-2 n}+2 \pi \\
& \sim \frac{12}{7} 2^{n}, \quad \text { for large } n
\end{aligned}
$$

Hence in the large $n$ limit we must have

$$
\frac{12}{7} 2^{n}=C\left(2^{-2 n}\right)^{-(D-1)}
$$

from which $C=12 / 7$, and the fractal dimension $D=3 / 2$.

### 7.8 Cracks revisited

We originally cited the experiments of [73] as evidence only for the anticrack theory; however, in the same paper, experiments are described concerning suction from the centre of an initially circular disc of glycerine surrounded by air, which is, of course, also an ill-posed problem. Timelapse photography is again used to study the evolution of the interface, which in this case is seen to be much more irregular than for the expanding bubble. Fingers of air are observed to form, but the larger ones impede the growth of the smaller, until: "Eventually, one finger dominates, and accelerates into the well". It is tempting to regard this instability as a crack; certainly from the photographs it is clear that during the latter stages of the motion the rest of the free boundary is not moving much relative to this dominant finger, as in the crack theory. We remind the reader again at this point of the numerical solutions of [55] for suction from an initially-circular disc of fluid. Although the suction point here was off-centre, a crack-like morphology was computed for small surface tension, and we might expect a similar result here.

The analysis that was carried out for the expanding bubble in $\S 7.6$ may be repeated for the problem of an approximately circular blob of viscous fluid, contracting under the action of a point sink (of strength $Q>0$ ) at the origin. With the same nondimensionalisations, the problem is as in (7.27)-(7.29), but with the sign of the right-hand side of (7.28) reversed. The asymptotic condition on $P$ is now

$$
P \sim \frac{1}{2} \log r, \quad \text { as } r \rightarrow 0
$$

The base state solution here is

$$
\begin{align*}
P_{0}(r, t) & =\frac{1}{2} \log \left(\frac{r}{R_{0}}\right)+\frac{T}{R_{0}}  \tag{7.49}\\
R_{0}(t) & =\sqrt{1-t} \tag{7.50}
\end{align*}
$$

We again assume that a perturbation $\epsilon \sin n \theta$ is imposed on the free boundary, and seek an asymptotic solution. With all notation as in $\S 7.6$, the solution for $R_{1}$ is found to be of the form

$$
\begin{equation*}
R_{1}(\theta, t)=C(t) \sin n \theta, \tag{7.51}
\end{equation*}
$$

where $C(t)$ satisfies the equation and boundary condition

$$
\frac{d C}{d t}=\frac{(n+1)}{2(1-t)} C\left\{1-\frac{2 T n(n-1)}{\sqrt{1-t}}\right\}, \quad C(0)=1
$$

This has solution

$$
\begin{equation*}
C(t)=(1-t)^{-(n+1) / 2} \exp \left\{\alpha\left(\frac{1}{\sqrt{1-t}}-1\right)\right\} \quad 0 \leq t<1 \tag{7.52}
\end{equation*}
$$

(recall that $\alpha:=2 \operatorname{Tn}\left(n^{2}-1\right)$ ). From the critical condition for a perturbation to the free boundary to be maintained, $d C / d t=0$, we find the values for $n_{c}$ and $\lambda_{c}$,

$$
\begin{aligned}
& n_{c}=\left(\frac{R_{0}}{2 T}+\frac{1}{4}\right)^{1 / 2}+\frac{1}{2} \\
& \lambda_{c}=2 \pi R_{0} /\left\{\left(\frac{R_{0}}{2 T}+\frac{1}{4}\right)^{1 / 2}+\frac{1}{2}\right\}
\end{aligned}
$$

Again, there is a minimum radius, below which the viscous blob is stable to perturbations, and above which perturbations will grow. (This critical radius $a_{c}$ can be found by solving $\lambda_{c}=2 \pi R_{0}$, with $R_{0} \equiv 1$, to get a critical value for the surface tension parameter $T$, then solving for $a_{c}$ from (7.26).) The maximal growth rate in an unstable situation corresponds to the wavenumber $n_{m}$ and wavelength $\lambda_{m}$ given by

$$
\begin{aligned}
& n_{m}=\frac{1}{\sqrt{3}}\left(\frac{R_{0}}{2 T}+1\right)^{1 / 2} \\
& \lambda_{m}=2 \sqrt{3} \pi R_{0} /\left(\frac{R_{0}}{2 T}+1\right)^{1 / 2}
\end{aligned}
$$

exactly as obtained for the expanding bubble problem. The observed free boundary shapes for this problem are much less regular than for the expanding bubble, as we expect from this analysis: an initially perturbed (unstable) boundary will begin to form the requisite number of fingers, $n_{m}$, but as the radius decreases, the number of unstable wavelengths which can be accommodated decreases also (contrast this with the bubble problem, where as the radius increases, more unstable wavelengths can be accommodated, and secondary bifurcations occur). Those fingers which are slightly larger will thus grow at the expense of the smaller ones, inevitably giving irregular patterns.

Agreement between the theory and experiment is not so good as for the anticrack case, with more fingers initially observed than predicted by $n_{m}$. We may still look at the theory of the smalltime evolution, however, to see what this suggests. Again there will be three different régimes of the instability within the scope of linear theory: the very early stage during which $\epsilon \ll T$, and the later stages, $1 \gg \epsilon \sim T$, and $1 \gg \epsilon \gg T$. Since the results are similar to the anticrack ones, we do not go into detail.

### 7.8.1 The case $\epsilon \ll T$

The solution to this problem has been carried out to order $\epsilon^{2}$; the terms $R_{0}$ and $R_{1}$ are given in (7.50), (7.51) and (7.52), and the term $R_{2}$ has the form

$$
R_{2}(\theta, t)=C(t) \cos 2 n \theta+D(t)
$$

The additive function $D(t)$ is of no consequence, and if we write

$$
C(t)=\hat{C}(t) \exp \left(-\frac{2 \alpha}{\sqrt{x}}\right)
$$

for $x:=1-t$, then we find

$$
\begin{aligned}
\hat{C}= & \frac{x^{-(n+1 / 2)}}{2}\left\{\frac{(2 n+1)}{\beta^{2}}\left[\frac{\beta}{\sqrt{x}}-1-(\beta-1) \exp \left(-\beta\left(\frac{1}{\sqrt{x}}-1\right)\right)\right]\right. \\
& \left.\quad-\frac{\left(8 n^{2}-3\right)}{3 n^{2} \beta^{2}}\left[\frac{\beta^{2}}{2 x}-\frac{\beta}{\sqrt{x}}+1-\left(\frac{\beta^{2}}{2}-\beta+1\right) \exp \left(-\beta\left(\frac{1}{\sqrt{x}}-1\right)\right)\right]\right\} .
\end{aligned}
$$

As for the anticrack problem, with the parameter values of [73], the size of $\epsilon^{2} C(t)$ is negligible compared with the size of $\epsilon R_{1}$, for all times for which the perturbations are small compared with $T$ (in fact, $C(t)$ itself is everywhere very small for times of interest, while the coefficient of $\sin n \theta$ in $R_{1}$ grows monotonically, causing this expansion to become invalid). Hence the perturbations are still approximately sinusoidal in the $\epsilon \sim T$ régime. When finding the value of the dimensionless surface tension to use, the value $a=7$ was used in (7.26), this being obtained visually from the photograph in figure 10 of [73]; the mobility $\mathcal{M}$ has the same value as in $\S 7.6$, and the suction rate $Q$ was $1.04 \mathrm{~cm}^{2} \mathrm{~s}^{-1}$.

### 7.8.2 The cases $1 \gg \epsilon \sim T, 1 \gg \epsilon \gg T$

If we assume the surface tension to be $T=\epsilon \chi$ for some order-one quantity $\chi$, then with the initial perturbation $\epsilon \sin n \theta$, the solution to the perturbation problem, to order $\epsilon^{2}$, is

$$
R=R_{0}+\epsilon R_{1}+\epsilon^{2} R_{2}+O\left(\epsilon^{3}\right)
$$

where

$$
\begin{gather*}
R_{0}=\sqrt{1-t}, \quad R_{1}=(1-t)^{-(n+1) / 2} \sin n \theta  \tag{7.53}\\
R_{2}=\frac{(2 n+1) t}{4(1-t)^{n+3 / 2}} \cos 2 n \theta+\frac{2 n \chi\left(n^{2}-1\right)}{(1-t)^{(n+1) / 2}}\left(1-\frac{1}{\sqrt{1-t}}\right) \sin n \theta  \tag{7.54}\\
+\frac{1}{2 \sqrt{1-t}}\left(1-\frac{1}{(1-t)^{n+1}}\right) . \tag{7.55}
\end{gather*}
$$

As for the anticrack problem, the ZST analysis is obtained from this merely by setting $\chi=0$. There will be some period during which evolution follows the $\chi \neq 0$ solution, but when the perturbations are larger than $T$, evolution will be according to (7.53), (7.55), with $\chi=0$.

A typical evolution sequence in this régime for $n=8$ (based on observations from [73]), is shown in figure 7.14 ; in this case we see the initial stages of $c r a c k$ formation. The value chosen for $\epsilon$ was 0.05 . When the amplitude of the perturbations becomes comparable to unity this analysis breaks down, and nonlinear theory is needed. In particular, as mentioned earlier, although these early stages of the instability may give a quite regular pattern, a slightly larger perturbation will grow at the expense of smaller ones as the blob radius decreases, and fewer unstable wavelengths are able to "fit in".

### 7.9 The "curvature conjecture"

The work of $\S \S 7.6$ and 7.8 provide the first real evidence for our conjecture regarding cracks and anticracks, and when they will form. We saw that the expanding bubble case led to anticracks, and that there is evidence that we expect cracks in the small surface tension limit of the contracting viscous blob problem. These "prototype problems" loosely suggest that, in a given suction problem, if the overall shape of the fluid domain is convex, we expect the instability to be manifested via crack formation, whereas if it is concave, we expect anticrack formation to occur.

