Ann. Rev. Fluid Mech. 1987. 19 : 271–311

VISCOUS FINGERING IN POROUS MEDIA

G. M. Homsy

Department of Chemical Engineering, Stanford University, Stanford, California 94305

INTRODUCTION

Scope

"Viscous fingering" generally refers to the onset and evolution of instabilities that occur in the displacement of fluids in porous materials. In most but not all cases, the mechanism of the instability is intimately linked to viscosity variations between phases or within a single phase containing a solute—hence the term "viscous fingering." Shown in Figure 1 are three examples of the complex and intriguing patterns that evolve as a result of this instability. Figure 1*a* shows the fingering pattern that occurs when a more viscous material is displaced by a less viscous one fully miscible with it by injecting from one corner and withdrawing from the diagonal corner of a horizontal square Hele-Shaw cell; in this case the fluid consists of water injected into glycerine. Although the mixture is fully miscible, it is obvious that less viscous material tends to penetrate and finger through the more viscous material. In this particular example, the patterns are driven by the difference in viscosity and influenced by the diffusive mixing between the fluids. In Figure 1*b*, we show an example of the extreme patterns that are formed when a less dense, less viscous fluid penetrates a more dense, more viscous fluid *immiscible* with it when a large Hele-Shaw cell is tipped into a vertical position. In this case both gravity and viscosity are important forces in driving the instability. Finally, in Figure 1*c* we show the pattern that results when a low-viscosity Newtonian fluid injected from a source penetrates a Hele-Shaw cell filled with a miscible but strongly non-Newtonian fluid. We discuss these examples in more detail below, but even though the experiments were done in the relatively simple geometry of Hele-Shaw cells, the detailed dynamics leading to the observed patterns

0066–4189/87/0115–0271$02.00

are not fully understood. However, one feature in common in these examples is the fact that the physical conditions of the experiments allow a wide spectrum of length scales to occur. Below we provide a detailed discussion of the mechanisms that govern these flows, which we refer to as *shielding*, *spreading*, and *splitting*, that will enable us to at least qualitatively understand these fascinating patterns.

Such phenomena are important in a wide variety of applications, including secondary and tertiary oil recovery, fixed bed regeneration in chemical processing, hydrology, and filtration. Indeed, the phenomena are expected to occur in many of the myriad of fields of science and technology in which fluids flow through porous materials, and thus the literature is a diverse one. Many combinations of configurations, important fluid-mechanical forces, and boundary conditions have been studied. Thus, we cannot provide an exhaustive review, and many important areas of research are omitted from our discussion. In addition, recent activity in the field has been explosive to such an extent that this review is destined to be out of date, perhaps seriously so, by the time it has appeared. Luckily, the editors of this series have provided me with both a time deadline and a page limit,

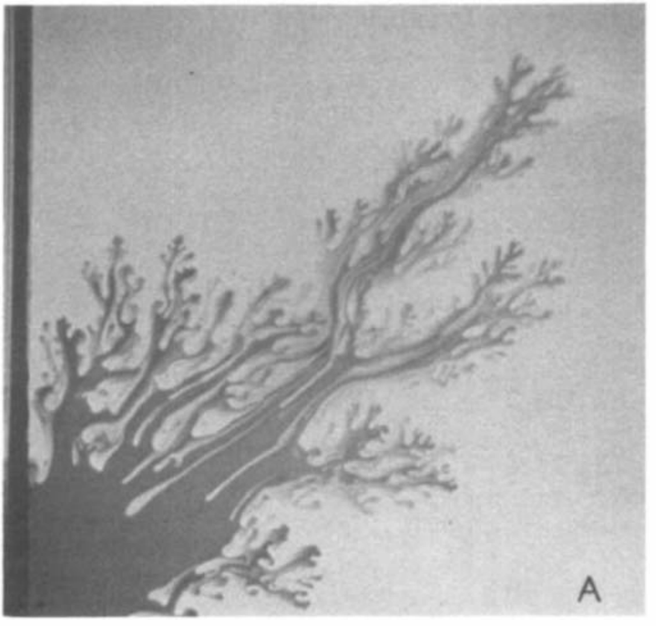
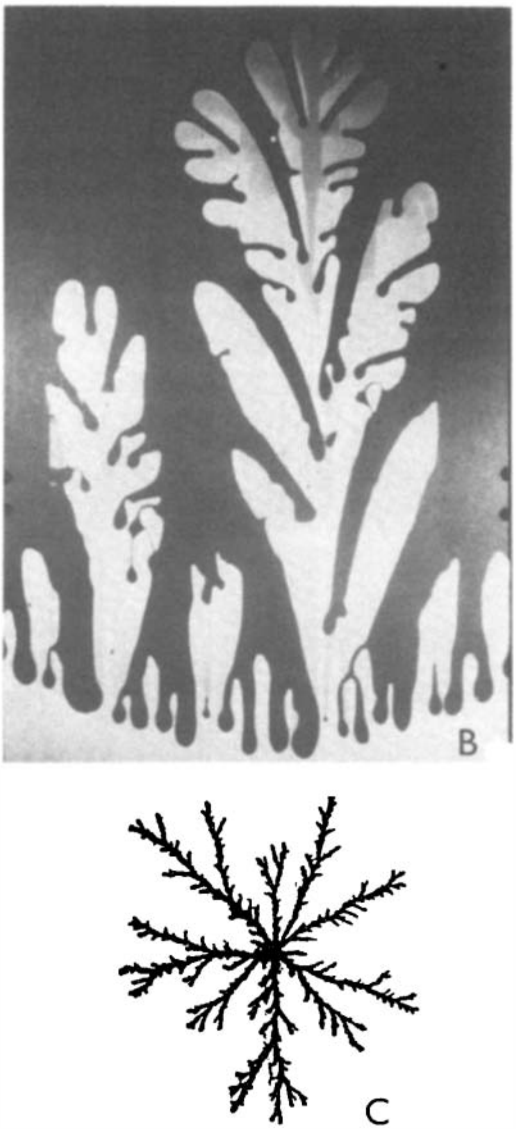


Figure 1 Examples of viscous fingering in Hele-Shaw cells: (*a*) miscible flow in a five-spot geometry (E. L. Claridge, personal communication, 1986); (*b*) immiscible flow in gravity-driven fingering (Maxworthy 1986); (*c*) miscible flow of a non-Newtonian fluid in radial source flow (Daccord et al. 1986). With permission.

behind which I take refuge against inevitable criticisms of the timeliness and scope of this review.

The areas to be covered are as follows: We are interested in viscous fingering in homogeneous porous materials and thus do not discuss the important and emerging area of the interaction between viscous fingering



and permeability heterogeneities. Furthermore, we cannot give a treatment of all possible geometries and boundary conditions. Thus, we deal exclusively with the three simple geometries of rectilinear displacement, radial source flow, and the so-called five-spot pattern, which are sketched in Figure 2. In each case there is a characteristic macroscopic length L, a characteristic velocity U, a permeability K, and a characteristic viscosity μ. We are primarily interested in two-dimensional flows, as little has been done on three-dimensional problems, especially nonlinear ones. We also confine ourselves to cases in which the orientation of gravity, should it be important, is colinear with the displacement direction. In limiting the review in this manner, we hope to emphasize mechanisms and current

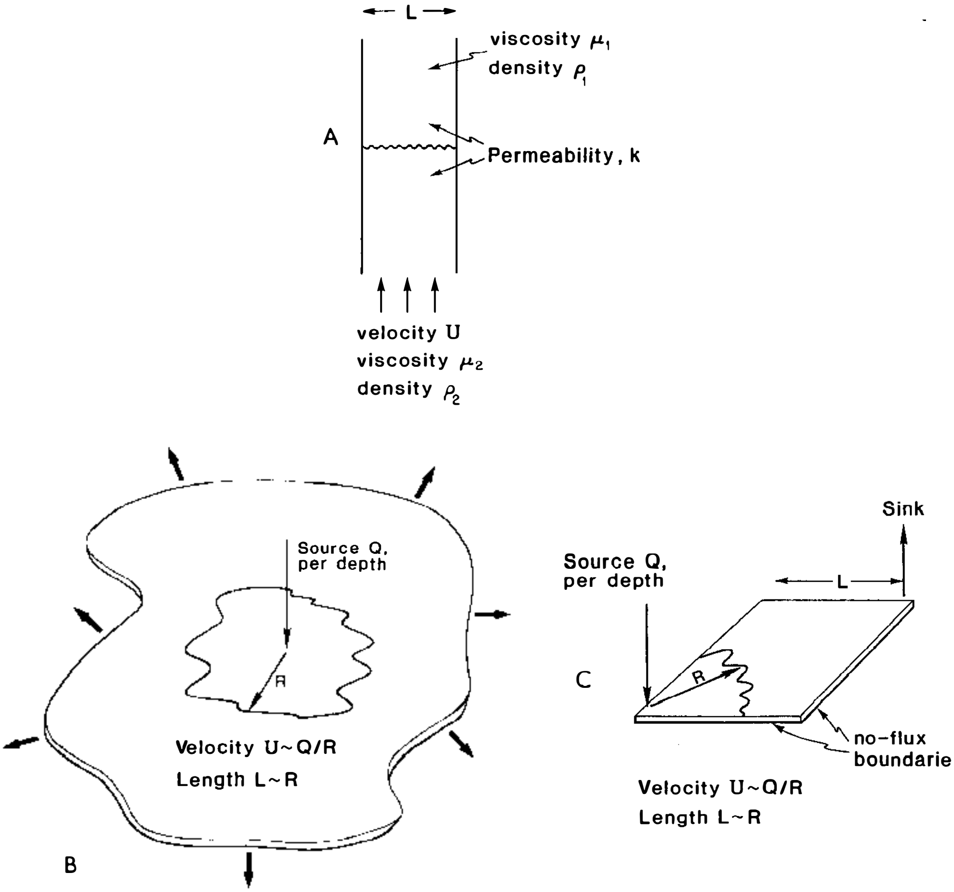


Figure 2 Common geometries and their defining parameters: (*a*) rectilinear flow; (*b*) radial source flow; (*c*) "five-spot" pattern.

research for this restricted class of problems. We deemphasize almost entirely the case in which instabilities are driven solely by buoyancy, although this is a field with an equally rich range of phenomena.

A subject of current focus is the extent to which viscous fingering may be analogous to other free-boundary problems that exhibit similar pattern formation, notably the growth of crystals from melt or solution. Unfortunately, this is a question that, while intriguing, is outside the scope of the present review, although many of the more recent references given here discuss this issue. Also, in the discussion of immiscible flow in porous materials, we touch only briefly on those aspects of displacement that are dominated by capillarity in the microscopic pore space, and that rightly form a subset of percolation phenomena, on which there exists an extensive literature. Some aspects of viscous fingering have been well reviewed elsewhere, and we simply refer the reader to those sources as appropriate. These include the review by Wooding & Morel-Seytoux (1976) on the more general subject of multiphase flow in porous media; that by Aref (1986) on modern approaches to simulations of such flows, and features these flows have in common with other well-studied nonlinear problems in fluid mechanics; the monograph edited by Ewing (1983), in which some of the numerical-analysis aspects of the simulation of displacement processes are discussed; and the reviews by Bensimon et al. (1986) and Saffman (1986), which deal exclusively with fingering in Hele-Shaw flows.

Mechanisms of Viscous Fingering

Consider a displacement in a homogeneous porous medium, characterized by a constant permeability K. The flow will typically involve the displacement of a fluid of viscosity μ_1 and density ρ_1 by a second of viscosity μ_2 and density ρ_2. These differences in physical properties may result from using two different, immiscible phases, or from injection of a solvent fully miscible with fluid 1. It is the *variation* of these properties across some front that is important. As noted above, we limit ourselves to the forces of gravity, viscosity, and (if the fluids are immiscible) surface tension. In the case of miscible systems in which differences in viscous forces are due to differences in solute concentration, we must also consider the molecular diffusion and mechanical dispersion of the solute.

The following simple argument may be made in order to understand the basic mechanism of the instability. Under suitable continuum assumptions, the flow may be taken to satisfy Darcy's law, which for a one-dimensional steady flow may be written

$$\frac{dp}{dx} = -\mu U/K + \rho g. \tag{1}$$

Now consider a sharp interface or zone where density, viscosity, and solute concentration all change rapidly, e.g. a zone such as that shown in Figure 2*a*. Then the pressure force $(p_2 - p_1)$ on the displaced fluid as a result of a virtual displacement δx of the interface from its simple convected location is

$$\delta p = (p_2 - p_1) = [(\mu_1 - \mu_2)U/K + (\rho_2 - \rho_1)g]\delta x. \tag{2}$$

If the net pressure force is positive, then any small displacement will amplify, leading to an instability. Thus we see that a combination of unfavorable density and/or viscosity ratios and flow *direction* can conspire to render the displacement unstable. For example, for downward vertical displacement of a dense, viscous fluid by a lighter, less viscous one, we have $(\mu_1 - \mu_2) > 0$, $(\rho_2 - \rho_1) < 0$, and $U > 0$. Thus, gravity is a stabilizing force, while viscosity is destabilizing, leading to a critical velocity U_c above which there is instability:

$$U_c = (\rho_1 - \rho_2)gK/(\mu_1 - \mu_2). \tag{3}$$

There are three other obvious cases depending upon the signs of $\Delta\rho$, U, and $\Delta\mu$: one in which gravity drives the instability and viscosity stabilizes it, and the two cases when both basic forces are either stabilizing or destabilizing.