This statement is very woolly, and obviously does not cover all eventualities (e.g. flat free boundaries). We now attempt to clarify matters a little. Firstly, when we refer to the "overall shape" of the free boundary, we mean that the free boundary be convex/concave on some lengthscale which is large compared to the crack/anticrack width, $\epsilon$ say. (Obviously, on the $O(\epsilon)$ lengthscale the free boundary must be concave where a crack is forming, and convex where an anticrack is forming, simply by the nature of the tip geometries-see $\S 7.5$ for the results of the local analysis in the limiting case $\epsilon \rightarrow 0$.) This convexity/concavity need not be global, however, as we may imagine a situation like figure 7.15 occurring. In both cases we have in mind the problem with small positive surface tension, which is necessary for both crack and anticrack problems, since although ZST solutions can give good agreement with observations for the anticrack problem, the parameter $T$ is still needed to select the right solution in a given situation.


Figure 7.14: Evolution of the free boundary of the contracting viscous blob in the régime $\epsilon \gg T$, so that the ZST theory is applicable. The early stages of crack formation are apparent prior to breakdown of the linear theory.

Free boundary convex on a lengthscale greater than the crack width


Figure 7.15: A free boundary with crack and anticrack development.

If this conjecture is true, then we would expect a solution like the travelling-wave Ivantsov parabola (an exact ZST steady solution, with pressure at large distances growing like the squareroot of the distance from the parabola tip) to destabilise via anticrack formation, since the free boundary is globally concave here. In fact there is an exact solution, which is essentially a perturbation of the Ivantsov parabola, and which does give rise to anticracks. ${ }^{2}$ The mapping function from the right-half plane onto the fluid domain is given by

$$
\begin{equation*}
z=w(\zeta)=\zeta^{2}+b_{0} \zeta+c_{0} \log (\zeta+d)+e \tag{7.56}
\end{equation*}
$$

where $b_{0}$ and $c_{0}$ are positive constants (with $c_{0}>0$, otherwise the solution undergoes finite time blow-up via cusp formation, just as we needed $\alpha_{i}>0$ for the solutions of $\S 7.3 .2$ ) and $d$ and $e$ are functions of time. This map gives a solution to the P-G equation (2.6) provided the conditions

$$
\begin{aligned}
b_{0} e(t)-2 c_{0} d(t) & =A t+k_{1}, \\
e(t)+d(t)^{2}+b_{0} d(t)+c_{0} \log d(t) & =k_{2},
\end{aligned}
$$

are satisfied, for some positive constant $A$ (the negative pressure gradient at infinity in the $\zeta$ plane). A large-time balance in these equations requires

$$
e(t) \sim \frac{A t}{b_{0}}, \quad d(t) \sim \exp \left(-\frac{A t}{b_{0} c_{0}}\right)
$$

(the Ivantsov parabola itself has $c_{0}$ and $d(t)$ identically zero, and $\left.e(t)=A t / b_{0}\right)$. The anticrack tip is at

$$
\begin{aligned}
z_{t i p}=w(0) & =c_{0} \log d(t)+e(t) \\
& \equiv k_{2}-d(t)^{2}+b_{0} d(t) \\
& \sim k_{2} \quad \text { as } t \rightarrow \infty,
\end{aligned}
$$

[^28]

Figure 7.16: The "Ivantsov" anticrack solution.
and is thus asymptotically stagnant, as we expect from an anticrack solution. Thus, if the parameter $c_{0}$ is chosen to be small, the free boundary shape (for large times) is approximately a travelling-wave parabola, with a stagnant, narrow strip of fluid left behind along the real axis (see figure 7.16). If we had a ZST solution with a convex parabolic free boundary, our conjecture suggests that this should lead to cusp formation within finite time, and hence crack formation (in the limit of small positive surface tension); however, there is no known exact solution of the P-G equation for this geometry.

We should point out that there are definite counter-examples to our conjecture, although it may be argued that we would not expect such solutions to be observed in practice. Firstly, in [46], Howison presents a specific case of the radial fingering solutions (7.22) of [45], which is an expanding bubble solution, yet which can lead to finite-time cusp formation (which we expect to be regularised by crack formation in the $T \rightarrow 0$ limit). This is a consequence of a judicious choice of mapping parameters and initial conditions, however. In particular, the number of fingers is chosen as $n=4$, whereas in reality the number of fingers which develops is determined by (7.26) and (7.43). For small values of surface tension such as we have in mind, this will lead to large values of $n$, and evolution as in the experiments of $[73,11]$.

There are also exact ZST "contracting blob" type solutions, driven by a point sink at the origin, which can give rise to anticrack-type structures. These are given by the family of mapping functions from the unit disc,

$$
\begin{equation*}
w(\zeta)=b \zeta+\alpha \sum_{k=1}^{N} \varpi^{-k} \log \left(1+c \varpi^{k} \zeta\right) \tag{7.57}
\end{equation*}
$$

where $\alpha$ is some positive constant, $b(t)$ and $c(t)$ are positive functions of time (where $c \in(0,1)$ ), $N$ is a positive integer, and $\varpi$ is an $N^{\prime}$ th root of unity. This map is a solution of the P-G equation (2.4) provided the following conditions are satisfied,

$$
\frac{d}{d t}(b(b+N \alpha c))=-\frac{Q}{\pi}
$$



Figure 7.17: Anticrack-type structure generated by the map (7.57) with $N=4$, $\alpha=0.2, b(0)=0.755, c(0)=0.887$. We see the onset of cuspidal blow-up, after which we expect continuation by crack or slit evolution towards the point sink.


Figure 7.18: The crack (a) and anticrack (b) geometries generated by maps (7.58) and (7.59) with $\epsilon=0.1$.

$$
b c+\alpha \sum_{k=1}^{N} \varpi^{-k} \log \left(1+c^{2} \varpi^{k}\right)=\kappa
$$

for some constant $\kappa$. This can generate evolving free boundaries of the kind in figure 7.17; however, these solutions always ultimately blow up via cusp formation (at which point we expect the crack regularisation to come into effect), and in any case the $T>0$ analysis of $\S 7.8$ suggests that crack formation is what we expect to observe in a real "contracting blob" experiment.

### 7.10 Extremal conformal maps

We now consider briefly crack and anticrack geometries (and their slit and "anti-slit" limits) in terms of extremal conformal maps, and univalent function theory. Consider mapping the right-half $\zeta$-plane conformally onto a fluid domain which we assume to contain either a single (stationary, for simplicity) crack, or anticrack. A crack can be realised as the image of the right-half $\zeta$-plane under the map

$$
\begin{equation*}
z=\sqrt{(\zeta+\epsilon)^{2}+1} \quad(0<\epsilon \ll 1) \tag{7.58}
\end{equation*}
$$

(figure 7.18 (a)) where $\epsilon$ is a measure of the crack thickness. This map has a uniform limit $\epsilon \rightarrow 0$, which gives a so-called extremal map from the right-half plane onto itself, but with the slit $0 \leq x \leq 1$ removed. The term "extremal" indicates that, although this map is not itself conformal on the boundary of the domain, $\Re(\zeta)=0$, it is a limit point of some set of maps which are all conformal on $\Re(\zeta) \geq 0$. Hence in terms of conformal maps, a slit may be viewed as a straightforward limit of a crack.

Anticracks, on the other hand, do not have a sensible anti-slit limit like this. A typical anticrack geometry is generated by the conformal map

$$
\begin{equation*}
z=\zeta+\epsilon \log (\zeta+\exp (-1 / \epsilon)) \quad 0<\epsilon \ll 1 \tag{7.59}
\end{equation*}
$$

where again, $\epsilon$ is a measure of the anticrack thickness (figure 7.18 (b)). This is the kind of free boundary shape we expect, from all the exact ZST anticrack solutions we found. However, it is easily seen that this map does not have a uniform limit as $\epsilon \rightarrow 0$. We can only interpret it when $\epsilon>0$, in which case it gives a smooth, narrow anticrack with tip at $x=-1$.

Geometrically, the two cases are fundamentally different. An anti-slit structure (of the kind envisaged in $\S 7.7$ ) cannot be realised as a limit point of some set of univalent functions in the


Figure 7.19: The "open sets" interpretation of the slit geometry.
way that a slit map can. To see why this should be so, consider the topological ${ }^{3}$ definition of continuity, in terms of open sets. A function between topological spaces, $f: X \rightarrow Y$, is continuous if and only if the inverse images (under $f$ ) of any open sets in $Y$ are also open in $X$. The topology of the fluid domain (which we identify with $Y$ ), and the right-half $\zeta$-plane (which we identify with $X$ ), is just the usual $\mathbb{R}^{2}$ topology.

Without assuming any particular form for a mapping function $f$ between $X$ and $Y$ (but hypothesising that it be at least continuous) we may make the following observations. For the slit geometry, given a point $z_{*}$ on the slit boundary, we can always find some semicircular open set $U_{*} \subset Y$ containing this point, the inverse image of which will also be open in $X$ (figure 7.19). Since the edges of the slit are the only potential problem points, the function $f$ will be continuous in this case. For the anti-slit geometry, however, given a point $z_{*}$ on the anti-slit itself, the only kind of open set $U_{*} \subset Y$ which also contains the point is an open interval of points along the slit. (The anti-slit has the topology of $\mathbb{R}$, rather than of $\mathbb{R}^{2}$.) Since boundary points map to boundary points, the inverse image of this interval under the map must also be a line segment along the imaginary axis in $X$. Such a set is not open in the topology of $X$, since we cannot fit an 'open ball', around an interior point, within $f^{-1}\left(U_{*}\right)$, and hence there can exist no continuous (and certainly no univalent) map between the right-half plane and an anti-slit domain-see figure 7.20.

We may also interpret this result in terms of the Carathéodory theorem of kernel convergence (see for instance [20], whose explanation we paraphrase below), a result of major importance in geometric function theory. This theorem was also used in relation to the Hele-Shaw problem by Hohlov \& Howison [32], to derive estimates for geometric properties of the fluid domain in the injection problem. Suppose we have some sequence of simply-connected domains $\left\{D_{n}\right\}$ in the complex plane, all containing some fixed point $z_{0}$ (and none of which is the entire complex plane). Let $z=f_{n}(\zeta)$ be the conformal mapping from the right-half $\zeta$-plane onto $D_{n}$, normalised by the conditions $f_{n}(1)=z_{0}, f_{n}^{\prime}(1)>0$. There are two possible cases distinguished by Carathéodory. First, suppose that $z_{0}$ is an interior point of the intersection of the $D_{n}$. Then the kernel of the sequence $\left\{D_{n}\right\}$ is defined as the largest domain $D$ which contains $z_{0}$, and which has the property that each compact subset of $D$ lies in all but a finite number of the domains $D_{n}$. The other possibility is that $z_{0}$ is not an interior point of the intersection. In this case the kernel is defined as $D=\{0\}$. In either case the sequence $\left\{D_{n}\right\}$ is said to converge to its kernel if every subsequence has the same kernel.

[^29]

Figure 7.20: The "open sets" interpretation of the anti-slit geometry. The 'set of boundary points' is not open in the topology of $X$.

The basic theorem is as follows [20]:
Theorem (Carathéodory): Let $\left\{D_{n}\right\}$ be a sequence of simply-connected domains, with $z_{0} \in D_{n}, n=1,2, \ldots$, and suppose $f_{n}$ maps the right-half plane $X$ conformally onto $D_{n}$, and satisfies $f_{n}(1)=z_{0}, f_{n}^{\prime}(1)>0$. Let $D$ be the kernel of $\left\{D_{n}\right\}$. Then $f_{n} \rightarrow f$ uniformly on each compact subset of $X$ if and only if $D_{n} \rightarrow D \neq \mathbb{C}$. In the case of convergence there are two possibilities.

1. If $D=\{0\}$, then $f=0$.
2. If $D \neq\{0\}$, then $D$ is a simply-connected domain, $f$ maps $X$ conformally onto $D$, and the inverse functions $f_{n}^{-1}$ converge uniformly to $f^{-1}$ on each compact subset of $D$.

With a little modification, the maps (7.58), (7.59) may be made to satisfy the conditions of the theorem. Taking $\epsilon$ to be some function of $n$ which tends to zero as $n \rightarrow \infty(e . g . \epsilon=1 / n$ will do) a suitable crack mapping sequence is found to be

$$
f_{n}(\zeta)=\left[\frac{(\zeta+\epsilon)^{2}+1}{(1+\epsilon)^{2}+1}\right]^{1 / 2}
$$

which has $f_{n}(1)=1$ for all $n$, and which for small $\epsilon$ (large $n$ ) describes an approximate half-space having a crack with tip at

$$
f_{n}(0)=\frac{1}{\sqrt{2}}\left(1-\frac{\epsilon}{2}+O\left(\epsilon^{2}\right)\right) .
$$

In the notation of the theorem then, $z_{0}=1$ (the fixed point), which lies within the domain $D_{n}$ for all $n$. The kernel of this sequence of domains is the right-half plane minus the slit along $(0,1 / \sqrt{2})$, which is exactly the limit of the image domains $D_{n}$ as $n \rightarrow \infty$. The mapping functions converge uniformly to

$$
f(\zeta)=\frac{\sqrt{1+\zeta^{2}}}{\sqrt{2}}
$$

which is the map from $X$ onto the kernel of the domain sequence, $D$ (defined above). Hence we have a straightforward example of case 2 of the theorem.

A suitable anticrack mapping sequence is

$$
f_{n}(\zeta)=\frac{1}{2}\left[\zeta-1+\epsilon \log \left(\frac{\zeta+\exp (-1 / \epsilon)}{1+\exp (-1 / \epsilon)}\right)\right]
$$

which has the fixed (interior) point $z_{0}=0\left(\right.$ since $f_{n}(1)=0$ for all $n$ ). For small $\epsilon$ (large $n$ ) these maps describe anticrack-type structures, having anticrack tips at the points

$$
f_{n}(0)=-\frac{1}{2}-\frac{\epsilon}{2} \log (1+\exp (1 / \epsilon))<-1 .
$$

The limiting domain $D_{\infty}$ (which is exactly the intersection of all the $D_{n}$ ) is the anti-slit structure; the right-half plane, with an outward-pointing slit along $(-1,0)$. The fixed point $z_{0}=0$ is not an interior point of this intersection, so the kernel for this sequence is just $D=\{0\}$. Hence by the theorem, there is no limiting function $f$ which will map $X$ onto the anti-slit structure. Recalling the analysis described in $\S 7.5$, this result is not surprising, since the results of that section strongly suggest that the correct "anti-slit" limit will in fact be an outward-pointing cusp, rather than an outward-pointing slit.

### 7.11 Summary

Our aim in this chapter has been to present a new theory of anticrack solutions for the HeleShaw problem, which complements the existing crack theory of [36, 62, 33], reviewed in §7.1. The complementary nature of the two is illustrated by pictures like figure 7.7. Both cracks and anticracks may be viewed as regularisations of the ill-posed suction (or retreating viscous boundary) problem, though of very different kinds.

Several exact ZST anticrack solutions were given in $\S \S 7.3 .1,7.3 .2,7.3 .3$, but what we termed the "generic anticrack" (7.9) captures all the essentials of the behaviour. Anticrack tips are asymptotically stagnant (the Schwarz function having a logarithmic singularity within the tip), while the rest of the anticrack cannot "feel" the tip, and behaves like the travelling-wave solution (7.14).

The analysis of the expanding bubble problem in $\S 7.6$ revealed that selection of the anticrack width occurs as soon as the instability sets in. Once the anticrack has formed, it is basically stagnant, and does not influence the rest of the free boundary. We summarise this, for a general situation, by the statement that surface tension effects at the anticrack root govern anticrack selection. Contrast this with the crack theory, where it is the surface tension effects in the tip which govern the behaviour.

We also saw the contrast between cracks and anticracks in $\S 7.10$. The results there suggest that, although extremal univalent maps provide a good framework for the crack/slit theory, they are not suited to dealing with anticrack structures.

We cited experimental, numerical, and theoretical evidence in support of the crack and anticrack theories. In addition, for the anticrack, there is supporting evidence from the formal asymptotics of $[56,57]$ ( $\S 7.5$ ). Based on the analyses of $\S \S 7.6$ and 7.8 , we made a conjecture to the effect that, whether crack or anticrack formation occurs may depend on the curvature of the retreating free boundary. We point out, though, that cracks must remain as the dominant stabilising mechanism, in that they are the only component of the irregular morphology that can affect the smooth part of the boundary. Anticracks do not affect it; the free boundary can 'sprout' them, and carry on its way otherwise unchanged.

The overall picture we have in mind in this chapter is that the free boundary for a general suction problem (with small positive surface tension) will be the union of some array of smooth components, cracks, and anticracks (figure 7.21). Note, however, that for such a free boundary configuration to occur, the anticracks must develop first, since there are three timescales in operation-anticracks are stagnant relative to the motion of the smooth components of the boundary, which are themselves stationary relative to the crack propagation.


Figure 7.21: The free boundary for a general suction problem (with small surface tension).

## Chapter 8

## Discussion and further work

### 8.1 Comparison of Hele-Shaw and Stokes flow

One of our aims in this thesis has been to highlight the similarities and differences between HeleShaw flow and Stokes flow. Although we have mentioned such points of comparison as they occurred, it is helpful to summarise and discuss them here.

- Both Hele-Shaw and Stokes flow are quasistatic free boundary problems, by which we mean that the time dependence enters the problem only through the kinematic boundary condition.
- Both are governed by elliptic p.d.e.'s; the pressure field for Hele-Shaw is harmonic, and the streamfunction for Stokes flow is biharmonic. (Note that in Hele-Shaw, since the pressure is a velocity potential for the flow, its harmonic conjugate is a streamfunction, which is also harmonic.) Both problems are thus amenable to attack by similar complex variable methods. In general, one might expect a second-order p.d.e. to be simpler than a fourth-order one, but as we have seen in chapters 2 and 3, the NZST Stokes flow problem is very much more tractable analytically than the NZST Hele-Shaw problem. Recall, in this context, the comments of $\S 3.6 .2$, where we stated that the mapping function which gives a solution to the ZST Stokes flow problem will also give a solution to the NZST problem, but the same is not true for the Hele-Shaw problem.

Although it is more usual to consider the surface tension driven Stokes flow problem, with a velocity field which is everywhere analytic (since this arises in real-world situations such as sintering), much of the work in this thesis has been for the singularity driven ZST problem, which is the usual Hele-Shaw scenario. We have thus been able to make direct comparison between the two.

- In §5.4.2 we have stated that with one technical assumption $\phi(0)=0$ (for Stokes flow), the same conformal map will yield solutions for both the Hele-Shaw and Stokes flow problems (ZST, and singularity-driven) in the same geometry. However, we also saw that if we have more than one singularity in the flow (including singularities at infinity), the Stokes flow solution will, in general, be difficult to realise in practice, having moving singularities. This is not the case for Hele-Shaw flow; the basic reason (as discussed in §5.1) is that the Schwarz function evolution for Stokes flow is determined in the $\zeta$-plane, while for Hele-Shaw it is determined in the physical plane. Hence only for Hele-Shaw flow can we expect to be able to match the singularities of the Schwarz function with those of the flow.
- Both ZST problems are time-reversible. From this one may deduce that for both problems a contracting viscous blob, unless it is a circle with suction from the origin, must undergo finite time blow-up before all fluid has been extracted.
- It is a simple matter to demonstrate that a ZST Hele-Shaw flow with an advancing/retreating viscous boundary, is stable/unstable, respectively (the latter, which we loosely refer to as the Hele-Shaw suction problem, is well known to be ill-posed). By contrast, Stokes flow is neutrally stable to rigid-body motions, and the question of stability of advancing/retreating viscous boundaries does not arise. However, we are able to analyse nearly-circular "bubbles" and "blobs" in ZST Stokes flow, and we find that an expanding/contracting bubble or blob is stable/unstable respectively (see appendix A). For Hele-Shaw flow, the above stability result means that a contracting/expanding bubble is stable/unstable respectively, while an expanding/contracting blob is stable/unstable respectively (see $\S \S 7.6,7.8$ ). The Hele-Shaw instability is much more dramatic than the Stokes flow, as comparison of (7.38) and (7.52) (with $\alpha=0$, and for large $n$ ) with (A.6) and (A.3) shows.
Our belief is that the ZST Hele-Shaw instability leads to either anticrack formation, or finitetime blow-up via cusp formation or self-overlapping of the free boundary. It is thought that the only possibilities for Stokes flow are the latter two; no examples of Stokes anticracks are known.
- In the case of $3 / 2$-power cuspidal blow up (which is always terminal for the ZST problems), the behaviour of the velocity fields at the cusp as the blow up time $t^{*}$ is approached is very different in the two cases. In Hele-Shaw, the velocity becomes unbounded; for a contracting viscous blob of the kind considered in [49], the speed at the cusp behaves like