A simpler statement may be made when the gravity force is absent, e.g. in a horizontal displacement. In this case, instability *always* results when a more viscous fluid is displaced by a less viscous one, since the less viscous fluid has the greater mobility. Thus we see that the two basic forces responsible for the instability are gravity and viscosity. More refined analyses, discussed below, will show that surface tension and/or dispersion can modify but not stabilize a flow characterized as unstable by this simple criterion.

As in many areas of fluid mechanics, interest lies in the behavior of the flow for conditions that exceed the critical limits, and perhaps the most fascinating behaviors are those that occur for highly supercritical conditions. In this respect, viscous fingering is no exception.

A Historical Note

It is interesting to trace the literature in order to establish priority for the discovery and understanding of viscous fingering in terms of the fluid mechanics involved. Despite the fact that this instability is discussed in many fluid-mechanics textbooks and literature papers as the "Saffman-Taylor Instability" and is attributed to Saffman & Taylor (1958), the phenomenon had been noted and recorded in many earlier works, although not always with a clear understanding of the mechanics and the basic mechanism. The first scientific study of viscous fingering can reasonably be attributed to Hill (1952), who not only published the simple "one-dimensional" stability analysis given above, but who also conducted a series of careful and quantitative experiments for both gravity-stabilized viscous fingering in vertical downflow and the curious counter-case of viscous stabilization of a gravitationally unstable configuration in the case of vertical downward displacement of a light, less viscous fluid by a heavy, more viscous one, which can be stabilized for velocities above U_c. In all cases, the critical displacement velocity measured via flow visualizations was in *quantitative* agreement with Equation (3). Those interested in either the history of viscous fingering or an example of concise scientific writing should consult this beautiful paper. The next significant development occurred in the late 1950s, when Chouke et al. (1959) and Saffman & Taylor (1958) published their now-classical papers. Both these papers, submitted within six months of one another, contain essentially identical linear-instability analyses of one-dimensional displacement, leading to Equation (7) below. Significantly, Saffman & Taylor state that "the result is not essentially new, and that mining engineers and geologists have long been aware of it," and they further attribute the inclusion of surface tension in the analysis to "Dr. Chouke." They then go on to study experimentally the evolution and shape of the now-famous single dominant finger and to discuss its theoretical description. The paper of Chouke et al., on the other hand, refers only to a presentation of the linear-instability analysis by Chouke at a technical meeting in 1958, with no mention of Saffman & Taylor, although they do reference Hill's work. Given the delays involved in publishing research from an industrial laboratory, it seems plausible that credit for the first rigorous stability *analysis* of viscous fingering be given to Chouke, but that the phenomenon under discussion should almost certainly be called the "Hill Instability." There is unfortunately little chance of this designation gaining general acceptance.

HELE-SHAW FLOWS

The Simplification of Hele-Shaw Flows

Since most porous materials are opaque, a convenient analogue to study is that of the Hele-Shaw model, the geometry of which is shown in the definition sketch in Figure 3. Flow takes place in a small gap of thickness b, and there is a second, macroscopic dimension L. Hele-Shaw theory and experiments seek to describe the two-dimensional features of the flow, *depth-averaged* over the thin gap. It is well known that in *single-phase* flow, these two-dimensional equations are

$$\nabla \cdot \bar{\mathbf{u}} = 0, \tag{4a}$$

$$\nabla \bar{p} = -12\mu\bar{\mathbf{u}}/b^2 + \rho\mathbf{g}, \tag{4b}$$

which hold in the limit of low-Reynolds-number flow, and $e \equiv b/L \to 0$. We see that the flow satisfies Darcy's law, in which the equivalent permeability of the medium is simply $b^2/12$. Thus, *single-phase* Hele-Shaw flow is analogous to two-dimensional incompressible flow in porous media. It might be supposed that the same analogy would hold for viscous fingering in porous materials. However, as we shall see, the analogy is imperfect. In the case of miscible fluids with concentration gradients, Taylor dispersion will occur due to the velocity profile in the thin dimension, making the mixing or dispersion characteristics in Hele-Shaw flow highly anisotropic in ways that porous materials may not be. In particular, if mechanical dispersion is important, it usually affects both the longitudinal and transverse dispersion coefficients in porous media, whereas in Hele-Shaw flow, only the longitudinal component is affected.

The analogy fails completely in the case of flow of immiscible fluids, since flow in porous materials in this case is truly *multiphase*, as opposed to two phase, and the forces associated with the propagation of menisci through the pore space comprising the medium cannot be neglected and are not modeled in the Hele-Shaw geometry. Furthermore, the geometry of the solid matrix can also influence the fingering patterns observed. Nevertheless, the case of Hele-Shaw flow is of fundamental interest in its own right and permits us to establish some useful concepts. Furthermore,

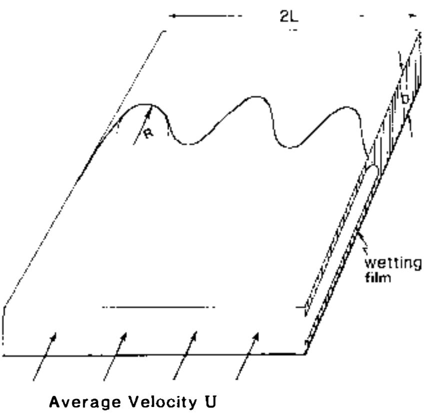


Figure 3 Detailed schematic of Hele-Shaw flow in the case where the displaced fluid wets the wall. The interface moves with normal velocity U, surface tension γ acts on the interface, and R is the radius of curvature in the plane.

it allows us to discuss the processes of *shielding*, *spreading*, and *splitting* alluded to in the Introduction.

Immiscible Displacements in Hele-Shaw Cells

In the Appendix of their paper, Saffman & Taylor (1958) showed that equations of the form of (4) also govern the flow of two *immiscible* phases in Hele-Shaw cells, provided that the film of displaced fluid left on the plates is of constant thickness, but that the effective viscosities and densities appearing therein depend upon the property ratios as well as the thickness ratio. Thus we adopt them as the field equations in each phase in regions away from the interface between the phases. It remains to give boundary conditions that apply at the location of the interface, taken as a *two-dimensional depth-averaged* surface. The only analyses of the details of the flow in the thin dimension, which must then be depth averaged in order to give jump conditions for the fields $\bar{\mathbf{u}}$ and $\bar{p}$, are those of Park & Homsy (1984) and Reinelt (1986). For the case in which the displaced fluid wets the wall, these conditions depend upon the local capillary number of the flow, $\mathrm{Ca} = \mu U/\gamma$, as well as on the magnitude of the surface tension γ. In terms of the quantities defined in Figure 3, these conditions are, for small Ca,

$$[|\bar{p}|] = \frac{2\gamma}{b}(1+3.8\ \mathrm{Ca}^{2/3}+\ldots)+\frac{\gamma}{R}[\pi/4+O(\mathrm{Ca}^{2/3})], \tag{5a}$$

$$[|n\cdot\bar{\mathbf{u}}|] = O(\mathrm{Ca}^{2/3}). \tag{5b}$$

Most analyses of Hele-Shaw flows pertain to the limit $\mathrm{Ca} \equiv 0$. In this case, the equations simplify to

$$[|\bar{p}|] = 2\gamma/b+\gamma\pi/(4R), \tag{6a}$$

$$[|n\cdot\bar{\mathbf{u}}|] = 0. \tag{6b}$$

Much of the literature on the subject uses these boundary conditions, with the constant 1.0 appearing in place of $\pi/4$, but a simple rescaling of surface tension allows these existing results to be carried over without change. This constant may be simply computed as a function of the contact angle in the thin dimension (Park 1985) and is set to unity in what follows. Furthermore, the leading constant in (6a) is often set to zero without loss of generality. Equations (4), together with these simpler jump conditions [Equations (6)], have been referred to as the *Hele-Shaw equations*, and they hold asymptotically in the limit of small capillary number and small ratios of the gap thickness to any macroscopic dimension. These equations have been the object of much study, and many features of their solutions have been discussed elsewhere (Aref 1986, Bensimon et al. 1986, Saffman 1986).

The Hele-Shaw equations may be scaled in a straightforward way by using a macroscopic length as the characteristic length and L/U as the characteristic time to show that solutions may be expected to depend upon three basic parameters:

$$\mathrm{Ca}' = \frac{12\mu_1 U}{\gamma}(L/b)^2 \quad \text{modified capillary number,}$$

$$A = \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} \quad \text{viscosity contrast,}$$

$$G = \frac{(\rho_1 - \rho_2)gb^2}{12(\mu_1 + \mu_2)U} \quad \text{modified Darcy-Rayleigh number.}$$

The modified capillary number measures the viscous forces relative to surface tension. Other equivalent definitions for this parameter are also in use, involving other numerical factors depending upon the choice of length and velocity scales, and the inverse of Ca′ is sometimes used, as it occurs naturally in the boundary conditions of the problem. Unfortunately no universal convention for this controlling parameter has been adopted in the literature, and care must be taken in comparing results of different investigators. The group G similarly measures the relative importance of buoyant to viscous forces. The property ratio A is self-explanatory.

The linear-stability analysis of one-dimensional rectilinear displacements with the action of surface tension is due to Chouke et al. (1959). In the usual fashion, if disturbances are taken to be of the form of normal modes proportional to $\exp(\sigma t + iky)$, one finds the following dispersion relation for the growth constant of the instability:

$$(\mu_1 + \mu_2)\sigma = \left[U(\mu_1 - \mu_2) + \frac{(\rho_1 - \rho_2)gb^2}{12}\right]k - \frac{\gamma b^2 k^3}{12}, \tag{7a}$$

or in dimensionless form,

$$\sigma = (A + G)k - \frac{(A+1)k^3}{2\,\mathrm{Ca}'}. \tag{7b}$$

It may be seen that the critical speed for which $\sigma > 0$ is given accurately by the simple analysis of Hill (1952), since it holds for long waves, but that for unstable situations, the inclusion of surface tension leads to a wave number of maximum growth rate and a cutoff wave number. For the simple case of both gravity and viscosity driving the instability, the maximum growth rate occurs for a wave number

$$k_{\mathrm{m}} = \left[\frac{2(A+G)\,\mathrm{Ca}'}{3(A+1)}\right]^{1/2}. \tag{8}$$

The simple physical interpretation of these results is that surface tension will damp short waves, whereas the basic mechanism favors them, leading to competing effects and the occurrence of a preferred mode in a manner similar to many other stability problems. The ratio of unstable length scales to the macroscopic scale decreases as Ca′ increases, which implies that many scales of motion are allowed in the limit of very large capillary number (small surface tension). Thus we expect the dynamics of fingering to become complex in this limit.

It has proven difficult to quantitatively verify the dispersion relation [Equation (7)], but the available body of information is in general agreement with it. Many of the experiments pertain to relatively simple measurements of the apparent wavelength of fingers in their early stages of growth, as typified by those of White et al. (1976). These measurements are found to compare favorably with those given by Equation (8). Park et al. (1984) have provided the only available measurements of growth rates and summarize previous investigations on the initial development of the instability. Recently, Schwartz (1986) has reanalyzed the linear-stability problem using Equation (5a) rather than (6a), which results in an improved agreement between theory and the experiments of Park et al., but the data are not sufficiently accurate to provide a critical test of the theory leading to (6a).