$$
\text { speed } \sim \frac{1}{\sqrt{t^{*}-t}} \quad \text { as } t \rightarrow t^{*}
$$

while for Stokes flow,

$$
\text { speed } \sim O(1) \quad \text { as } t \rightarrow t^{*}
$$

- For 5/2-power cuspidal blow-up it is possible for the solution to evolve through the cusped configuration. An example of this behaviour for ZST Hele-Shaw flow arose in $\S 5.3$ (and many others exist in the literature); for NZST Stokes flow see [85]-we do not give an example in this thesis. Note though, that in the cubic polynomial example of $\S 6.2,5 / 2$-power cusp formation is terminal for the ZST Stokes flow problem, whereas for the analogous Hele-Shaw problem, it is not [50]. Our results there show that in the $T \rightarrow 0$ limit, the $5 / 2$-power cusp becomes a $3 / 2$-power cusp, although it is a borderline case. It is possible that nonzero surface tension is necessary to have continuable cuspidal solutions (meaning that the free boundary be nonanalytic only for an instant, before smoothing again) for the Stokes flow problem.
- For both problems, a $T \rightarrow 0$ regularisation may be considered. For Stokes flow we saw that this leads to solutions which have persistent cusps in the free boundary (chapter 6 , the "weak solution" concept). For Hele-Shaw, the conjectured scenario is the slit model (chapter 7), with an air slit propagating into the fluid domain from the cusp, moving infinitely fast relative to the rest of the free boundary. Note that "weak" Stokes flow solutions are not time reversible, as the cusped "similarity" solutions of [49] demonstrate. A time-reversal for slit solutions can be proposed, however, since we would expect the slit to contract back along its length to the cusp, after which the free boundary would instantly smooth, and classical theory take over.

One of the most striking similarities between the two problems is the existence of the "moments", which obey the same evolution equations for the point sink (or source) problem, although they are differently defined ( $\S \S 2.5,3.6,3.6 .2$ ).

Underlying this difference in definition is the fact that the singularities of the Schwarz function for Hele-Shaw are determined in the physical domain, whereas for Stokes flow (in the case $\phi(0)=0$ ) they are determined within the unit disc in $\zeta$-space. Recall the repercussions that this had for
many-singularity problems: the ZST Hele-Shaw results are readily extended to cope with such problems, but the Stokes flow results are not.

Another analogy between the two problems is the existence of a Baiocchi transform for each ( $\S 22.6,3.9 .1)$. Again these are differently defined; like the "moments", the integrals in the definitions are carried out in physical space for Hele-Shaw, and in $\zeta$-space for Stokes flow.

### 8.2 Further work

At some points in this thesis we have been unable to complete the solution to particular problems. In other places we chose to leave a problem at a certain stage, when the work could have been extended in various directions. We list below some of the more interesting and important areas for future study related to the work of the thesis.

1. We have seen that, unlike the ZST Hele-Shaw theory, the ZST Stokes flow theory of chapter 3 can only deal with many driving singularities (or singularity-driven problems on unbounded domains) if we allow them to move, leading to solutions which are somewhat contrived. Essentially the same difficulty arose in $\S 5.4 .2$ where, to avoid a moving singularity, we required the solution to have $\phi(0) \neq 0$. This led to a very complicated formula for the conformal map, and we were unable to solve the problem fully. It is possible that there is a better way of attacking such problems (perhaps not involving complex variable methods?). Further investigation is needed if we are to develop a comprehensive theory to deal with time-dependent problems of this kind (note that the existing methods are perfectly adequate for steady problems).
2. It was mentioned in $\S 8.1$ above that although Hele-Shaw slit solutions are reversible in time, the "weak" similarity solutions of [49], with persistent cusps, are not-would injection into a cusped configuration give an expanding cusped shape of similarity type, or would the free boundary smooth instantly? This is an interesting open question, and may be compared with the Hele-Shaw results of [56], where weak solutions to the injection problem (with an acute-angled corner in the initial free boundary) are found to exhibit "waiting time" behaviour. In such solutions, the corner persists for the waiting time, at which the corner angle jumps to its supplement, then instantaneously smooths.
3. Recall the comment in footnote (3) of chapter 3, that the governing equations (3.18) have the form of a general conservation law. This enables a possible weak formulation to be written down, which we did not pursue. It would be interesting to investigate this point further, in particular, to see if it can be linked to the "weak solution" theory of chapter 6 . It is possible that the ZST breakdown time $t=t^{*}$ may be associated with a shock surface, across which the form of the solution changes, and that Rankine-Hugoniot conditions may be associated with persistent cusps. This suggestion is highly speculative as yet.
4. The crack/anticrack theory of chapter 7 is incomplete. More work needs to be done on the crack and slit theories, which were already known to be ill-posed and under-determined, respectively (and which we did not attempt to extend). Although we saw how the anticrack "selection at the root" worked for the expanding bubbles and contracting blobs of Paterson's experiments [73], we do not yet have a full understanding of the role played by surface tension in general anticrack solutions, and in particular, we do not have a clear idea of what we expect in the $T \rightarrow 0$ limit (if cracks become slits, what do anticracks become? Section 7.10 implies that the theory of extremal univalent maps cannot help us in this case). Finally, although in $\S 7.9$ we conjecture that the curvature of the free boundary may determine which particular instability is observed in a given situation, this is still a very tentative suggestion, which needs backing up with some hard evidence.

## Appendix A

## Stability of blobs and bubbles in Stokes flow

In this appendix we show how complex variable methods may be used to perform a simple linear stability analysis of expanding and contracting circular blobs and bubbles in ZST Stokes flow. We use known, exact solutions, in a suitable small parameter limit, to deduce our stability results.

## A. 1 The perturbed circular blob

Howison \& Richardson [49] presented exact solutions for the source or sink driven evolution of a viscous blob, described by the conformal map

$$
z=w(\zeta)=a\left(\zeta-\frac{b}{n} \zeta^{n}\right)
$$

for positive integer $n$, and time-dependent parameters $a$ and $b$ which may both be assumed real and positive by a suitable choice of axes. The equations governing the ZST evolution of $a$ and $b$ are given as

$$
\begin{array}{r}
a^{2} b=k \\
\frac{d S}{d t}=\frac{d}{d t}\left\{\pi a^{2}\left(1+\frac{b^{2}}{n}\right)\right\}=-Q \tag{A.2}
\end{array}
$$

for some positive constant $k$. Here, $S(t)$ denotes the area of the fluid domain, so $Q>0$ for a point sink at the origin, and $Q<0$ for a source. If we consider the case $b=\epsilon \ll 1$, then on the free boundary,

$$
|z|=a\left(1-\frac{\epsilon}{n} \cos (n-1) \theta+O\left(\epsilon^{2}\right)\right),
$$

which is just a sinusoidal perturbation to an expanding or contracting circular blob (i.e. linear stability theory).

To lowest order, (A.1) and (A.2) give the solution for $\epsilon(t)$ as

$$
\begin{equation*}
\epsilon(t)=\frac{\pi k}{S(t)} \tag{A.3}
\end{equation*}
$$

Hence we see that $\epsilon$ is growing in time for a point $\operatorname{sink}(S(t)$ decreasing), which means an unstable situation, and decreasing in time for a point source, i.e. a stable situation, the growth or decay being algebraic in $t$. Thus for viscous blobs, we have the same situation as for Hele-Shaw flow (see §7.8).

## A. 2 The perturbed circular bubble

The analogous bubble problem was solved by Tanveer \& Vasconcelos [96]. The conformal map for this case is

$$
z=w(\zeta)=a\left(\frac{1}{\zeta}-\frac{\epsilon}{n} \zeta^{n}\right)
$$

for positive integer $n$, where again $a>0$ and $0<\epsilon \ll 1$. The free boundary here is such that

$$
|z|=a\left(1-\frac{\epsilon}{n} \cos (n+1) \theta+O\left(\epsilon^{2}\right)\right)
$$

and the ZST equations governing the parameters are

$$
\begin{array}{r}
a^{2} \epsilon=k \\
\frac{d B}{d t}=\frac{d}{d t}\left\{\pi a^{2}\left(1-\frac{\epsilon^{2}}{n}\right)\right\}=Q \tag{A.5}
\end{array}
$$

for some positive constant $k$, where now $B(t)$ denotes the bubble area, so $Q>0$ for a sink at infinity, i.e. a growing bubble, and $Q<0$ for a shrinking bubble. Combining (A.4) and (A.5) gives the evolution of $\epsilon(t)$ (to lowest order) as

$$
\begin{equation*}
\epsilon(t)=\frac{\pi k}{B(t)} \tag{A.6}
\end{equation*}
$$

so a growing bubble is stable ( $\epsilon$ decreasing), while a shrinking bubble is unstable. This result is in direct contrast to the corresponding Hele-Shaw result of $\S 7.6$, where we saw that expanding bubbles are unstable (giving rise to anticracks), while contracting bubbles are stable.

## Appendix B

## Results used for the cubic polynomial map

In this Appendix we give the exact and asymptotic expressions for the functions $f_{+}(0, t)$ and $f_{+}^{\prime}(0, t)$ which we omitted from the text in chapter 6.