The linear instability of radial source flow was first treated by Wilson (1975), and thereafter by Paterson (1981). The velocity field in radial source flow is given by

$$u = Q/r, \tag{9}$$

where Q is the two-dimensional source strength. The characteristic scalings change somewhat for this case, but a modified capillary number still determines the cutoff scales. Since both the velocity and characteristic length change with time, the analysis of the linear instability is based upon a quasi-static analysis that treats the velocity U at a radius R as locally constant. The results, in dimensionless form, are

$$\sigma = Am - 1 - \mathrm{Ca}'m(m^2-1)(A+1)/2, \tag{10}$$

where A and Ca′ are defined above using an instantaneous velocity and radius, and m is a discrete azimuthal wave number. Analysis of Equation

(10) is simple only when $A = 1$, in which case all waves with wave numbers satisfying

$$m \leq 1/2[1 + (1 + 4\mathrm{Ca}')^{1/2}] \tag{11}$$

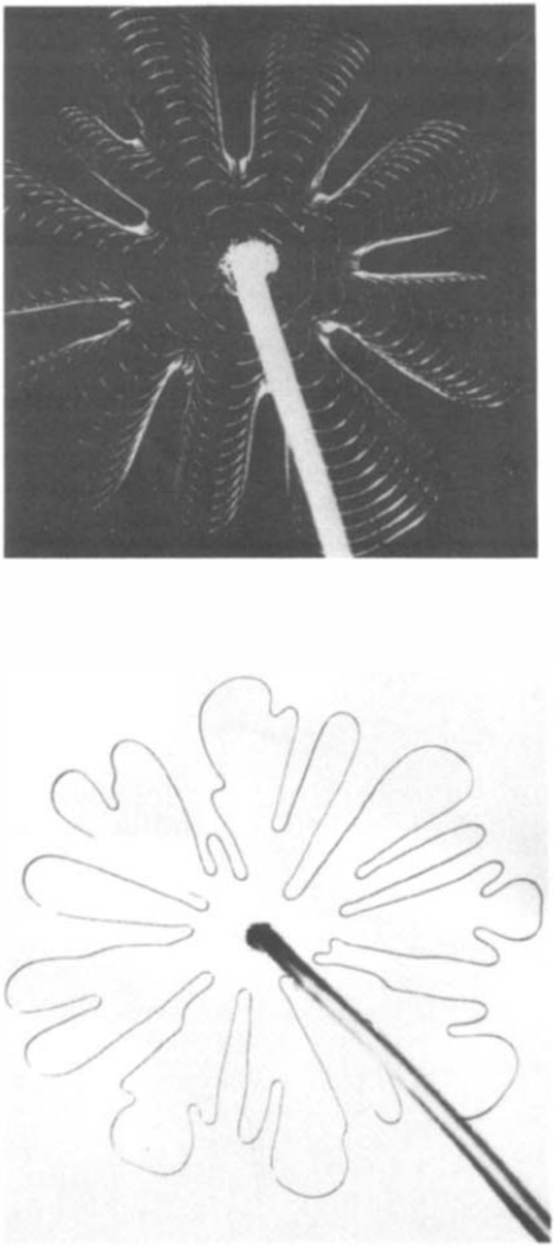
are unstable, with a wave number of maximum growth rate, as before.

Experiments by Paterson (1981), shown in Figure 4*a*, show the evolution of fingering patterns in source flow for displacement of glycerine with air. For short times, there is reasonable agreement with the expectations of the linear theory. Radial source flow is also a good short-time approximation to the flow in a five-spot pattern, and the above theoretical results should hold for that case as well.

The classical experimental study of Saffman & Taylor focused on the nonlinear evolution of fingers in the limit $A = 1$, $G = 0$. They found that after the initial instability, a single finger appeared to dominate the flow. The reason for this can be understood simply in terms of *shielding*. Since the tendency is for fingers of mobile fluid to grow in the direction of the pressure gradient in the more viscous fluid, a finger slightly ahead of its neighbors quickly outruns them and shields them from further growth. (An equivalent argument can be made using the fact that the pressure in the less mobile phase is harmonic and the interface is nearly isopotential, leading to a larger flux of fluid near the tip of any finger that is ahead of neighboring ones.)

Steady solutions describing finite-amplitude solutions in rectilinear flow have been much discussed in the recent literature, which we briefly summarize here. Saffman & Taylor (1958) sought solutions of the Hele-Shaw equations in the limit $A = 1$, $\mathrm{Ca}' = \infty$, $G = 0$. They found that there was a continuous family of solutions, all of which were linearly unstable (Taylor & Saffman 1958). The traditional parameter characterizing these solutions is λ, the ratio of the finger width to the characteristic macroscopic length. Thus λ is not uniquely determined for $\mathrm{Ca}' = \infty$. McLean & Saffman (1981) and, more recently Vanden Broeck (1983) solved the steady free-boundary problem numerically for finite Ca' and examined the limit $\mathrm{Ca}' \gg 1$. There exists an apparently infinite family of discrete solutions, $\lambda_n(\mathrm{Ca}')$, all of which tend toward the same shape with $\lambda_n = 1/2$ as $\mathrm{Ca}' \rightarrow \infty$.

Large-scale numerical simulations of the evolution of such finger shapes have been provided by Tryggvason & Aref (1985), DeGregoria & Schwartz (1986), and Liang (1986). Tryggvason & Aref (1983, 1985) applied vortex-in-cell methods to study a number of problems in viscous fingering, and their simulations show the typical growth of the dominant finger by shielding. The results they obtain for the steady shape, for the cases in which they have good numerical accuracy ($\mathrm{Ca}' \leq 50$), are in agreement with the



A

B

Figure 4 Fingering in immiscible radial source flow into a Hele-Shaw cell. (*a*) Multiple exposure showing growth at early time; (*b*) tip splitting at later time (Paterson 1981). With permission.

branch calculated by McLean & Saffman (1981). DeGregoria & Schwartz (1986) developed a solution technique based on the fact that from Equations (4), the pressure satisfies Laplace's equation if the viscosity is constant almost everywhere. Thus, boundary integral techniques are appropriate. They too reproduce the McLean & Saffman branch, extending the range of numerical solutions to $Ca' \cong O(10^3)$. Finally, Liang (1986) has recently discussed a numerical method based upon random-walk simulations of the solution of Laplace's equation, which is claimed to be superior to conventional techniques for very large capillary numbers and which goes over to solutions of diffusion-limited aggregation (DLA), discussed below, when $Ca' = \infty$. The physical significance of certain key features of this method are obscure, however. His simulations show shielding, of course, and evolve to the McLean & Saffman branch of solutions. Thus, at present the other branches of steady shapes have not been realized in any initial-value calculation, which has led to speculation that these branches are unstable; calculations in support of this speculation are given by Kessler & Levine (1986c). Most intriguing in the simulations of DeGregoria & Schwartz and of Liang is the fact that as Ca' becomes large, the steady finger shapes are not always stable. We discuss these observations in more detail below.

We also mention the recent analysis of Kessler & Levine (1986b), who show that a periodic array of steady fingers is unstable to any spanwise modulation. They further speculate that the preferred mode is one corresponding to an annihilation of nearest neighbors, which leads to a pairing process that, if the resulting finger is stable, would persist until only one finger from the array remained. The mechanism of this spanwise instability can be easily understood in terms of the shielding effect discussed above.

Thus we see that surface tension, *however weak*, acts to spread the dominant finger to a particular width. This will be important in interpreting patterns of fingering at high Ca'. The mathematical description of how this *selection mechanism* can remain present in the limit of extremely small surface tension has been treated very recently by Shraiman (1986), Hong & Langer (1986), and Combescot et al. (1986).

Experimental attempts to measure the shape of the steady dominant finger as a function of Ca' are not conclusive. Pitts (1980) reports a curious experimental finding that the shapes at finite capillary number form a self-similar family, a result not shared by the theoretical shapes. There also exists a discrepancy between the experimental shapes and any of the theoretical branches of solutions. Recently, Tabeling & Libchaber (1986) have suggested that this may be due to the fact that experiments have not been done in the regime of applicability of the Hele-Shaw equations. With reference to Equations (5) and (6), the Hele-Shaw equations pertain to the double limit $e \to 0$, $\mathrm{Ca} \to 0$, $\mathrm{Ca}' = e^{-2}\,\mathrm{Ca}$ finite. Uniform neglect of the additional pressure jump due to the existence of a wetting film, the $O(\mathrm{Ca}^{2/3})$ term in Equation (5a), requires $e\,\mathrm{Ca}^{-2/3} \gg 1$. It is difficult to satisfy all of these restrictions in experiments. Generally, the experiments of Saffman & Taylor and of Pitts were carried out for *finite* Ca, with e moderately small. The experiments of Tabeling et al. (1986), in which e was varied systematically, were again generally for small e but for values of Ca as large as 1.0. The experiments of Park & Homsy (1985), which utilized a very small aspect-ratio cell, were done for small Ca and $e\,\mathrm{Ca}^{-2/3} \approx O(0.5)$. Thus we see that many of the widely quoted experimental studies have not been conducted in the region of validity of the Hele-Shaw equations, and comparison between experiment and theory must be approached with some caution. Numerical solutions of the equations that include the parameters e and Ca separately instead of the combination $\mathrm{Ca}' = e^{-2}\,\mathrm{Ca}$ are not available, but the ability of Tabeling & Libchaber (1986) to collapse available data on finger width using a modified scaling parameter based upon Equation (5a), is encouraging. Quantitative investigation of the effect of the wetting layer must await solution to a set of equations that take explicit account of its existence.

There are also unanswered questions regarding the stability of the single dominant finger. Taylor & Saffman (1958) and McLean & Saffman (1981) provided stability analyses of the steady nonlinear shapes that indicated that such shapes should be *unstable* for infinite and finite Ca′, respectively. However, the experiments of Saffman & Taylor and of Pitts indicated stability. Recently, a number of workers have shown experimentally that the single finger is subject to a tip-splitting instability (Nittmann et al. 1985, Park & Homsy 1985, Tabeling et al. 1986). Partial rationalization of this paradox between the observation of stable fingers on the one hand and prediction of instability on the other has been provided by Kessler & Levine (1986a,c), who suggest that previous analyses of the stability of the single finger are inaccurate. [The stability calculations of McLean & Saffman are apparently incorrect (Saffman 1986).] Kessler & Levine analyze the full spectrum of the linear operator for instability modes that grow in space, and they solve the resulting eigenvalue problem by numerical means. The range of accurate numerical results is limited to $\mathrm{Ca}' \leq O(10^3)$, for which steady fingers are stable. Unfortunately, the numerical solution of the linear-stability problem becomes ill conditioned at high Ca′, in a manner similar to the Orr-Sommerfeld equation at high Reynolds number. Thus reliable results for very large Ca′ are not available, but Kessler & Levine speculate that stability persists to $\mathrm{Ca}' = \infty$. The numerical results of DeGregoria & Schwartz (1986) show that the McLean & Saffman branch of steady solutions is apparently stable for $\mathrm{Ca}' < 10^3$

but unstable to a symmetrical tip-splitting mode for $Ca' > 10^3$. They also provide numerical evidence that the onset of the tip-splitting instability depends upon the amplitude of the imposed perturbation, indicating that the instability is a *subcritical bifurcation.* They further argue that the bifurcation, in addition to being subcritical, is from infinity, i.e. fingers are stable to infinitesimal perturbations for all but infinite Ca′, a point of view that is only partially supported by their calculations. DeGregoria & Schwartz (1986), using an improved version of their algorithm that allows asymmetrical solutions, have observed tip splitting above $Ca' = 1.25 \times 10^3$. Liang (1986), using random-walk simulations, has also observed both steady fingers and tip splitting. However, as noted above, several aspects of the algorithm used are not clear. As an apparent result of insufficient spatial resolution, he observes splitting for very low Ca′, for which the dominant finger should be linearly stable. An attempt to analyze the nature of the bifurcation has been made by Bensimon (1986). His analysis clearly indicates the subcritical nature of the instability, but because of the qualitative nature of the model, in which he considers perturbations that are not dynamically admissible, further work is needed to settle the issue of the nature of the bifurcation.

Careful experimental studies of the tip-splitting instability are only now beginning. Park & Homsy (1985) were the first to attempt to measure the critical Ca′ for onset of splitting, which they report to be $Ca' = 600$. Since the instability is subcritical, this value may not stand the test of time; indeed, Tabeling et al. (1986) report delay in the onset of splitting to $Ca' = 2\text{–}3 \times 10^3$ by reducing the noise level in the experiment. Much more experimental work needs to be done in order to characterize the critical conditions for onset of tip splitting, as well as the regime of flow in the postinstability regime. Figure 5 shows a comparison between the experiments of Park (1985) and the calculations of DeGregoria & Schwartz (1985) and of Liang (1986). The qualitative similarity between the patterns observed is encouraging in the sense that the experimental behavior appears to be contained within the Hele-Shaw equations.

The mechanism by which such patterns are formed can be understood in the following way. As surface tension becomes weak, the front of the steady finger is susceptible to a viscous-fingering instability by the basic mechanism associated with a less viscous fluid displacing a more viscous one. After a split, each of the new lobes of the finger is stable as a result of their being thinner than the finger from which they split. As a result of *shielding*, one of these lobes will eventually outgrow the other and, owing to surface tension, will then *spread* to occupy the appropriate width of the cell. In the process, it reaches a width that is again unstable to *splitting*, and the pattern repeats. Thus, surface tension plays a subtle but essential dual role; it must be weak enough for the tip front to be unstable, but it is also the physical force causing the spreading and ensuing repeated branching. In this sense, it is analogous to the dual role of viscosity in the instability of parallel shear flows. We thus arrive at a scenario in which *shielding*, *spreading*, and *splitting* are all important processes in determining the dynamics of viscous fingering.

Once the existence of the tip-splitting instability is known and the mechanism understood at least qualitatively, more complicated patterns such as those observed by Maxworthy (1986), shown in Figure 1*b*, and those of Paterson (1981), in Figure 4, can be understood. Maxworthy's experiments (which incidentally are far outside the region of validity of the Hele-Shaw equations) were done at extremely large Ca′ (Ca′ = 5×10^4), values for which tip splitting takes place over a range of length scales, leading to a cascade of splittings, including secondary splittings of the side branches. Such primary and secondary events may be clearly identified in the sequence of photographs leading to Figure 1*a* (Maxworthy 1986; see also the photographs published by Reed 1985). This cascade continues in time, leading to geometrical shapes that are not simple and that may have statistical features describable by a fractal geometry [see Maxworthy (1986) and Nittmann et al. (1985) for a further exposition of this idea].

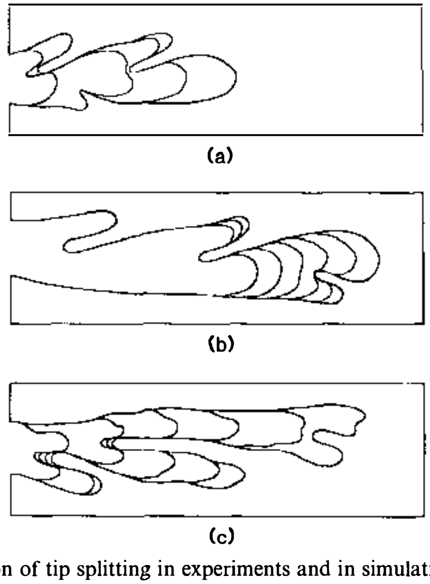


Figure 5 Comparison of tip splitting in experiments and in simulations of the Hele-Shaw equations: (*a*) tracings from experiments for Ca′ = 1.3×10^3 (Park 1985); (*b*) simulations for Ca′ = 1.3×10^3 (DeGregoria & Schwartz 1985); (*c*) simulations for Ca′ = 600 (Liang 1986). With permission.

The observations of Paterson (1981), shown in Figure 4*b*, as well as similar ones by Ben-Jacob et al. (1985) in radial source flow may be similarly understood. For a given Ca′, there will be a preferred azimuthal wave number from Equation (10) when the critical radius is reached. Following growth of the preferred mode, the lobes of the pattern again spread laterally as a result of surface tension until they exceed a local stability limit and undergo another fingering instability at the tip. Although this will presumably continue indefinitely, little experimental work past the first one or two generations of splittings has been reported, and there is no evidence one way or the other for a regime of fractal splitting.

Few studies have been made to characterize the effect of the other two dynamic groups, G and A, on fingering. Maher (1985) reports observations in small Hele-Shaw cells. By using two fluids near a thermodynamic critical point, he is able to achieve small A while maintaining immiscible conditions. Because the mobilities of both phases are nearly equal, the strong shielding present when one phase essentially controls the pressure drop is reduced or eliminated, with the result that all fingers grow, at least for short times, and the pattern is not characterized by the emergence of the single dominant finger. It is not clear if this regime persists for long times and large displacements, but at short times it is clearly qualitatively different from the $A = 1$ case. Furthermore, the experimental observations are in good qualitative agreement with the calculations of Tryggvason & Aref (1983) for small A. Without the growth of the dominant finger and its subsequent splitting instability, the patterns of fingering are also correspondingly different.

Although the effect of the gravity group G is superficially similar to that of an imposed flow in that density differences impose a vertical pressure gradient far from the boundary of the finger, the dynamic effects of gravity and viscous forces are not interchangeable except when the interface is nearly flat and perpendicular to the motion, as indicated in Equation (7b) above. Thus, much work remains to be done to fully characterize the parametric dependence of nonlinear viscous fingering upon the remaining groups G and A.

Miscible Displacements in Hele-Shaw Cells

An important consequence of immiscibility in the case of Hele-Shaw flow is that the viscosity is constant in each phase, with a jump at the interface. This is not necessarily the case for miscible displacements. In miscible displacements, there is no interface, and as a result a *single-phase* Darcy's law [Equation (4)] holds throughout the domain. Viscous fingering may still be driven, however, by variations in viscosity that result from variations in the concentration of a chemical component in the fluid. If we let the concentration be denoted by c, scaled between zero and one, then the equations governing the system are

$$\nabla \cdot \mathbf{u} = 0, \tag{12a}$$

$$\nabla p = -\mu(c)\mathbf{u}/K + \rho g, \tag{12b}$$

$$\frac{Dc}{Dt} = \nabla \cdot (\mathbf{D} \cdot \nabla c), \tag{12c}$$

$$\mu = \mu(c). \tag{12d}$$

(Note that although we are discussing Hele-Shaw flow, we have written the permeability as K rather than $b^2/12$, as the theory for homogeneous porous media, treated in the next section, is *identical* to Hele-Shaw theory in the special case of isotropic dispersion.) The concentration is taken to obey a convection-diffusion equation, perhaps with an anisotropic dispersion tensor, and the relation between concentration and viscosity must also be given. If we select a macroscopic length L, a viscosity μ_1, and a velocity U as scaling parameters, then the solution to these equations depends upon the following dimensionless parameters:

$\mathrm{Pe} = UL/D_0$	Peclet number,
$A = \dfrac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$	Viscosity contrast,
$G = \dfrac{\Delta\rho g K}{(\mu_1 + \mu_2)U}$	Gravity group,
$\mathbf{D}^*(u)$	Dimensionless dispersion function,
$\mu^*(c)$; $M^{-1} < \mu^* < 1$	Dimensionless viscosity function, where M is the viscosity, or mobility, ratio μ_1/μ_2.

In these definitions, μ_1, μ_2 are the viscosities without solvent and with the maximum solvent concentration, respectively; D_0 is a reference value of the dispersion coefficient, usually taken as the zero-velocity diffusion limit or the longitudinal dispersion at the reference velocity; and $\mathbf{D}^*$, μ^* are dimensionless functions describing the material behavior of the medium and the solute/solvent mixture, respectively.

Discussion of the linear stability theory for this case is complicated by the fact that, unlike the case of miscible displacements, there is no simple *steady* solution to the relevant equations, since dispersion will always act to render the concentration profile and hence the viscosity profile time dependent. Thus we must deal with the stability of time-dependent base states, a subject of some difficulty. However, consider for the moment the case of infinite Peclet number (zero dispersion). For rectilinear flow, the solvent concentration will be independent of time in a convected coordinate system, and the above stability results, specialized for the case of zero surface tension, apply. We see that

$$\sigma = (A+G)k, \tag{13}$$

a nonphysical result that indicates that smaller and smaller wavelengths are increasingly unstable. Clearly, dispersion or some other physical effect will act on such short-wavelength disturbances, leading to a cutoff. The first analysis of the effect of dispersion was given by Chouke almost 30 years ago, and reported recently in the Appendix to Gardner & Ypma (1982). Expressed in the present variables, for the case of a jump in viscosity (i.e. a base-state profile corresponding to zero axial dispersion), but allowing both axial and transverse dispersion to act on disturbances, Chouke's result reads

$$\sigma = 1/2[Ak - \mathrm{Pe}^{-2}k^3 - k(\mathrm{Pe}^{-2}k^2 + 2A\mathrm{Pe}^{-1}k)^{1/2}] \tag{14}$$

with a cutoff wave number

$$k_c = \mathrm{Pe}A/4, \tag{15}$$

while the growth rate is a maximum at

$$k_m = \mathrm{Pe}(2\sqrt{5}-4)A/(4) \approx 0.12\mathrm{Pe}A. \tag{16}$$

The effect of gravity on these results may be simply included through the parameter G, as in the above discussion. As in the case of immiscible displacements, there is a physical parameter that leads to a cutoff length scale, in this case the Peclet number. Correspondingly, we again expect complex behavior when the Peclet number becomes very large. Thus we see that transverse dispersion is responsible for controlling the length scales of fingers, even when axial dispersion has not yet distorted the concentration profile.

There have been many subsequent attempts to analyze the stability characteristics of miscible displacements, including the dispersive widening of the zone of viscosity variation. Heller (1966) approximated the profiles by straight-line segments and invoked a quasi-static approximation in neglecting the time dependence of the base state relative to the growth of disturbances. Schowalter (1965) dealt with fingering driven by both density and viscosity variations and used a constant-thickness diffusive zone to describe the mobility profile. He assumed a *steady* base state (which is not allowed by the equations), special boundary conditions, and a competition between viscosity and density stratifications. Both these works give dispersion relations with cutoff wave numbers, but the questionable assumptions make their general validity suspect. Wooding (1962) made one of the few attempts to treat the stability of a time-dependent base state. He considered buoyancy-driven fingering and expressed the disturbance quantities as a Hermite expansion. By analyzing a one-term truncation, he concluded that the dispersion relation has the general characteristics of Equation (14) above, but that the cutoff length shifts with time to larger values, corresponding to the dispersive widening of the profile, and that ultimately all disturbances must decay if dispersion is given an infinite time to act. Recently, Tan & Homsy (1986) have discussed this problem in the limit $G = 0$, $\mu^*(c) = \exp(-c \ln M)$ for fingering in rectilinear flow in unbounded domains. They treat both isotropic and highly anisotropic media and solve for the stability of the time-dependent mobility profiles by utilizing both a quasi-static assumption and a numerical solution of the initial-value problem for the growth of small-amplitude fingers. They find that for the isotropic case, the analysis of Chouke is essentially correct in predicting the magnitudes of the growth rates and preferred wave numbers, but that as time proceeds dispersion acts to shift the scales to larger wave lengths and to stabilize the entire flow somewhat. Figure 6 shows typical results for the quasi-static growth rates for fingering in infinite domains, which indicate these general trends. Tan & Homsy also utilize Chouke's technique to determine how anisotropic dispersion characteristics influence the linear-stability characteristics. Not surprisingly, small transverse dis-

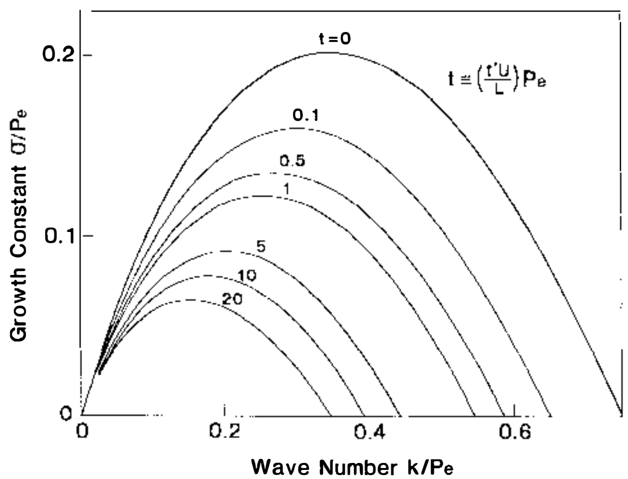


Figure 6 Linear-stability results for fingering in miscible displacements with isotropic dispersion $M = 20$ (Tan & Homsy 1986). With permission.

persion will result in a shift to smaller length scales. Recently, Hickernell & Yortsos (1986) have analyzed the stability of rectilinear displacements for the case when there is a zone of mobility variation of finite thickness but zero dispersion. This corresponds to the situation in which the solvent is injected in varying amounts over time, resulting in a spatially varying mobility profile. In this case, the finite thickness of the zone of viscosity stratification provides a cutoff scale, similar to the "regularization" of other ill-posed problems [see Aref (1986) for examples].

Curiously, there are no experimental studies of fingering in rectilinear flows in Hele-Shaw cells with which the theory can be *quantitatively* compared, although there have been a large number of experiments, characterized by those of Benham & Olson (1963), which record results of engineering importance, such as the time history of the production of solute. Experiments in porous media, which are closely related, are discussed below. Buoyancy-driven fingering, while outside the primary scope of this review, deserves limited mention here. Wooding (1969) has reported observations of such instabilities for $A \ll 1$, $G \neq 0$ in Hele-Shaw cells and has discussed the evolution of nonlinear fingers. Some of his photographs are reproduced in Figure 7. Here dispersion plays a dual and subtle role similar to that of surface tension in the immiscible case. Transverse dispersion sets the initial length scale of fingering, but it also leads to the lateral *spreading* of fingers in the nonlinear regime. As spreading occurs, the tips of fingers may become unstable as their characteristic breadth exceeds the cutoff scale. Thus *tip splitting* may occur in *miscible* as well as in *immiscible* flows, and it is the primary mechanism of pattern formation. The similarity lies in the fact that physical phenomena—dispersion and surface tension, respectively—are responsible both for determining the allowable scales of *splitting* and for causing the tip *spreading*. An important difference, at least as far as these experiments are concerned, is that substantial *shielding* does not take place, as it does when $A = 1$. In Figure 7*a*, the Peclet number is apparently below a critical value, *which is presently unknown*, and tip splitting does not occur, whereas Figure 7*b* shows the tip splitting that occurs at higher Peclet number. We mention here the simulations of Tryggvason & Aref (1983) in the limit of small A and large Ca′, which bear a superficial resemblance to the fingering pattern of Wooding. However, as they discuss, these simulations cannot describe spreading and the continuous change in the length scales caused by dispersion. Numerical methods capable of accurately describing the long-time effects of weak dispersion at very high Peclet numbers are not presently available. In fact, the pioneering attempt at such a simulation by Peaceman & Rachford (1962) failed because of the dominance of numerical errors and has led to a misconception in the petroleum-engineering literature that