Using the definition (3.8) we find

$$
\begin{aligned}
f_{+}(0) & \equiv \frac{1}{\pi a} \int_{0}^{\pi} \frac{d \theta}{\left[b^{2}+(c-1)^{2}+2 b(1+c) \cos \theta+4 c \cos ^{2} \theta\right]^{1 / 2}}, \\
\text { and } f_{+}^{\prime}(0) & \equiv \frac{1}{\pi a} \int_{0}^{\pi} \frac{\cos \theta d \theta}{\left[b^{2}+(c-1)^{2}+2 b(1+c) \cos \theta+4 c \cos ^{2} \theta\right]^{1 / 2}},
\end{aligned}
$$

the forms of which functions change as we cross the curve $b^{2}=4 c$ in $V$ (according to whether the denominator has real or complex roots as a function of $\cos \theta) ; f_{+}(0)$ itself is continuous across this curve, however. In $b^{2}<4 c$ we use formula 3.145.2 in Gradshteyn \& Ryzhik [30] (henceforth $\mathrm{G} \& \mathrm{R}$ ), and also the asymptotic result

$$
\begin{equation*}
K(1-\epsilon) \sim-\frac{1}{2} \log (\epsilon / 8) \sim-\frac{1}{2} \log \epsilon \quad \text { as } \epsilon \rightarrow 0 \tag{B.1}
\end{equation*}
$$

where $K(\cdot)$ is the complete elliptic integral of the first kind. We find that

$$
\begin{aligned}
f_{+}(0) & =\frac{2 K\left(k_{1}\right)}{\pi a \sqrt{(c+1)^{2}-b^{2}}}, \quad \text { where } \quad k_{1}^{2}:=\frac{4 c-b^{2}}{(c+1)^{2}-b^{2}} \\
& \sim \frac{-2}{\pi a \sqrt{4-b^{2}}} \log (1-c) \quad \text { as } c \uparrow 1
\end{aligned}
$$

$c=1$ being the only singularity within this part of $V$. In $b^{2}>4 c$ we need formulae 3.147 .6 and 3.147.4 of $\mathrm{G} \& \mathrm{R}$ (in regions $c>0, c<0$ respectively) together with (B.1) to deduce that

$$
\begin{array}{rlrl}
f_{+}(0) & =\frac{2 K\left(k_{2}\right)}{\pi a\left(\sqrt{b^{2}-4 c}+(1-c)\right)} & & \text { where } \quad k_{2}^{2}:=\frac{4(1-c) \sqrt{b^{2}-4 c}}{\left[\sqrt{b^{2}-4 c}+(1-c)\right]^{2}} \\
& \sim \frac{-1}{\pi a(1-c)} \log (1+c-b) & \text { as }(1+c-b) \downarrow 0,
\end{array}
$$

$b=1+c$ now being the only line of singularities within $V$. Explicit formulae for $f_{+}^{\prime}(0)$ are much more complicated; in $b^{2}<4 c$ we find

$$
\begin{align*}
f_{+}^{\prime}(0)=\frac{-b(1+c) K\left(k_{1}\right)}{2 \pi a c \sqrt{(1+c)^{2}-b^{2}}}+ & \frac{1}{\sqrt{c}}\left\{E\left(k_{1}\right) F\left(\psi, k_{1}^{\prime}\right)+K\left(k_{1}\right)\left(E\left(\psi, k_{1}^{\prime}\right)-F\left(\psi, k_{1}^{\prime}\right)\right)\right\}, \\
& \text { where } \quad\left(k_{1}^{\prime}\right)^{2}=1-k_{1}^{2}, \quad \psi=\sin ^{-1}\left(\frac{b}{2 \sqrt{c}}\right) \tag{B.2}
\end{align*}
$$

here $k_{1}$ is as previously defined, $E(\cdot), E(\cdot, \cdot)$ denote the complete and incomplete (respectively) elliptic integrals of the second kind, and $F(\cdot, \cdot)$ denotes the incomplete elliptic integral of the first kind (so $K(\cdot) \equiv F(\pi / 2, \cdot)$ ). The key formulae used in finding this expression were 259.07 and 410.02 in Byrd \& Friedman [8] (henceforth B \& F), along with various properties of elliptic integrals and Jacobian elliptic functions, all of which may be found in B \& F.

In $b^{2}>4 c$ we find

$$
\begin{array}{r}
f_{+}^{\prime}(0)=\frac{2}{\pi a\left(\sqrt{b^{2}-4 c}+(1-c)\right)}\left\{A K\left(k_{2}\right)-(1+A) \Pi\left(\frac{2}{1-A}, k_{2}\right)\right\} \\
\text { where } \quad A=\frac{1}{4 c}\left(-b(1+c)+(1-c) \sqrt{b^{2}-4 c}\right) \tag{B.3}
\end{array}
$$

again, $k_{2}$ is as previously defined, and $\Pi(\cdot, \cdot)$ denotes the complete elliptic integral of the third kind. In finding this expression the formulae used were G \& R 3.148.6 and 3.148.4 (in regions $c>0$ and $c<0$ respectively).

In using these two books, care was necessary to account for slight differences in definitions. Likewise, when carrying out numerical checks on the analysis, care was needed due to different inbuilt definitions in the software package Mathematica. The above assumes the definitions:

$$
\begin{align*}
F(\phi, k) & =\int_{0}^{\phi} \frac{d \theta}{\sqrt{1-k^{2} \sin ^{2} \theta}}=\int_{0}^{\sin \phi} \frac{d x}{\sqrt{\left(1-x^{2}\right)\left(1-k^{2} x^{2}\right)}}  \tag{B.4}\\
E(\phi, k) & =\int_{0}^{\phi} \sqrt{1-k^{2} \sin ^{2} \theta} d \theta=\int_{0}^{\sin \phi} \frac{\sqrt{1-k^{2} x^{2}}}{\sqrt{1-x^{2}}} d x \\
\Pi\left(\alpha^{2}, k\right) & =\int_{0}^{\pi / 2} \frac{d \theta}{\left(1-\alpha^{2} \sin ^{2} \theta\right) \sqrt{1-k^{2} \sin ^{2} \theta}} \\
& =\int_{0}^{1} \frac{d x}{\left(1-\alpha^{2} x^{2}\right) \sqrt{\left(1-x^{2}\right)\left(1-k^{2} x^{2}\right)}}
\end{align*}
$$

The results of $\S 6.2$ require the asymptotic evaluation of the ratio $f_{+}^{\prime}(0) / f_{+}(0)$ near each of the lines $c=1$ and $b=1+c$. This is not too bad for the case $c \uparrow 1$, and fairly nasty for the case $b \downarrow(1+c)$; we give only brief details.

In $b^{2}<4 c$ results of $\mathrm{B} \& \mathrm{~F} \S \S 111-112$ are used, together with (B.1) above, to deduce that as $c \uparrow 1$, the term in curly brackets in $G_{+}^{\prime}(0)$ (B.2) is everywhere negligible compared to the first term. Hence we see that the asymptotic behaviour here is

$$
\begin{equation*}
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \sim-\frac{b}{2} \tag{B.5}
\end{equation*}
$$

To study the behaviour of $f_{+}^{\prime}(0)$ as $b \uparrow(1+c)$ in the region $b^{2}>4 c$ we must consider the cases $c>0$ and $c<0$ separately, since these give different types of behaviour in (B.3). We write $\epsilon=1+c-b$ and eliminate $b$ to work with $c$ and $\epsilon$, so that letting $\epsilon \rightarrow 0$ corresponds to approaching the univalency boundary $\partial V$. We also define the auxiliary parameter $\delta:=\epsilon^{2} /\left(4(1-c)^{2}\right)$; this will always be small since we do not consider the elliptical part of $\partial V$ corresponding to blow-up via overlapping of the free boundary, so $c$ lies in the range $-1<c<3 / 5$. We find:

$$
\begin{array}{r}
A=-1-2 \delta+\cdots, \quad-(1+A)=2 \delta+\cdots, \\
k_{2}^{2}=1-\delta\left(\frac{1+c}{1-c}\right)^{2}+\cdots, \quad \alpha^{2} \equiv \frac{2}{1-A}=1-\delta+\cdots
\end{array}
$$

We know the asymptotic behaviour of the first term in curly brackets in $f_{+}^{\prime}(0)$ (B.3), from (B.1). The term outside, multiplying the curly bracket, is also straightforward. Hence we only need to find the behaviour of the second term within curly brackets, which to lowest order is

$$
-(1+A) \Pi\left(\frac{2}{1-A}, k_{2}\right) \sim 2 \delta \Pi\left(\alpha^{2}, k_{2}\right)
$$

Suppose first that $c \in(0,3 / 5)$. Then by the above expressions, $0<\alpha^{2}<k_{2}^{2}<1$, and so according to the classifications of B \& F (p. 223) we have a case II elliptic integral of the third kind (a circular case). ${ }^{1}$ Formula 412.01 in B \& F thus applies, giving the result in terms of the Heuman Lambda function. Results from $\S 150$ of the book may then be used to arrive at the approximation

$$
2 \delta \Pi\left(\alpha^{2}, k_{2}\right)=\frac{1-c}{\sqrt{c}}\left(\frac{\pi}{2}-\sin ^{-1}\left(\frac{1-c}{1+c}\right)\right)+O(\delta \log \delta)
$$

which gives excellent agreement when checked numerically. This term will thus be everywhere negligible compared to the first term in the curly brackets ( $K\left(k_{2}\right)$ being singular as $k_{2} \rightarrow 1$ ), hence we get the approximation

$$
f_{+}^{\prime}(0) \sim \frac{1}{\pi a(1-c)} \log (1+c-b)
$$

It follows that for $c$ in this parameter range we will have

$$
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \sim-1
$$

as we approach the boundary.
For $c \in(-1,0)$ (still using (B.3)) we have $0<k_{2}^{2}<\alpha^{2}<1$, which is a case III elliptic integral of the third kind (a hyperbolic case). Thus formula 414.01 of B \& F applies, and it is relatively easy to see that

$$
2 \delta \Pi\left(\alpha^{2}, k_{2}\right) \simeq \frac{1-c}{\sqrt{-c}} K\left(k_{2}\right) Z\left(\beta, k_{2}\right) \quad \text { for } \quad \beta=\sin ^{-1}\left(\frac{\alpha}{k_{2}}\right)=\frac{\pi}{2}-\frac{2 \sqrt{-c \delta}}{1-c}+\cdots
$$

where $Z(\cdot, \cdot)$ denotes the Jacobi Zeta function (discussed in $\S 140$ of B \& F). Then

$$
f_{+}^{\prime}(0) \sim \frac{-K\left(k_{2}\right)}{\pi a(1-c)}\left(1-\frac{1-c}{\sqrt{-c}} Z\left(\beta, k_{2}\right)\right)
$$

so that

$$
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \sim-1+\frac{1-c}{\sqrt{-c}} Z\left(\beta, k_{2}\right)
$$