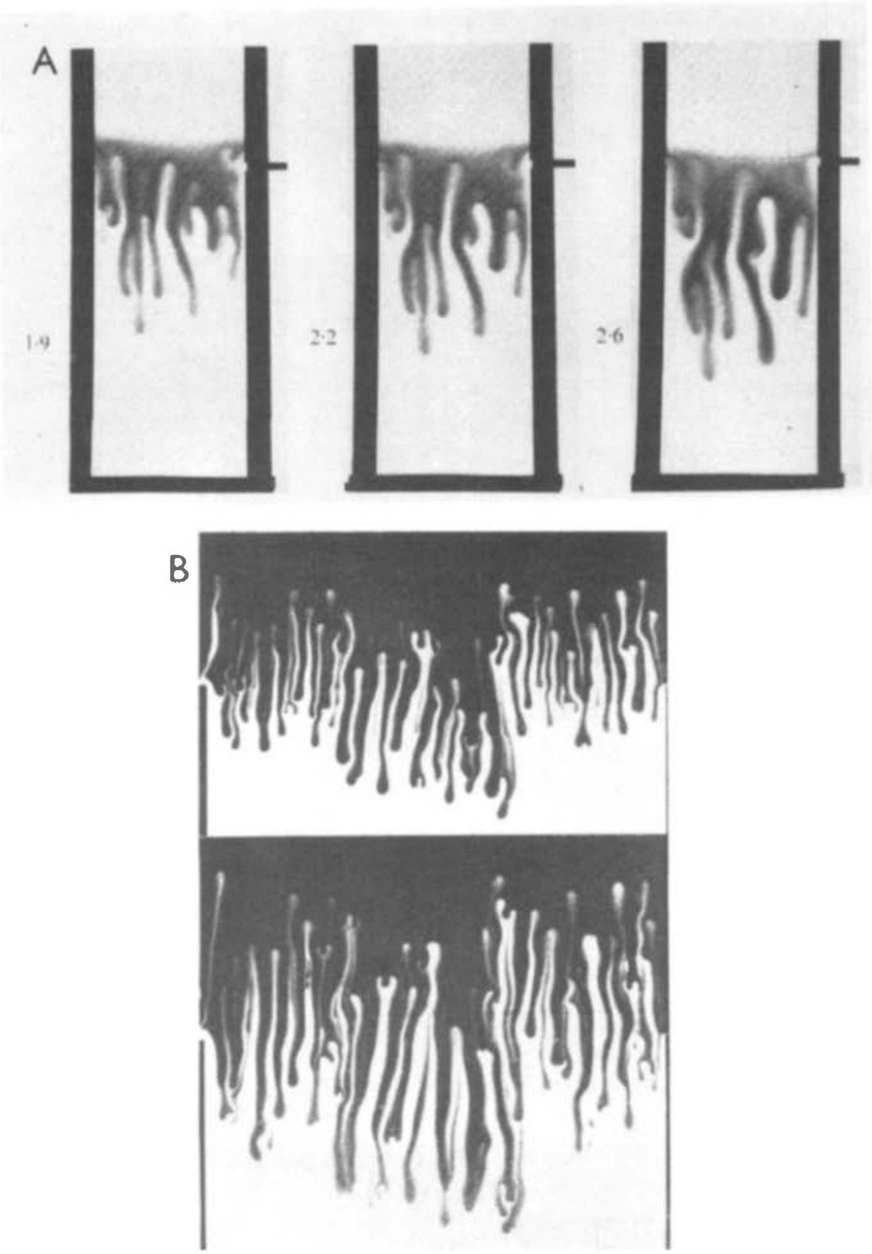


Figure 7 Fingering in gravity-driven miscible displacements showing (*a*) nonlinear fingering with transverse spreading at moderate Peclet number, and (*b*) tip splitting at high Peclet number (Wooding 1969). The different panels represent a time sequence. With permission.

permeability heterogeneities are *essential* in causing fingering instabilities. For a further discussion of the difficulty in accurate simulation at high Peclet number, see Claridge (1972) and Christie & Bond (1986). Similar considerations of the accurate resolution of dispersion in numerical simulations of miscible displacements also pertain to other geometries; for a discussion of five-spot patterns, see Ewing (1983).

The linear-stability characteristics of radial source flow in the absence of dispersion can be obtained directly from the analysis of Wilson (1975) and have been summarized by Paterson (1985). Not surprisingly, we again obtain a dispersion relation that has no cutoff azimuthal wave number, i.e. from Equation (10) we obtain

$$\sigma = Am - 1. \tag{17}$$

Paterson has suggested that as a result, fingers will occur in Hele-Shaw cells on all scales down to a scale comparable to the gap width. If this is true, then in experiments, a cutoff occurs because of phenomena not included in the two-dimensional Hele-Shaw equations leading to Equation (17) above. He gives a heuristic argument based upon energy dissipation in the gap, by which this cutoff, which scales with the gap width b, may be computed. Shown in Figure 8 is a set of photographs from his paper that display fingering in miscible displacement of glycerine by water. The process of areal spreading followed by tip splitting is a familiar one, being reminiscent of that occurring in immiscible radial source flow. The small tick marks on the photographs indicate the cutoff wavelength estimated by the heuristic argument, and it is seen that the estimate is qualitatively correct. No theory for the stability of radial source flow that incorporates dispersion is currently available by which one could distinguish cutoff by dispersion vs. cutoff by three-dimensional effects in these experiments, but it seems clear that the latter predominate, at least in this range of Peclet numbers.

Let us return to Figure 1*a*, which shows the fingering observed in

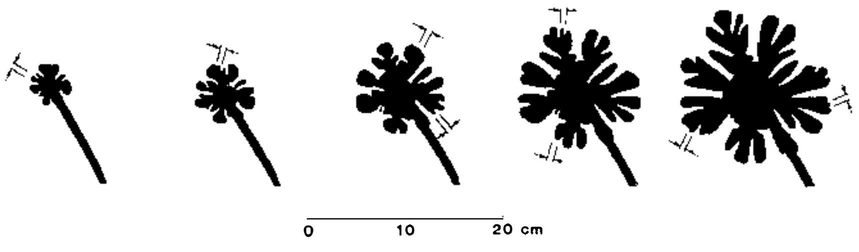


Figure 8 Fingering in miscible radial Hele-Shaw source flow (Paterson 1985). The tick marks indicate the scale of the gap width. With permission.

extremely thin-gap Hele-Shaw cells (E. L. Claridge, personal communication, 1986). [Numerous tracings of similar experiments, which unfortunately do not capture the finer details of these patterns, are to be found in Stoneberger & Claridge (1985).] Examination of a time sequence of such patterns reveals that fingers grow, spread by transverse dispersion, and split at the tips, in the same manner as described by Wooding (1969) for buoyancy-driven fingering. In this case, fingering does not occur down to the scale of the gap width, this length being exceedingly thin in these experiments (0.01 cm). This would seem to indicate that the heuristic argument of Paterson (1985) that fingering occurs at all scales down to the Hele-Shaw gap width is not generally applicable. Unfortunately, too few experiments that specifically study the effect of the Peclet number are available, and in many cases insufficient data exist to allow a calculation of Peclet numbers for the conditions of a given experiment. This is a curious situation, given the key role dispersion plays in fingering in miscible systems, and it is hoped that careful experiments at high Peclet numbers might be reported in the near future. In summary, we have seen that the phenomena of *shielding*, *spreading*, and *splitting* are all important in determining the dynamics of fingering in miscible displacements, that the parameter analogous to the modified capillary number is the Peclet number, and that interesting and complex behavior occurs at high Peclet number.

We close this subsection by discussing the intriguing work of Nittmann et al. (1985) and Daccord et al. (1986). They conducted experiments in both rectilinear flow and radial source flow of water into a Hele-Shaw cell initially containing a non-Newtonian, viscoelastic fluid. The fingering patterns observed are qualitatively different from those observed with conventional fluids—compare Figure 1*c* with Figures 1*b* and 8. In these experiments the pattern grows by extremely localized tip splitting, and the shielding is much stronger than in the case of Newtonian fluids. They present data suggesting that the geometry of the patterns has a fractal structure, and the measured fractal dimension is close but not equal to that obtained from simulations of diffusion-limited aggregation (DLA), discussed below. Visual comparison with DLA clusters reveals a lesser degree of side branching in the experiments. They argue that such a choice of fluid pairs is necessary in order to obtain such structures, since Newtonian fluids do not have sufficiently high viscosity contrast. Such a claim does not stand up under close scrutiny, since the viscosity ratio enters the problem as the parameter A, not the absolute ratio, and for large viscosity ratios, the dominant length scales become sensibly independent of this ratio. Thus, systems of any reasonable viscosity contrast, such as those for the systems used by Paterson and by Claridge, have $A = 1$, but do not exhibit such ramified patterns. It is quite probable that these results, while fascinating, are linked to the viscoelastic nature of the displaced fluid used in their experiments. It is tempting to speculate on the mechanism that leads to dramatic structures such as those shown in Figure 1*c*, but until more work is done, these structures will remain unexplained and in sharp contrast to the results available for Newtonian fluids.

POROUS MEDIA FLOWS

There is a distinction to be drawn, where appropriate, between viscous fingering in model systems like Hele-Shaw cells and "real" porous materials, since some of the phenomena are distinctly different and some of the mathematical models used are less fundamentally based. Many of the mechanisms discussed above still pertain; however, in many cases it becomes necessary to recognize the particulate nature of the medium. In this section we review the available understanding of fingering in real media, with the anticipation that this knowledge will be incomplete in significant ways. The description of fingering will in turn depend strongly on the length scales at which we choose to view the flow. Thus we face at the outset a fundamental issue of the level of detail of description of pore-level fluid mechanics that is desirable or even possible. There is at present no general agreement on this issue, and therefore we briefly discuss both continuum approaches, which seek to describe the fingering on scales large compared with typical pore dimensions, and discrete approaches, which are essentially discontinuous descriptions. As we shall see, both approaches involve large elements of modeling and simulation, as opposed to fundamental prediction obtained as a result of solving an agreed-upon set of equations.

Miscible Displacements in Porous Media

If we begin by adopting a continuum description of miscible displacements in which we average over scales comparable to the scales of pores or grains of the medium, then the single-phase Darcy equations with variable viscosity hold [Equations (12)]. Furthermore, we recognize that the effective dispersion tensor of the medium arises as a result of mixing and mechanical dispersion on the pore scale. Thus we have in effect accomplished whatever averaging is necessary to obtain the continuum description, which will certainly be valid in describing any instability that is smooth on the continuum length scale. The theoretical studies of fingering in miscible systems reviewed above pertain *equally* to Hele-Shaw cells and to porous media as long as the dispersion tensor is evaluated appropriately. The dynamic dimensionless groups are as we have indicated above, with the Peclet number the main determinant of the scale of fingering. We expect, therefore, to find a close similarity between Hele-Shaw results and porous-media results in the limit of isotropic dispersion.

As we have seen, both transverse and longitudinal dispersion lead to preferred wavelengths for fingering, and that in the strict absence of dispersion, fingering will occur on all scales, with growth rates that increase with decreasing scale. Scales will ultimately be reached for which the continuum hypothesis no longer holds, and a pore-level description is then appropriate. Thus the nature of fingering at very high Peclet number may be more conveniently described in noncontinuum terms. As noted above, there are very few continuum analyses available of either the linear-stability characteristics or the nonlinear miscible fingering. Those that exist indicate that as dispersion acts to spread the mobility profile, there is a shift to longer wavelength fingers. No analyses to date have been able to describe tip splitting in the nonlinear regime.

There have been many experimental studies of fingering in porous materials, most of which have appeared in the petroleum-engineering literature. In these studies, great care has typically been taken to ensure the homogeneity of the media by constructing them from sand or other unconsolidated materials. Since porous materials are generally opaque, very few visualizations of fingers exist in thick media. We take special note of some early X-ray studies by Slobod & Thomas (1963) and Perkins et al. (1965). As an illustrative example, we show in Figure 9 two of the X-ray pictures of Slobod & Thomas taken during the same experiment in quasi-two-dimensional horizontal displacement. Viscous fingers whose characteristic length is many times the characteristic pore size are clearly evident, as is the trend toward longer scales as time progresses. Note the similarity between these patterns and those of Wooding (1969) in Hele-Shaw cells, shown in Figure 7*a*. Unfortunately, insufficient time-resolved information is available to determine the mode of finger growth in these experiments, i.e. whether it occurs by tip splitting or simply by spreading due to transverse dispersion. Since the transverse Peclet number is comparatively low, the latter seems the more probable. Tan & Homsy (1986) have compared the apparent wavelengths in visualizations like Figure 9*a* to the linear-stability calculations shown in Figure 6, with reasonable agreement between theory and experiment.

Although the X-ray visualizations apparently did not show tip splitting, there are other experiments that do, most notably the celebrated five-spot experiments of Habermann (1960), done in relatively thin media and shown in Figure 10*a*. Although the Peclet number is not known for these experiments and no time sequences are shown, it is obvious that significant growth of the pattern has occurred at the tips due to shielding, and that repeated tip splitting has also occurred, resulting in the characteristic pattern shown. Note the similarity with the Hele-Shaw experiments of Claridge (Figure 1*a*), also done at high Peclet numbers. It is also interesting to note that the fractional recovery of the displaced fluid as a function of mobility ratio M is nearly equal in the porous-media experiments of Habermann and the Hele-Shaw experiments of Stoneberger & Claridge (1985), which indicates the similarity of the mechanics in the two cases. There are many empirical correlations of the fractional recovery that indicate that this quantity is relatively insensitive to the Peclet number for sufficiently large Pe, which implies that the large-scale *geometric* properties of the pattern become independent of Pe as $\mathrm{Pe} \rightarrow \infty$. This is a concept familiar to researchers who study the large-scale features of turbulent flows at high Reynolds numbers.

In spite of the geometrical complexity of Figures 1*a* and 10*a*, in both cases the dominant length scales are much larger than any microscopic pore scale. This is not true when one studies fingering at ever-increasing Peclet numbers at ever-decreasing scales. Paterson et al. (1982) have studied miscible fingering in packed beds that shows patterns similar to those occurring in DLA. Obviously, for a given pore scale there is a crossover between the continuum and discrete descriptions, but very little quantitative information is available to indicate when this crossover occurs.

A challenging and interesting question pertains to the structure of nonlinear fingering in the limit in which phenomena leading to a cutoff scale are entirely missing, i.e. the case of zero dispersion or molecular diffusion in the examples above. Because of the properties of the linear-stability theory in this limit, the initial-value problem is ill posed, but we may seek solutions that have discontinuities or other singularities. Neglecting dispersion preserves sharp jumps in viscosity profiles for all times, with the result that the pressure obeys Laplace's equation, with prescribed boundary conditions at the moving boundary. Various types of singularities can occur in the solution as a result, and very little work exists on

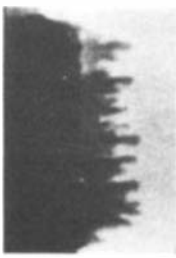
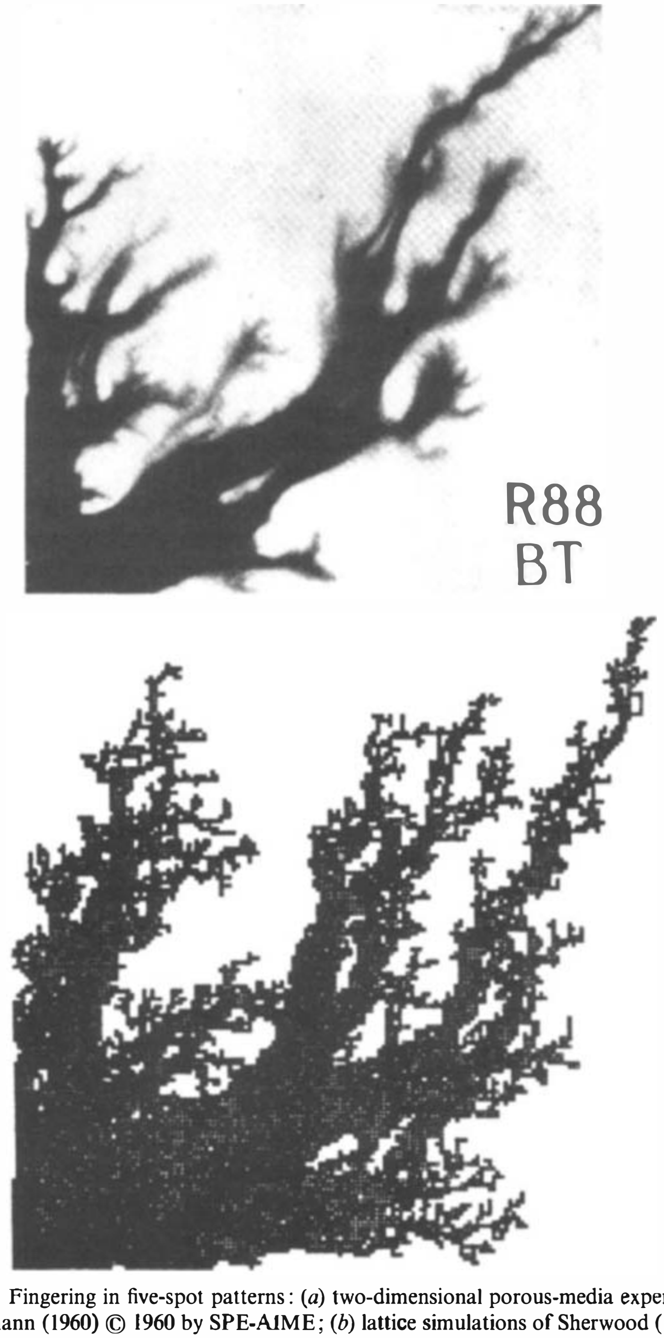
(a)  (b) 

Figure 9 Fingering in miscible displacement in opaque media, at two different times, as visualized by X-ray absorption (Slobod & Thomas 1963). Compare Figure 9*b* with Figure 7*a*. © 1963 by SPE-AIME: with permission.



(a)

(b)

Figure 10 Fingering in five-spot patterns: (*a*) two-dimensional porous-media experiments of Habermann (1960) © 1960 by SPE-AIME; (*b*) lattice simulations of Sherwood (1986b). Compare these two with Figure 1*a*. With permission.

the subject. However, we can anticipate that different nonuniformities will appear in different boundary-value problems. In this regard, we mention the work of Shraiman & Bensimon (1984) and Howison (1986a,b) on cusp formation in fingering problems.

Also of interest are the large number of recent papers that attempt to construct discontinuous solutions by random-walk calculations of the type used in DLA. The first observation of the relevance of DLA to fingering in porous media is due to Paterson (1984), who commented on the formal equivalence between DLA and fingering in miscible systems at infinite Peclet numbers and $A = 1$, and who reported simulations in which he claimed to describe fingering in five-spot patterns. Similar approaches, but modified to account for different geometries, boundary conditions, and finite mobility ratios, have been discussed by Sherwood & Nittmann (1986), DeGregoria (1985), and Sherwood (1986a,b). For $A = 1$, these approaches capture the effect of shielding quite accurately, and growth of fingers is seen to occur primarily at the tips. Figure 10*b* shows the results of a two-dimensional simulation in a five-spot pattern at the point of breakthrough for $A = 1$ (Sherwood 1986a). When this simulation is compared with the experiments of E. L. Claridge (personal communication, 1986) and of Habermann (1960), a striking geometrical similarity is observed. The resulting pattern, since it can access all scales above the scale of the lattice on which the simulation is performed, has the geometrical properties of a fractal object. The analogy with DLA is not so clear in the case of finite mobility, as the rules for sticking the random walkers to the aggregates must be modified to account for the finite mobility of the "aggregate" phase. However, Sherwood & Nittmann (1986) and DeGregoria (1985) have suggested rules for simulations at finite mobility ratio that advance the interface at a point with a probability proportional to the local pressure gradient. Sherwood & Nittmann argue that such rules model physical dispersion, but the physical significance of some aspects of these algorithms, including the rules for trapping of the displaced phase, is not clear.

While these approaches to discrete simulations of viscous fingering seem to hold much promise, there are at least two drawbacks to their quantitative accuracy. First, at their present state of development, they do not account for the *spreading* of fingers due to weak dispersion, and as a result they cannot capture all of the physics of tip splitting that are so important in determining much of the fascinating structure of fingering patterns. Second, many parameters of scientific and engineering interest, e.g. the fractional recovery of the displaced fluid, depend upon the lattice size, sometimes sensitively so (see, e.g., DeGregoria 1985). Thus, these simulations involve an artificial cutoff related to a lattice size, which at present cannot be related to any quantity of physical significance and which must be used as a fitting parameter in order to obtain results other than those represented solely by the *geometry* of the fingered pattern, i.e. other than the fractal dimension. DeGregoria (1985) has argued that once this size is fit to a given physical system, the effect of varying the mobility ratio may be *quantitatively* predicted, and he gives a calculation of the fractional recovery as a function of mobility ratio that agrees well with experiments. Further research needs to be done to connect these discrete simulations to smooth but steep solutions of the continuum equations.

Immiscible Displacements in Porous Media

The problem of viscous fingering in immiscible systems is arguably one of the most difficult problems pertaining to porous media flow. This is so because of the wide variety of phenomena occurring that involve a myriad of pore-scale phenomena, including the details of wetting behavior and wetting films, the movement of contact lines and dynamic contact angles, the static stability of capillary bridges, the dynamic instability of blobs of immiscible, nonwetting fluid, the propagation of these blobs, the phase transitions during flow, the heat and mass transfer across interfaces, and many others. Some of these areas have been touched on in the review by Wooding & Morel-Seytoux (1976), and some of them are sufficiently rich that they have been the subject of specialized reviews in this series [e.g. the review on "ganglia" mechanics by Payatakes (1982), that on contact lines and contact angles by Dussan V. (1979), and that on capillary instabilities by Michael (1981). Further compounding the complexity of the subject is the fact that the distributions of menisci within the material, together with the configuration of phases within the medium, are under some circumstances strongly coupled to the topology of the underlying solid matrix, the description of which is a subject of much current research in the field of disordered media. It is clear that we cannot give a detailed account of all these issues, so our discussion is necessarily limited to those situations in which *viscosity* plays a major role. As we have commented above, this focus excludes a large and interesting emerging literature on the slow, capillary-dominated propagation of menisci on lattices and within disordered materials, which finds a natural setting within the framework of percolation phenomena.

Once again, we face the major problem that "real" media are opaque, and as a result very few visualizations are available, certainly not enough for us to understand the detailed dependence of viscous fingering upon the parameters of the process. As in the case of miscible fingering, instructive observations are again ones that were done some years ago, in this instance by van Meurs (1957). Some of these visualizations, accomplished by working with displaced fluids of the same index of refraction as the medium, have already been reproduced in the article by Wooding & Morel-Seytoux (1976), and the interested reader is urged to consult that reference. This beautiful time sequence shows that the processes of *shielding*, *spreading*, and *splitting* are present in real porous materials and determine the patterns of fingering in essential ways. Shown in Figure 11 are similar observations of fingering from Chouke et al. (1959), which show the variation of the length scales of fingering with increasing velocity and viscosity contrast, i.e. with increasing capillary number. Fingering apparently takes place on many scales, including a macroscopic one, and thus one might assign a characteristic macroscopic length scale (if not a wavelength) to what is seen. It is obvious from these and other visualizations that the scale of fingering changes, becoming smaller when either the velocity or the viscosity of the displaced fluid is increased. From these and other observations on both three- and quasi-two-dimensional media (see, e.g., Peters & Flock 1981, Paterson et al. 1984a,b, White et al 1976, Måløy et al. 1986, and the references quoted therein), the following may be observed:

1. Wetting properties are important: There is a qualitative difference in fingering when the invading fluid does or does not wet the medium. In the former case fingering is characterized by some macroscopic continuum scale, whereas in the latter fingering is more likely to be confined to the pore scale, with shielding dominating over spreading, as might be expected.
2. Characteristic macroscopic scales, if present, decrease with increasing capillary number $\mathrm{Ca} = \mu_1 U/\gamma$.
3. When the invading fluid is nonwetting, the pattern is a probe of the topology of the microstructure and is characteristic of percolation behavior, with a backbone that may have a fractal character.
4. There is simultaneous flow of both phases in a zone behind a displacement front, assuming such a front may be identified.

The above observations are oversimplifications of complex behavior, but they will help to fix our ideas.

The first attempt to provide a theoretical analysis of the onset of fingering was by Chouke et al. (1959). They assumed that there was *complete* displacement of one fluid by the other and ignored the zone of partial saturation or volume concentration of the displacing fluid behind the front. This reduces the equations in each phase to the single-phase Darcy equations [Equations (4)]. In order to avoid the short-wave catastrophe that, as we have seen, occurs, Chouke took the bold step of applying the jump condition [Equation (6a)] at the front, but with a constant γ^* that is different from the molecular surface tension. He provides a heuristic justification based upon energy arguments, which is incorrect. This has been referred to as "the Chouke boundary condition," and the resulting predictions will be referred to as Hele-Shaw-Chouke theory. This boundary condition has been used in similar contexts by White et al. (1976), Peters & Flock (1981), Paterson et al. (1984a), and many others. It is by no means clear how surface tension, acting as it must on menisci at the