Results of $\S 140$ and $\S 100$ in B \& F show that for small $\delta$,

$$
Z\left(\beta, k_{2}\right) \sim \frac{1}{K\left(k_{2}\right)} \log \left(\frac{2 \sqrt{-c}}{1+c}+\left(1-\frac{4 c}{(1+c)^{2}}\right)^{1 / 2}\right)
$$

Note that $c=0$ is not a problem point, despite the factor $1 / \sqrt{-c}$ in the above, because for small $c$ we may expand the logarithmic term appearing in the expression for $Z\left(\beta, k_{2}\right)$. The only problem is at $c=-1$; away from this point we can see that

$$
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \sim-1
$$

Near $c=-1$, the function $Z\left(\beta, k_{2}\right)$ will no longer be negligible according to the above. Here we have

$$
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \sim-1+2\left(1+\frac{\log \epsilon}{\log (1+c)}\right)^{-1}=-1+2\left(1+\frac{\log \epsilon}{\log (\epsilon+b)}\right)^{-1}
$$

[^30]for $c$ close to -1 (or, $b$ small and positive). So, for instance, if we take $b=\lambda \epsilon$ for some order one quantity $\lambda$ we will have
$$
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \rightarrow 0 \quad \text { as } \epsilon \rightarrow 0
$$

In particular, this will be the case as we approach the univalency boundary along the phase path $b \equiv 0$. We thus have a nonuniform limit, with

$$
\begin{equation*}
\frac{f_{+}^{\prime}(0)}{f_{+}(0)} \rightarrow-1 \quad \text { as } \epsilon \rightarrow 0 \tag{B.6}
\end{equation*}
$$

everywhere except $c=-1$ ( or $b=0$ ); at this point the limit is zero.

## Appendix C

## The Stokes flow velocity field in terms of $w(\zeta, t)$

It is useful to have an expression for the Stokes flow velocity field in terms of the mapping function, so that we can easily check on the behaviour at "problem points", such as at infinity, for problems on unbounded domains, or near "cusps", in the $T \rightarrow 0$ problem. We now derive such an expression, starting from (3.41).

The form of this expression will depend on the behaviour of $\Phi$ at the origin. Consider first the case $\Phi(0)=0$, so that equations (3.14) and (3.18) apply. Substitution from these equations into (3.41) yields, after a certain amount of manipulation, the result

$$
\begin{align*}
2(u & -i v)=\bar{w}_{t}(1 / \zeta)+\overline{w_{t}(\zeta)}+\frac{w_{t}^{\prime}(\zeta)}{w^{\prime}(\zeta)}(\bar{w}(1 / \zeta)-\overline{w(\zeta)}) \\
& -\frac{T}{2 \mu}(\bar{w}(1 / \zeta)-\overline{w(\zeta)})\left\{\frac{\partial}{\partial \zeta}\left[\zeta\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right]+\frac{w^{\prime \prime}(\zeta)}{w^{\prime}(\zeta)} \zeta\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right\} \\
& +\frac{T}{2 \mu}\left\{\overline{\zeta w^{\prime}(\zeta)}\left(2 \overline{f_{+}(\zeta)}-f_{+}(0)\right)-\frac{1}{\zeta} \bar{w}^{\prime}(1 / \zeta)\left(2 f_{+}(\zeta)-f_{+}(0)\right)\right\} \tag{C.1}
\end{align*}
$$

This has some symmetry, but is still very cumbersome, save in the ZST case, when most of the terms vanish. Clearly, the analogous expression for $\Phi(0)=A \neq 0$ will be even worse, but note that by equations (3.15) and (3.19), terms in $A$ and $T$ are quite separate. Hence if we just find the ZST version of (C.1) for this case, the NZST version (should we need it) will follow by adding on the surface tension terms from (C.1). Setting $T=0$ in (3.15) and (3.19) and substituting into the formula (3.41) gives the result

$$
\begin{align*}
2(u-i v) & =\bar{w}_{t}(1 / \zeta)+\overline{w_{t}(\zeta)}+\frac{2 A}{w^{\prime}(0)}\left\{\bar{w}^{\prime}(1 / \zeta)\left(1-\frac{1}{\zeta^{2}}\right)+\overline{w^{\prime}(\zeta)}\left(1-\bar{\zeta}^{2}\right)\right\} \\
& +\frac{1}{w^{\prime}(\zeta)}(\bar{w}(1 / \zeta)-\overline{w(\zeta)})\left\{w_{t}^{\prime}(\zeta)+\frac{2 A}{w^{\prime}(0)} \frac{\partial}{\partial \zeta}\left[w^{\prime}(\zeta)\left(1-\zeta^{2}\right)\right]\right\} \tag{C.2}
\end{align*}
$$

or, in terms of the scaled time variable $\tau$ introduced in $\S 3.8$,

$$
\begin{align*}
\frac{w^{\prime}(0)}{A}(u-i v) & =\bar{w}_{\tau}(1 / \zeta)+\overline{w_{\tau}(\zeta)}+\left\{\bar{w}^{\prime}(1 / \zeta)\left(1-\frac{1}{\zeta^{2}}\right)+\overline{w^{\prime}(\zeta)}\left(1-\bar{\zeta}^{2}\right)\right\} \\
& +\frac{1}{w^{\prime}(\zeta)}(\bar{w}(1 / \zeta)-\overline{w(\zeta)})\left\{w_{\tau}^{\prime}(\zeta)+\frac{\partial}{\partial \zeta}\left[w^{\prime}(\zeta)\left(1-\zeta^{2}\right)\right]\right\} \tag{C.3}
\end{align*}
$$