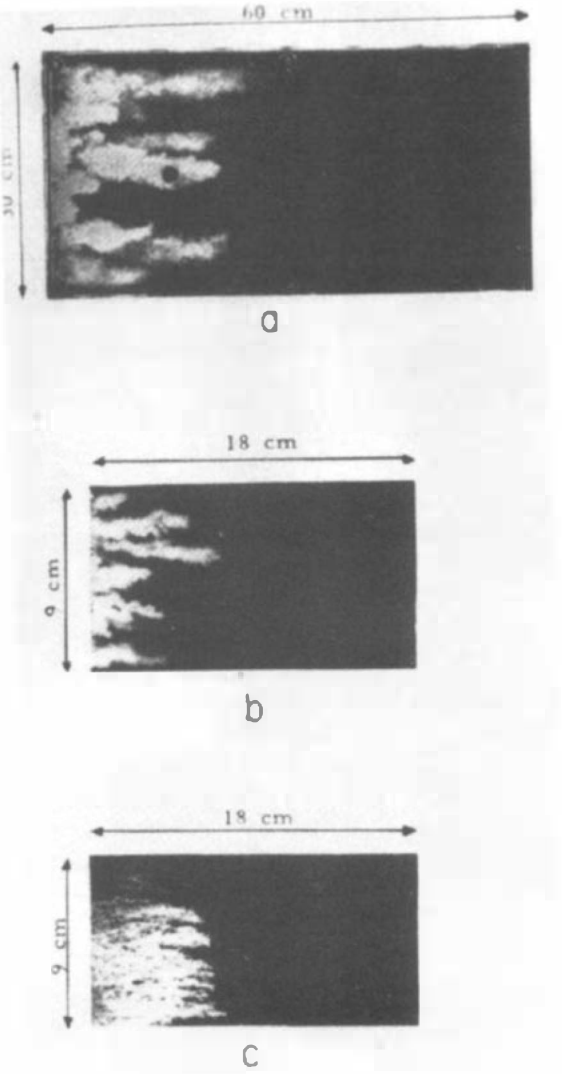


Figure 11 Fingering in immiscible displacement in porous media (Chouke et al. 1959): (*a*) Ca = 2×10^{-5}; (*b*) Ca = 1.2×10^{-4}; (*c*) Ca = 3×10^{-3}. © 1959 by SPE-AIME: with permission.

pore level, can provide a restoring force proportional to macroscopic curvature, as implied by Equation (6a). However, it is found that estimates of the characteristic macroscopic length from observations such as those shown in Figure 11 compare well with those given by Equation (8) if the empirical "effective surface tension" γ^*, which is found to depend upon wetting conditions, is fit at one value of the capillary number. This is essentially equivalent to the statement that the length scale of fingers scales with the modified capillary number $\mathrm{Ca}' = (\mu_1 UL^2/\gamma^* K)$ as

$$k \sim O(\mathrm{Ca}'^{1/2}). \tag{18}$$

Alternative theoretical descriptions of fingering attempt to model the fact that variations in the saturation behind the front can reduce the mobilities of both phases there, and that because of these saturation gradients, there will be a tendency, depending upon the wetting conditions, for one fluid to displace the other by the action of capillary imbibition. This leads to a process of *spreading*, which appears in the equations [Equations (22) below] as a diffusive effect.

While it is clear that the continuum description of such pore-scale events of simultaneous flow of both phases and capillary invasion is not well understood, the set of equations that is thought to describe these events involves modifications of Darcy's law as follows:

$$\nabla p_i = -\lambda_i^{-1}\mathbf{u}_i, \qquad i = 1,2, \tag{19}$$

where

$$\lambda_i = K_i(c)/\mu_i \tag{20}$$

is the mobility of phase i, and the quantity K_i is the permeability of phase i, dependent upon the local concentration or saturation of the phases, c. Without loss of generality, we take c to be the relative saturation of the invading phase. It is common to factor K_i as the intrinsic permeability times the relative permeability. The difference between the pressures in the phases, which models the invasion by the wetting phase, is also dependent upon saturation and is taken to scale with a characteristic pore scale and the molecular surface tension, i.e.

$$p_1 - p_2 \equiv \gamma P_c(c)/K^{1/2}, \tag{21}$$

which is the defining equation for P_c, the dimensionless capillary pressure. These equations must be augmented by continuity equations for the individual phases. With suitable manipulations, the problem reduces to an evolution equation for the saturation c of the general form

$$\frac{\partial c}{\partial t} + \nabla\cdot\left(\frac{\mathbf{u}_0}{1+M^{-1}}\right) = \mathrm{Ca}^{*-1}\nabla\cdot[G(c)\nabla c] \tag{22a}$$

with

$$M(c) = \lambda_2(c)/\lambda_1(c) \tag{22b}$$

and

$$G(c) = -\left(\frac{dP_c}{dc}\right)\left(\frac{1}{1+M^{-1}}\right). \tag{22c}$$

Here $\mathbf{u}_0$ is the volume average velocity of the mixture, and the modified capillary number Ca* is defined as

$$\mathrm{Ca}^* \equiv \mu_1 UL/\gamma K^{1/2}, \tag{23}$$

which differs from that appearing in Hele-Shaw theory by the factor involving the ratio of macroscopic to microscopic length scales. As noted, these equations have some physical intuition embodied in them and are routinely used in the petroleum-engineering literature, but they cannot be rigorously justified and have never been derived from a pore-level description, even for simple geometries. For a further discussion of these equations, see Wooding & Morel-Seytoux (1976) and the references therein.

Solutions of these equations will depend upon the modified capillary number, the mobility function $M(c)$, and the capillary pressure function $P_c(c)$. It will be important later to note that models from percolation theory and data suggest that $M(c)$ vanishes at a finite value of c, related to the percolation threshold. Analyses of these equations in the limit of $\mathrm{Ca}^* = \infty$ go under the general designation of Buckley-Leverett theory. Since Equation (22a) with $\mathrm{Ca}^* = \infty$ is a nonlinear hyperbolic system, it is not surprising that the solutions develop shocks at a finite time, with a rarefaction wave of saturation that disperses as time proceeds. For specific choices of the function $M(c)$, simple one-dimensional solutions are well known. For large but finite Ca*, the problem is a singularly perturbed one, and Wooding (1975) has analyzed the continuous structure of the front region using matched expansions. He finds that this region is $O(\mathrm{Ca}^{*-1})$ in thickness, as opposed to the more probable $O(\mathrm{Ca}^{*-1/2})$ for nonlinear diffusion problems.

Stability analyses of these one-dimensional solutions have only just recently begun (Yortsos & Huang 1984, Huang et al. 1984, Jerauld et al. 1984a,b). There are several conceptual as well as practical difficulties in performing such analyses. The first is that the profiles are inherently time dependent, as in the case of miscible displacements, and thus one must discuss the stability of time-dependent flows. This has been circumvented in some cases by applying inflow boundary conditions corresponding to injection of both phases at saturations that allow steady solutions to the

Buckley-Leverett equations. Even then, the predictions are specific to the choices of the material functions $M(c)$ and $P_c(c)$, so only a few features of a general nature are known. The first is that, unlike Hele-Shaw-Chouke theory, it is incorrect to infer stability or instability from evaluation of the *viscosity* ratio of the fluids; rather, it is the *mobility* contrast at the front, together with the saturation profile behind the front, that determines stability. In this sense the stability characteristics share much in common with graded mobility processes analyzed by Gorell & Homsy (1983) for immiscible processes, and by Hickernell & Yortsos (1986) for miscible displacements at infinite Peclet number. Thus, depending upon the mobility function, a displacement that has an unfavorable viscosity ratio may still be linearly stable, even at infinite Ca*. Yortsos & Huang (1984) have analyzed the linear stability by using a quasi-static approximation for the saturation profiles, and straight-line approximations to the mobility profiles, in the spirit of Heller (1966). They find that without capillary pressure, there is no cutoff scale, which is a familiar property of sharp fronts with mobility jumps but with no dissipative mechanism. With capillary pressure, they find a cutoff wave number

$$k_c \sim O(\mathrm{Ca}^*), \tag{24}$$

as opposed to that given by Hele-Shaw-Chouke theory [Equation (18)]. Huang et al. (1984) have analyzed more general spatially varying profiles. They find certain profiles that are neutrally stable even in the absence of capillary pressure. They also solve the eigenvalue problem by numerical means, a process that becomes difficult at high Ca*, in a manner similar to the ill conditioning that occurs in Hele-Shaw theory at high Ca′, in miscible displacements at high Peclet number, and in Orr-Sommerfeld theory at high Reynolds numbers. They find that the critical viscosity ratio below which displacement is stabilized by the graded mobility profile may be simply estimated from the inlet saturations and the mobility jump from Buckley-Leverett theory. Chikhliwala & Yortsos (1985) have provided long-wave expansions of the eigenvalue problem that they claim capture most of the qualitative features of the numerical calculations. Jerauld et al. (1984a,b) have analyzed a class of steady solutions of the Buckley-Leverett equations in a manner similar to that of Yortsos & Huang (1984) and Chikhliwala & Yortsos (1985), including long-wave expansions and numerical solutions to the related variable-coefficient eigenvalue problem, and find similar results of stabilization due to spatially varying mobility profiles, and stability at viscosity ratios greater than one. They also comment on the differences in predicted scales given in Equations (18) and (24). The linear-stability calculations done to date must be considered as providing only preliminary understanding of the solution properties of a set of *model* equations, which must be further tested and validated by comparison with more extensive experimentation.

As can be seen, there is a formal similarity between immiscible displacements, described by the fractional flow equations (22) above, and miscible displacements, described by Equations (12). Differences appear in the material behavior of the function $M(c)$ or $\mu^*(c)$, in the fact that $M(c)$ has a zero, and in the fact that the dispersive term in each continuity equation has different functionalities, being isotropic and dependent upon saturation on the one hand, and anisotropic and dependent upon velocity on the other. However, it is probable that these differences become only ones of detail in the limits of $\mathrm{Pe} \rightarrow \infty$ and $\mathrm{Ca}^* \rightarrow \infty$, respectively.

Insufficient experimental evidence exists to adequately test any of these predictions. Both approaches provide scalings, given in Equations (18) and (24), which capture the qualitative trends shown, e.g., in Figure 11 of decreased scales of fingers with increasing velocity and displaced fluid viscosity, but careful experiments that vary all the parameters over wide ranges and that distinguish the differences between Equations (18) and (24) have yet to be reported. However, the observations by Chouke et al. (1959), White et al. (1976), Peters & Flock (1981) and Paterson et al. (1984a) provide compelling evidence in favor of the Hele-Shaw-Chouke theory, which in this author's opinion is presently unexplained.

Qualitative experimental studies on rectilinear displacements in two-dimensional etched networks by Lenormand & Zarcone (1985) have suggested that continuum model equations have a restricted range of validity. The authors conducted displacement experiments for both favorable and unfavorable mobility ratios as a function of capillary number $\mathrm{Ca} = \mu_1 U/\gamma$, where U is the average interstitial velocity. For favorable mobility ratios, they found that below $\mathrm{Ca} \sim 10^{-4}$–10^{-3}, displacement was by invasion percolation, for which a continuum description is not possible. For $\mathrm{Ca} > 10^{-3}$, displacement proceeds in a fashion appropriate to Buckley-Leverett theory. For unfavorable mobility ratios for which viscous fingering is possible, they observed a crossover at $\mathrm{Ca} = 10^{-8}$–10^{-7}, from invasion percolation behavior to that characterized by DLA, without observing a continuum regime. Lenormand (1985) has proposed a "phase diagram" of these different regimes that involves mobility ratio and capillary number, which can at present be considered to be only qualitative and specific to the apparatus used, and most certainly dependent upon wetting conditions.

Fingering resulting from invasion of nonwetting fluids into porous media remains a poorly understood and complicated subject. Recently, Måløy et al. (1986) have reported fingering in radial source flow at high capillary number, defined using the superficial velocity, in which the pattern is claimed to have a fractal geometry with dimension appropriate to DLA. Since this is an immiscible displacement of nonwetting fluid, it is not clear why DLA should apply at all, and since their apparatus consisted of only a monolayer of particles between solid plates, the results cannot be considered to have applicability to porous materials.

Recently, King (1985) and King & Scher (1986) have presented some probabilistic lattice simulations for both rectilinear and five-spot flow, similar in spirit to those of Sherwood and of DeGregoria discussed above, but for the Buckley-Leverett equations [Equations (22) in the singular limit of $Ca^* \rightarrow \infty$]. As stability analyses show, this is in general an ill-posed problem unless the mobility function and inlet saturations are specially chosen, and a cutoff due to the finite lattice spacing exists. The patterns produced are as intriguing and suggestive as those shown in Figure 10*b*, but as the authors point out, fractional recovery is sensitively dependent upon the lattice spacing. Again, this spacing must be connected to some parameter of physical significance before this approach can be generally useful. Finally, we note briefly the comments by Chen & Wilkinson (1985), who point out that the probabilistic rules used by DeGregoria, Sherwood, and King can be considered as models for the natural variation of pore-scale features such as pore radius, since patterns in deterministic network models with variable geometrical features are directly comparable to those observed in probabilistic simulations.

ACKNOWLEDGMENTS

Our work on viscous fingering and the preparation of this article were sponsored by the US Department of Energy, Office of Basic Energy Sciences, whose support is gratefully acknowledged. I am also pleased to acknowledge helpful discussions of viscous fingering with H. Aref, M. King, T. Maxworthy, P. G. Saffman, L. Schwartz, J. Sherwood, and Y. Yortsos.

Literature Cited

Aref, H. 1986. Finger, bubble, tendril, spike. *Pol. Acad. Sci. Fluid Dyn. Trans.* 13: In press

Benham, A. L., Olson, R. W. 1963. A model study of viscous fingering. *Soc. Pet. Eng. J.* 3: 138–44

Ben-Jacob, E., Godbey, R., Goldenfeld, N. D., Koplik, J., Levine, H., et al. 1985. Experimental demonstration of the role of anisotropy in interfacial pattern formation. *Phys. Rev. Lett.* 55: 1315–18

Bensimon, D. 1986. On the stability of viscous fingering. *Phys. Rev. A* 33: 1302–8

Bensimon, D., Kadanoff, L. P., Liang, S., Shraiman, B., Tang, C. 1986. Viscous flows in two dimensions. *Rev. Mod. Phys.* In press

Chen, J.-C., Wilkinson, D. 1985. Pore-scale viscous fingering in porous media. *Phys. Rev. Lett.* 55: 1892–95

Chikhliwala, E. D., Yortsos, Y. C. 1985. Theoretical investigations on finger growth by linear and weakly nonlinear stability analysis. *SPE 14367*, Soc. Pet. Eng., Dallas, Tex.