## References

[1] Acrivos, A. The breakup of small drops and bubbles in shear flows. Ann. N.Y. Acad. Sci. 404, 1-11 (1983).
[2] Antanovskil, L.K. Influence of surfactants on a creeping free boundary flow induced by two counter-rotating horizontal thin cylinders. Eur. J. Mech. B/Fluids 13 (1), 73-92 (1994).
[3] AntanovskiI, L.K. A plane inviscid incompressible bubble placed within a creeping viscous flow: formation of a cusped bubble. Eur. J. Mech. B/Fluids 13 (4), 491-509 (1994).
[4] Antanovskir, L.K. Mathematical modelling of formation of a pointed drop in a four-roll mill. Preprint (1995).
[5] Baiocchi, C. Ann. Mat. Pura. Appl. 92, 107-127 (1972).
[6] Batchelor, G.K. An introduction to Fluid Dynamics. C.U.P. (1967).
[7] Buckmaster, J.D. Pointed bubbles in slow viscous flow. J. Fl. Mech. 55, 385-400 (1972).
[8] Byrd, P.F., Friedman, M.D. Handbook of Elliptic Integrals for Engineers and Scientists. Springer-Verlag (1971).
[9] Caginalp, G. Stefan and Hele-Shaw type models as asymptotic limits of the phase field equations. Phys. Rev. A 39, 5887 (1989).
[10] Carrier, G.F., Krook, M., Pearson, C.E. Functions of a Complex Variable. Hod Press: Ithaca, New York (1983).
[11] Chen, J.-D. Growth of radial viscous fingers in a Hele-Shaw cell. J. Fl. Mech. 201, 223-242 (1989).
[12] Chen, X., Hong, J., Yi, F. Existence, uniqueness and regularity of classical solutions of the Mullins-Sekerka problem. Preprint (1996).
[13] Combescot, R., Hakim, V., Dombre, T., Pomeau, Y., Pumir, A. Shape selection for Saffman-Taylor fingers. Phys. Rev. Lett. 56, 2036-2039 (1986).
[14] Couder, Y., Gérard, N., Rabaud, M. Narrow fingers in the Saffman-Taylor instability. Phys. Rev. A - General Physics 34 (6), 5175-5178 (1986).
[15] Cowling, V.F., Royster, W.C. Domains of variability for univalent polynomials. Proc. Amer. Math. Soc. 19, 767-772 (1968).
[16] Crowdy, D.G., Tanveer, S. A theory of exact solutions for plane viscous blobs. Preprint (1996).
[17] Davis, P.J. The Schwarz function and its applications. Carus Math. Monographs 17, Math. Assoc. of America (1974).
[18] DiBenedetto, E., Friedman, A. The ill-posed Hele-Shaw model and the Stefan problem for supercooled water. Trans. Am. Math. Soc. 282, 183-204 (1984).
[19] Duchon, J., Robert, R. Interface evolution by capillarity and volume diffusion (1). Existence locally in time. Annales de l'Institute Henri Poincarè-Analyse non-lineaire 1 (5), 361-378 (1984).
[20] Duren, P.L. Univalent functions. Springer-Verlag (1983).
[21] Elliott, C.M., Janovsky, V. A variational inequality approach to Hele-Shaw flow with a moving boundary. Proc. Roy. Soc. Edin. A88, 93-107 (1981).
[22] Elliott, C.M., Ockendon, J.R. Weak and variational methods for moving boundary problems. Pitman, London (1982).
[23] Elliott, C.M., Gardiner, A.R. Double obstacle phase field computations of dendritic growth. Preprint (1996).
[24] Entov, V.M., Etingof, P.I., Kleinbock, D.Ya. Hele-Shaw flows with a free boundary produced by multipoles. Euro. J. Appl. Math. 4, 97-120 (1993).
[25] Entov, V.M., Etingof, P.I., Kleinbock, D.Ya. On nonlinear interface dynamics in Hele-Shaw flows. Euro. J. Appl. Math. 6 (5), 399-420 (1996).
[26] Escher, J., Simonett, G. On Hele-Shaw models with surface tension. Math. Research Letters (to appear).
[27] Escher, J., Simonett, G. Classical solutions of multi-dimensional Hele-Shaw models. S.I.A.M. Jl. Math. Anal. (to appear).
[28] Galin, L.A. Unsteady filtration with a free surface. Dokl. Akad. Nauk. S.S.S.R. 47, 246-249 (1945).
[29] Garabedian, P.R. Free boundary flows of a viscous liquid. Comm. Pure \& Appl. Math. XIX (4), 421-434 (1966).
[30] Gradshteyn, I.S., Ryzhik, I.M. Table of Integrals, Series and Products (corrected and enlarged edition). Academic Press, inc. (1980).
[31] Hele-Shaw, H.S. The flow of water. Nature, 58 (1489), 34-36 (1898).
[32] Hohlov, Yu.E., Howison, S.D. On the classification of solutions to the zero surface tension model for Hele-Shaw free boundary flows. Quart. Appl. Math. 51 (4), 777-789 (1993).
[33] Hohlov, Yu.E., Howison, S.D., Huntingford, C., Ockendon, J.R., Lacey, A.A. A model for non-smooth free boundaries in Hele-Shaw flow. Q. Jl. Mech. Appl. Math. 47 (1), 107-128 (1994).
[34] Hong, D.C., Langer, J.S. Analytic theory for the selection of Saffman-Taylor fingers. Phys. Rev. Lett. 56, 2032-2035 (1986).
[35] Howison, S.D., Ockendon, J.R., Lacey, A.A. Singularity development in moving boundary problems. Q. Jl. Mech. Appl. Math. 38 (3), 343-360 (1985).
[36] Howison, S.D., Lacey, A.A. Ockendon, J.R. Hele-Shaw free boundary problems with suction. Q. Jl. Mech. Appl. Math. 41, 183-193 (1988).
[37] Hopper, R.W. Coalescence of two equal cylinders: Exact results for creeping viscous plane flow driven by capillarity. J. Am. Ceram. Soc. (Comm.) 67, C262-264. Errata, ibid. 68, C138 (1985).
[38] Hopper, R.W. Plane Stokes flow driven by capillarity on a free surface. J. Fl. Mech. 213, 349-375 (1990).
[39] Hopper, R.W. Stokes flow of a cylinder and half-space driven by capillarity. J. Fl. Mech. 243, 171-181 (1992).
[40] Hopper, R.W. Capillarity-driven plane Stokes flow exterior to a parabola. Q. Jl. Mech. Appl. Math. 46, 193-210 (1993).
[41] Howard, G.C., Fast, C.R. Hydraulic Fracturing. Soc. Petrol. Eng., AIME, Dallas, TX. (1970).
[42] Howell, P.D. Extensional thin layer flows. D. Phil. Thesis, University of Oxford (1994).
[43] Howell, P.D. The evolution of a slender bubble with elliptical cross-section in an extensional flow. (In preparation) (1996).
[44] Howison, S.D. Private communication (1995).
[45] Howison, S.D. Fingering in Hele-Shaw cells. J. Fl. Mech. 167, 439-453 (1985).
[46] Howison, S.D. Cusp development in Hele-Shaw flow with a free surface. S.I.A.M. Jl. Appl. Math. 46 (1), 20-26 (1986).
[47] Howison, S.D. Bubble growth in porous media and Hele-Shaw cells. Proc. Roy. Soc. Edin. A. 102, 141-148 (1986).
[48] Howison, S.D. Complex variable methods in Hele-Shaw moving boundary problems. Europ. J. Appl. Math. 3, 209-224 (1992).
[49] Howison, S.D., Richardson, S. Cusp development in free boundaries, and twodimensional slow viscous flows. Europ. J. Appl. Math. (1995).
[50] Huntingford, C. An exact solution to the one-phase zero surface-tension Hele-Shaw freeboundary problem. Computers Math. Applic. 29 (10), 45-50 (1995).
[51] Ivantsov, G.P. Dokl. Akad. Nauk. S.S.S.R. 58, 567 (in Russian) (1947).
[52] Jeong, J., Moffatt, H.K. Free-surface cusps associated with flow at low Reynolds number. J. Fl. Mech. 241, 1-22 (1992).
[53] Joseph, D.D., Nelson, J., Renardy, M., Renardy, Y. Two-dimensional cusped interfaces. J. Fl. Mech. 223, 383-409 (1991).
[54] Kelly, E.D., Hinch, E.J. Numerical solutions of Entov's multipole problem in the HeleShaw cell. Submitted to Europ. J. Appl. Math. (1995).
[55] Kelly, E.D., Hinch, E.J. Numerical simulations of sink flow in the Hele-Shaw cell with small surface tension. Submitted to Europ. J. Appl. Math. (1996).
[56] King, J.R., Lacey, A.A., Vazquez, J.L. Persistence of corners in free boundaries in Hele-Shaw flow. Europ. J. Appl. Math. 6 (5), 455-490 (1995).
[57] King, J.R. Development of singularities in some moving boundary problems Europ. J. Appl. Math. 6 (5), 491-507 (1995).
[58] King, J.R. Private communication (1995).
[59] Kopf-Sill, A.R., Homsy, G.M. Narrow fingers in a Hele-Shaw cell. Phys. Fluids 30 (9), 2607-2609 (1987).
[60] Lacey, A.A. Moving boundary problems in the flow of liquid through porous media. Jl. Austral. Math. Soc. B24, 171-193, (1982).
[61] Lacey, A.A. I.M.A.J. Appl. Math. 35, 357-364 (1985).
[62] Lacey, A.A., Howison, S.D., Ockendon, J.R., Wilmott, P. Irregular morphologies in unstable Hele-Shaw free boundary problems. Q. Jl. Mech. Appl. Math. 43, 387-405 (1990).
[63] Langlois, W.E. Slow Viscous Flow. Macmillan (1964).
[64] Mallet-Paret, J. Generic unfoldings and normal forms of some singularities arising in the obstacle problem. Duke Math. J. 46, 645-683 (1979).
[65] McLean, J.W., Saffman, P.G. The effect of surface tension on the shape of fingers in a Hele-Shaw cell. J. Fl. Mech. 102, 455-469 (1981).
[66] Mineev-Weinstein, M.B., Ponce-Dawson, S. Class of nonsingular exact solutions for Laplacian pattern formation. Phys. Review E 50 (1), 24-27 (1994).
[67] Morgan, J.D. Codimension-Two free boundary problems. D. Phil. Thesis, University of Oxford (1994).
[68] Nehari, Z. Conformal Mapping. Dover (1975).
[69] Nie, Q., Tian, F.R. Singularities in Hele-Shaw flows. Submitted to SIAM J. Appl. Math. (1996).
[70] Nie, Q., Tanveer, S. The evolution of an axisymmetric Stokes bubble with volumetric change. Submitted to Phys. Fluids (1996).
[71] Ockendon, H., Ockendon, J.R. Viscous Flow. C.U.P. (1995).
[72] Pamplin, B.R. Crystal Growth. Pergamon, Oxford (1975).
[73] Paterson, L. Radial fingering in a Hele-Shaw cell. J. Fl. Mech. 113, 513-529 (1981).
[74] Pitts, E. Penetration of fluid into a Hele-Shaw cell: the Saffman-Taylor experiment. J. Fl. Mech. 97, 53-64 (1980).
[75] Polubarinova-Kochina, P.Ya. On the motion of the oil contour. Dokl. Akad. Nauk. S.S.S.R. 47, 254-257 (in Russian) (1945).
[76] Polubarinova-Kochina, P.Ya. Theory of groundwater movement. University Press, Princeton (1962).
[77] Rallison, J.M. The deformation of small viscous drops and bubbles in shear flows. Ann. Rev. Flu. Mech. 16, 45-66 (1984).
[78] Richardson, S. Two dimensional bubbles in slow viscous flows. J. Fl. Mech. 33 (3), 476-493 (1968).
[79] Richardson, S. Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. J. Fl. Mech. 56, 609-618 (1972).
[80] Richardson, S. Two dimensional bubbles in slow viscous flows. Part 2 J. Fl. Mech. 58 (1), 115-127 (1973).
[81] Richardson, S. Some Hele-Shaw flows with time-dependent free boundaries. J. Fl. Mech. 102, 263-278 (1981).
[82] Richardson, S. Two-dimensional slow viscous flows with time-dependent free boundaries driven by surface tension. Europ. J. Appl. Math. 3, 193-207 (1992).
[83] Richardson, S. Hele-Shaw flows with time-dependent free boundaries involving injection through slits. Studies in Appl. Math. 87, 175-194 (1992).
[84] Richardson, S. Hele-Shaw flows with time-dependent free boundaries involving a concentric annulus. Submitted to Phil. Trans. Roy. Soc. (1995).
[85] Richardson, S. Two-dimensional Stokes flows with time-dependent free boundaries driven by surface tension. Preprint (1996).
[86] Rubinstein, L.I. The Stefan Problem. American Mathematical Society, Providence (1971).
[87] Saffman, P.G., Taylor, G.I. The penetration of fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. Proc. Roy. Soc. Lond. A 245, 312-329 (1958).
[88] Saffman, P.G. Exact solutions for the growth of fingers from a flat interface between two fluids in a porous medium or Hele-Shaw cell. Q. Jl. Mech. Appl. Math. 12, 146-150 (1959).
[89] Saffman, P.G. Selection Mechanisms and stability of fingers and bubbles in Hele-Shaw cells. I.M.A. Journal of Applied Mathematics. 46 (1-2), 137-145 (1991).
[90] Schaeffer, D.G. Some examples of singularities in a free boundary. Ann. Scuol. Norm. Sup. Pisa 4, 133-144 (1977).
[91] Shraiman, B.I. Velocity selection and the Saffman-Taylor problem. Phys. Rev. Lett. A 56, 2028-2031 (1986).
[92] Sutherland, W.A. Introduction to metric and topological spaces. O.U.P (1975).
[93] Tanveer, S. New solutions for steady bubbles in a Hele-Shaw cell. Phys. Fluids 30 (3), 651-658 (1987).
[94] Tanveer, S. Analytic theory for the selection of a symmetric Saffman-Taylor finger in a Hele-Shaw cell. Phys. Fluids 30 (6), 1589-1605 (1987).
[95] Tanveer, S., Vasconcelos, G.L. Bubble breakup in two dimensional Stokes flow. Phys. Rev. Lett. 73 (21), 2845-2848 (1994).
[96] Tanveer, S., Vasconcelos, G.L. Time-evolving bubbles in two dimensional Stokes flow. J. Fl. Mech. 301, 325-344 (1995).
[97] Turcotte, D.L. Fractals and chaos in geology and geophysics. C.U.P. (1992).
[98] Vanden-Broeck, J.M. Fingers in a Hele-Shaw cell with surface tension. Phys. Fluids 26, 2033-2034 (1983).
[99] Vasconcelos, G.L., Kadanoff, L.P. Stationary solutions for the Saffman-Taylor problem with surface tension. Phys. Rev. A 44 (10), 6490-6495 (1991).
[100] Vorst, G.A.L. Van de. Modelling and numerical simulation of viscous sintering. $\operatorname{PhD}$ Thesis, Eindhoven University of Technology (1994).
[101] Youngren, G.K., Acrivos, A. On the shape of a gas bubble in a viscous extensional flow. J. Fl. Mech. 76 (3), 433-442 (1976).