Chouke, R. L., van Meurs, P., van der Poel, C. 1959. The instability of slow, immiscible, viscous liquid-liquid displacements in permeable media. *Trans. AIME* 216: 188–94

Christie, M. A., Bond, D. J. 1986. Detailed simulation of unstable processes in miscible flooding. *SPE/DOE 14896*, Soc. Pet. Eng., Dallas, Tex.

Claridge, E. L. 1972. Discussion of the use of capillary tube networks in reservoir performance studies. *Soc. Pet. Eng. J.* 12: 352–61

Combescot, R., Dombre, T., Hakim, V., Pomeau, Y. 1986. Shape selection of Saffman-Taylor fingers. *Phys. Rev. Lett.* 56: 2036–39

Daccord, G., Nittmann, J., Stanley, H. E. 1986. Radial viscous fingers and diffusion-limited aggregation: fractal dimension and growth sites. *Phys. Rev. Lett.* 56: 336–39

DeGregoria, A. J. 1985. A predictive Monte Carlo simulation of two-fluid flow through porous media at finite mobility ratio. *Phys. Fluids* 28: 2933–35

DeGregoria, A. J., Schwartz, L. W. 1985. Finger break-up in Hele-Shaw cells. *Phys. Fluids* 28: 2313–14

DeGregoria, A. J., Schwartz, L. W. 1986. A boundary-integral method for two-phase displacement in Hele-Shaw cells. *J. Fluid Mech.* 164: 383–400

Dussan V., E. B. 1979. On the spreading of liquids on solid surfaces: static and dynamic contact lines. *Ann. Rev. Fluid Mech.* 11: 371–400

Ewing, R. E., ed. 1983. *The Mathematics of Reservoir Simulation.* Philadelphia: SIAM. 186 pp.

Gardner, J. W., Ypma, J. G. J. 1982. An investigation of phase behavior–macroscopic bypassing interaction in CO_2 flooding. *SPE 10686*, Soc. Pet. Eng., Dallas, Tex.

Gorell, S., Homsy, G. M. 1983. A theory of the optimal policy of oil recovery by secondary displacement processes. *SIAM J. Appl. Math.* 43: 79–98

Habermann, B. 1960. The efficiency of miscible displacement as a function of mobility ratio. *Trans. AIME* 219: 264–72

Heller, J. P. 1966. Onset of instability patterns between miscible fluids in porous media. *J. Appl. Phys.* 37: 1566–79

Hickernell, F. J., Yortsos, Y. C. 1986. Linear stability of miscible displacement processes in porous media in the absence of dispersion. *Stud. Appl. Math.* 74: 93–115

Hill, S. 1952. Channelling in packed columns. *Chem. Eng. Sci.* 1: 247–53

Hong, D. C., Langer, J. S. 1986. Analytic theory of the selection mechanism in the Saffman-Taylor problem. *Phys. Rev. Lett.* 56: 2032–35

Howison, S. D. 1986a. Cusp development in Hele-Shaw flow with a free surface. *SIAM J. Appl. Math.* 46: 20–26

Howison, S. D. 1986b. Fingering in Hele-Shaw cells. *J. Fluid Mech.* In press

Huang, A. B., Chikhliwala, E. D., Yortsos, Y. C. 1984. Linear stability analysis of immiscible displacement including continuously changing mobility and capillary effects: Part II—general basic flow profiles. *SPE 13163*, Soc. Pet. Eng., Dallas, Tex.

Jerauld, G. R., Davis, H. T., Scriven, L. E. 1984a. Stability fronts of permanent form in immiscible displacement. *SPE 13164*, Soc. Pet. Eng., Dallas, Tex.

Jerauld, G. R., Nitsche, L. C., Teletzke, G. F., Davis, H. T., Scriven, L. E. 1984b. Frontal structure and stability in immiscible displacement. *SPE 12691*, Soc. Pet. Eng., Dallas, Tex.

Kessler, D. A., Levine, H. 1986a. Stability of finger patterns in Hele-Shaw cells. *Phys. Rev. A* 32: 1930–33

Kessler, D. A., Levine, H. 1986b. Coalescence of Saffman-Taylor fingers: a new global instability. *Phys. Rev. A* 33: 3625–27

Kessler, D. A., Levine, H. 1986c. Theory of the Saffman-Taylor finger. I & II. *Phys. Rev. A* 33: 2621–33, 2634–39

King, M. J. 1985. Probabilistic stability analysis of multiphase flow in porous media. *SPE 14366*, Soc. Pet. Eng., Dallas, Tex.

King, M., Scher, H. 1986. A probability approach to multi-phase and multi-component fluid flow in porous media. *Phys. Rev. A*. In press

Lenormand, R. 1985. Différents mécanismes de désplacements visqueux et capillaires en milieu poreaux: diagramme de phase. *C. R. Acad. Sci. Paris Ser. II* 310: 247–50

Lenormand, R., Zarcone, C. 1985. Two-phase flow experiments in a two-dimensional permeable medium. *PhysicoChem. Hydrodyn.* 6: 497–506

Liang, S. 1986. Random walk simulations of flow in Hele-Shaw cells. *Phys. Rev. A* 33: 2663–74

Maher, J. V. 1985. Development of viscous fingering patterns. *Phys. Rev. Lett.* 54: 1498–1501

Måløy, K. J., Feder, J., Jøssang, T. 1986. Viscous fingering fractals in porous media. *Phys. Rev. Lett.* 55: 2688–91

Maxworthy, T. 1986. The non-linear growth

of a gravitationally unstable interface in a Hele-Shaw cell. *J. Fluid Mech.* In press
McLean, J., Saffman, P. G. 1981. The effect of surface tension on the shape of fingers in a Hele-Shaw cell. *J. Fluid Mech.* 102: 455–69
Michael, D. H. 1981. Meniscus stability. *Ann. Rev. Fluid Mech.* 13: 189–215
Nittmann, J., Daccord, G., Stanley, H. E. 1985. Fractal growth of viscous fingers: quantitative characterization of a fluid instability phenomenon. *Nature* 314: 141–44
Park, C.-W. 1985. *Theory and experiment for instabilities in Hele-Shaw cells.* PhD thesis. Stanford Univ., Stanford, Calif. 123 pp.
Park, C.-W., Homsy, G. M. 1984. Two-phase displacement in Hele-Shaw cells: theory. *J. Fluid Mech.* 139: 291–308
Park, C.-W., Homsy, G. M. 1985. The instability of long fingers in Hele-Shaw flows. *Phys. Fluids* 28: 1583–85
Park, C.-W., Gorell, S., Homsy, G. M. 1984. Two-phase displacement in Hele-Shaw cells: experiments on viscously driven instabilities. *J. Fluid Mech.* 141: 257–87. Corrigendum: 144: 468–69
Paterson, L. 1981. Radial fingering in a Hele-Shaw cell. *J. Fluid Mech.* 113: 513–29
Paterson, L. 1984. Diffusion-limited aggregation and two-fluid displacements in porous media. *Phys. Rev. Lett.* 52: 1621–24
Paterson, L. 1985. Fingering with miscible fluids in a Hele-Shaw cell. *Phys. Fluids* 28: 26–30
Paterson, L., Hornof, V., Neale, G. 1982. A consolidated porous medium for the visualization of unstable displacements. *Powder Technol.* 33: 265–68
Paterson, L., Hornof, V. Neale, G. 1984a. Water fingering into an oil-wet porous medium saturated with oil or connate water saturation. *Rev. Inst. Fr. Pet.* 39: 517–21
Paterson, L., Hornof, V., Neale, G. 1984b. Visualization of a surfactant flood of an oil saturated porous medium. *Soc. Pet. Eng. J.* 24: 325–27
Payatakes, A. C. 1982. Dynamics of oil ganglia during immiscible displacement in water-wet porous media. *Ann. Rev. Fluid Mech.* 14: 365–93
Peaceman, D. W., Rachford, H. H. Jr. 1962. Numerical calculation of multidimensional miscible displacement. *Soc. Pet. Eng. J.* 2: 327–39
Perkins, T. K., Johnston, O. C., Hoffman, R. N. 1965. Mechanics of viscous fingering in miscible systems. *Soc. Pet. Eng. J.* 5: 301–17
Peters, E. J., Flock, D. L. 1981. The onset of instability during two-phase immiscible displacement in porous media. *Soc. Pet. Eng. J.* 21: 249–58
Pitts, E. 1980. Penetration of fluid into a Hele-Shaw cell. *J. Fluid Mech.* 97: 53–64
Reed, H. 1985. Gallery of fluid motion. *Phys. Fluids* 28: 2631–40
Reinelt, D. A. 1986. Interface conditions for two-phase displacement in Hele-Shaw cells. Submitted for publication
Saffman, P. G. 1986. Viscous fingering in Hele-Shaw cells. *J. Fluid Mech.* In press
Saffman, P. G., Taylor, G. I. 1958. The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London Ser. A* 245: 312–29
Schowalter, W. R. 1965. Stability criteria for miscible displacement of fluids from a porous medium. *AIChE. J.* 11: 99–105
Schwartz, L. 1986. Stability of Hele-Shaw flows: the wetting layer effect. *Phys. Fluids.* In press
Sherwood, J. D. 1986a. Island size distribution in stochastic simulations of the Saffman-Taylor instability. *J. Phys. A* 19: L195–200
Sherwood, J. D. 1986b. Unstable fronts in a porous medium. *J. Comput. Phys.* In press
Sherwood, J. D., Nittmann, J. 1986. Gradient-governed growth: the effect of viscosity ratio on stochastic simulations of the Saffman-Taylor instability. *J. Phys.* 47: 15–22
Shraiman, B. I. 1986. Velocity selection and the Saffman-Taylor problem. *Phys. Rev. Lett.* 56: 2028–31
Shraiman, B., Bensimon, D. 1984. Singularities in nonlocal interface dynamics. *Phys. Rev. A* 30: 2840–42
Slobod, R. L., Thomas, R. A. 1963. Effect of transverse diffusion on fingering in miscible-phase displacement. *Soc. Pet. Eng. J.* 3: 9–13
Stoneberger, M., Claridge, E. L. 1985. Graded viscosity bank design with pseudoplastic fluids. *SPE 14230*, Soc. Pet. Eng., Dallas, Tex.
Tabeling, P., Libchaber, A. 1986. Film draining and the Saffman-Taylor problem. *Phys. Rev. A* 33: 794–96
Tabeling, P., Zocchi, G., Libchaber, A. 1986. An experimental study of the Saffman-Taylor instability. Submitted for publication
Tan, C.-T., Homsy, G. M. 1986. Stability of miscible displacements in porous media: rectilinear flow. *Phys. Fluids.* In press
Taylor, G. I., Saffman, P. G. 1958. Cavity flows of viscous fluids in narrow spaces. *Symp. Nav. Hydrodyn., 2nd*, pp. 277–91
Tryggvason, G., Aref, H. 1983. Numerical experiments on Hele-Shaw flow with a sharp interface. *J. Fluid Mech.* 136: 1–30

Tryggvason, G., Aref, H. 1985. Finger-interaction mechanisms in stratified Hele-Shaw flow. *J. Fluid Mech.* 154: 287–301
Vanden Broeck, J.-M. 1983. Fingers in a Hele-Shaw cell with surface tension. *Phys. Fluids* 26: 2033–34
van Meurs, P. 1957. The use of transparent three-dimensional models for studying the mechanism of flow processes in oil reservoirs. *Trans. AIME* 210: 295–301
White, I., Colombera, P. M., Philip, J. R. 1976. Experimental study of wetting front instability induced by gradual change of pressure gradient and by heterogeneous porous media. *Soil Sci. Soc. Am. J.* 41: 483–89
Wilson, S. D. R. 1975. A note on the measurement of dynamic contact angles. *J. Colloid Interface Sci.* 51: 532–34
Wooding, R. A. 1962. The stability of an interface between miscible fluids in a porous medium. *ZAMP* 13: 255–66
Wooding, R. A. 1969. Growth of fingers at an unstable diffusing interface in a porous medium or Hele-Shaw cell. *J. Fluid Mech.* 39: 477–95
Wooding, R. A. 1975. Unsaturated seepage flow from a horizontal boundary. *Q. Appl. Math.* 33: 143–60
Wooding, R. A., Morel-Seytoux, H. J. 1976. Multiphase fluid flow through porous media. *Ann. Rev. Fluid Mech.* 8: 233–74
Yortsos, Y. C., Huang, A. B. 1984. Linear stability analysis of immiscible displacement including continuously changing mobility and capillary effects: Part I—simple basic flow profiles. *SPE 12692*, Soc. Pet. Eng., Dallas, Tex.