[^0]:    ${ }^{1}$ See for example [71] for the details.

[^1]:    ${ }^{2}$ This dimensionless parameter is defined by $\operatorname{Re}=\rho U L / \mu$, where $U$ is a typical flow speed, $L$ is a typical lengthscale of the flow, $\rho$ is the fluid density, and $\mu$ is the viscosity. It is a measure of the ratio of inertial effects to viscous effects in the flow.
    ${ }^{3}$ But see Richardson [80]: ". . . the [two-dimensional] solutions derived show remarkable similarities with the observed behaviour of [the three-dimensional bubbles encountered in practice]..., suggesting that often the essential physics is retained, even if one is solving the 'wrong' problem!"

[^2]:    ${ }^{4}$ Cuspidal blow-up is not the only possibility; for instance, solutions can break down via corners forming in the free boundary [56]. Breakdown may also occur via the free boundary beginning to "overlap" itself (see §2.7), so that at the instant of breakdown, the fluid domain changes from simply to multiply connected. With the current theory the solution cannot be continued; however if one has a theory applicable to multiply connected domains, the possibility exists of continuing the solution beyond blow-up time - see Richardson [84].
    ${ }^{5}$ The term "viscosity" refers to the kind of solution, and should not be confused with physical viscosity.

[^3]:    ${ }^{1}$ We refer forward to $\S 2.7$ for more discussion of what exactly we mean by "univalency"; for the moment it is enough to note that we require $w(\zeta, t)$ analytic (except possibly at a single point which maps to infinity in the case that we have an unbounded fluid domain), and that the free boundary it describes must be smooth and simple.

[^4]:    ${ }^{2}$ If the coefficients $a_{1}, a_{2}$ are non-real initially, we can always rotate the co-ordinates so that the fluid domain is symmetric about the $x$-axis, which will ensure that the map relative to the new co-ordinates has real coefficients (with the singularity still at the origin); this symmetry will clearly then persist for $t>0$. Moreover, if we have $a_{2}<0$, then the transformation $\zeta \rightarrow-\zeta, a_{1} \rightarrow-a_{1}$ sets both coefficients to be of the same sign, so that we may assume them both to be positive without loss of generality.

[^5]:    ${ }^{3}$ This theorem states that for a function $f$ analytic on a domain $D, \iint_{D} f(z) d x d y=\frac{1}{2 i} \int_{\partial D} \bar{z} f(z) d z$, and is a trivial consequence of the usual Green's theorem in the plane.

[^6]:    ${ }^{4}$ These ideas are also generalised to the case of a multipole singularity at the origin by Entov et al. in [24].

[^7]:    ${ }^{5}$ The free boundary can always be written as $f(\mathbf{x}, t)=0$. In principle, this can be solved to give $t$ as a function of ( $x, y$ ), but this function will only be single-valued provided the free boundary never "moves back on itself", and also, will only be defined on the set of points crossed by the free boundary.

[^8]:    ${ }^{6}$ If we are mapping to an unbounded fluid domain then we will have a single point $\zeta_{\infty}$, within the unit disc, mapping to infinity, i.e. an isolated singularity of $w(\zeta)$, but this is a special case and the theory still holds.

[^9]:    ${ }^{1}$ The main difference in notation is that we define $\mathcal{X}(\zeta)=\chi(w(\zeta))$ (equation (3.5)), whereas in [82], X( $\zeta$ ) := $\chi^{\prime}(w(\zeta))$, which we consider confusing.

[^10]:    ${ }^{2}$ There may be certain unusual problems where we wish to specify a given, nonzero momentum for the fluid domain; this does not invalidate our discussion, since the choice $\Phi(0) \neq 0$ may not give us the desired value for the momentum. We emphasise the zero-net-momentum case only because this is most likely to be the physicallyrelevant one.

[^11]:    ${ }^{3}$ It is interesting to note that these evolution equations have the form of a general conservation law, that is, $\partial P / \partial t+\partial Q / \partial \zeta=0$ (a fact which has also been noted by [16]). The weak formulation of such a law takes the form $\int_{S}(P d \zeta-Q d t)=0$, where the region of integration $S$ will be the surface of some cylinder in $(\zeta, t)$-space ( $P$ and $Q$ are evident from (3.18), (3.19)).

[^12]:    ${ }^{4}$ These are measured so that $u_{n}>0$ if the motion is along the outward normal, and $u_{t}>0$ if the velocity is along the anticlockwise tangent vector.

[^13]:    ${ }^{5}$ We really need the proviso here that we consider only classical solutions in which the free boundary is analytic for all times less than the breakdown time, since in chapter 6 we shall see "weak" solutions where a nonanalytic free boundary is permitted, and time-reversibility cannot be inferred.

[^14]:    ${ }^{6}$ It will in fact be sufficient for our purposes to ensure that we start off at time $t=0$ with a univalent map, since the polynomial solution is bound to break down within finite time anyway by the observation of $\S 3.5$. The coefficients evolve smoothly with time, so we are at least guaranteed local existence of the solution in time by doing this.

[^15]:    ${ }^{7}$ This is just equation (2.11) reformulated in the $\zeta$-plane for direct comparison with the Stokes flow result. The conserved quantities for Hele-Shaw can be derived without recourse to complex variable theory, and extension to higher spatial dimensions is straightforward-see for example [35]-which seems to be not the case for Stokes flow.

[^16]:    ${ }^{8}$ Note also that the concept of "exterior" cannot be sensibly applied to the functions in the $\zeta$-plane, since although the domain $\{|\zeta| \leq 1\}$ maps to the interior of the fluid domain, it is not the case that $\{|\zeta| \geq 1\}$ maps to the exterior of the fluid domain. This is being rather pedantic, however; the important thing is that the functions $H_{e}(\zeta)$ and $G_{e}(\zeta)$ contain all the singularities of $H(\zeta)$ and $G(\zeta)$ within the unit disc.

[^17]:    ${ }^{1}$ In the work of [82] and [42], this result is only found as a by-product of the analysis for each specific example considered, by suitably manipulating the p.d.e.'s satisfied by the parameters of the mapping function. In [42] the result is found by elementary considerations elsewhere, but our method has the advantage of not relying on any extraneous analysis.

[^18]:    ${ }^{1}$ We shall use the term "fully-infinite" to denote an unbounded flow domain containing a finite air bubble (or more than one finite bubble if the domain is multiply connected, but we shall not study such cases). "Semi-infinite" will refer to domains where the area of air present is also unbounded.

[^19]:    ${ }^{2}$ Note that the system (5.12)-(5.15) could have been obtained working wholly in the $\zeta$-plane, using the ZST version of (3.38).

[^20]:    ${ }^{3}$ We note in passing that the analysis is easily generalised to the case of a general multipole singularity at the origin, for which the function $\mathcal{X}(\zeta)$ has the behaviour

    $$
    \begin{equation*}
    \mathcal{X}(\zeta)=\frac{M}{\zeta^{n} w^{\prime}(0)^{n}}+O\left(\frac{1}{\zeta^{n-1}}\right), \quad \text { as } \zeta \rightarrow 0 . \tag{5.28}
    \end{equation*}
    $$

[^21]:    ${ }^{4}$ As commented there (and again in footnote (3)), we could equally well consider a multipole singularity at the origin. For a multipole of order $n$ the function $\mathcal{X}(\zeta)$ has the behaviour of (5.28); this would also lead us to the same map as for Hele-Shaw, namely (5.24).

[^22]:    ${ }^{5}$ As an aside, we note that if we consider the simpler problem of a point sink singularity in this geometry, the same unphysical result is obtained.

[^23]:    ${ }^{1}$ These solutions are not "weak" in the true sense of the definition; we use the terminology because the solutions are not classical. To avoid cumbersome notation we now drop the inverted commas when referring to them.
    ${ }^{2}$ In this paper (which was discussed in detail in $\S 5.4$ ), the radius of curvature at the near-cusp is found to be proportional to $\exp (-32 \pi C a)$.

[^24]:    ${ }^{3}$ Note, though, that we are actually considering the projection of a "univalency cylinder" in ( $a, b, c$ ) space, onto $a=1$, with (6.6) providing the extra information about the variation of $a$ with time.

[^25]:    ${ }^{4}$ Note that $B$ and $A$ are proportional to the 2 nd and 3rd Stokes flow "moments" of the map (6.5), from the definition (3.36).

[^26]:    ${ }^{5}$ Such continuable solutions can be found for the NZST Stokes flow problem, however; see [85].

[^27]:    ${ }^{1}$ Solutions to the ZST suction problem can be constructed in which the free boundary reaches a singularity of the Schwarz function within finite time, but such solutions are the time-reversals of blowing problems with nonanalytic initial data (see for instance [32]), and therefore "pathological" examples.

[^28]:    ${ }^{2}$ We owe this observation to Dr K. Kornev of Moscow.

[^29]:    ${ }^{3}$ For more details on precise topological definitions see, for instance, [92].

[^30]:    ${ }^{1}$ The case $c=0$ is the special case $\alpha^{2}=k^{2}$, and provides a check on the analysis in both regions $c>0, c<0$.

