Linear Stability Analysis of two phase Hele-Shaw Flow

Instructor: Lou Kondic; 451H, Spring 2014

February 2, 2014

Consider the following boundary value problem in the domain D

$$\nabla^2 P = 0 \tag{1}$$

With these boundary conditions:

$$P|_{\Gamma_1} = -\gamma\kappa; \quad P|_{\Gamma_2} = 0, \tag{2}$$

Where Γ_1 is the inner boundary, specified by the radius vector

$$\mathbf{R}_1(\theta, t) = R_1(t)(1 + \epsilon\eta(\theta, t))\hat{\mathbf{r}} \tag{3}$$

and the outer boundary is a circle of radius R_2. Assume that $R_2 \gg R_1$, and that epsilon is a small quantity. Velocity of the fluid everywhere in D is given by the Darcy's law, $\mathbf{u} = -\nabla P$; in particular, this says that the boundaries move with the local velocity of the fluid.

We want to analyze how the perturbation $\eta(\theta, t)$ evolves in time; in particular we consider perturbations of the form $\eta(\theta, t) = N(t)\cos(m\theta)$, where $N(t)$ is the amplitude and m is the wave number. If, for a given m, $N(t)$ grows in time, that particular wave number is unstable; if $N(t)$ decreases, that wave number is stable.

Base problem: Put $\epsilon = 0$. From (1) and (2), we obtain the solution for the base pressure

$$P_0 = -\frac{\gamma}{R_1}\left(1 - \frac{\ln r/R_1}{\ln R_2/R_1}\right) \tag{4}$$

The Darcy's law then gives the base radial velocity

$$u_{0r} = \frac{\gamma}{r} \frac{1}{\ln R_2/R_1} \tag{5}$$

Linear stability analysis: Now, allow for $\epsilon \neq 0$. However, since ϵ is small, neglect all the terms which are proportional to ϵ^n, $n > 1$. Assume that all quantities of interest can be expanded as follows:

$$P = P_0 + \epsilon P_1(r, \theta) \tag{6}$$

$$\mathbf{u} = u_0 \hat{\mathbf{r}} + \epsilon \mathbf{u}_1(r, \theta) \tag{7}$$

$$\kappa = \kappa_0 + \epsilon \kappa_1; \quad \kappa_1 = -\frac{\eta + \eta_{\theta\theta}}{R_1} \tag{8}$$

(κ_1 includes only leading order terms). Since P_0 satisfies Laplace equation, P_1 will satisfy the Laplace equation as well. Therefore, write the solution for P_1 in the form

$$P_1 = F_m \cos m\theta \; r^{-m} \tag{9}$$

where F_m is a constant. Then, the boundary condition requires that

$$(P_0 + \epsilon P_1)|_{R_1(1+\epsilon\eta)} = -\gamma\kappa|_{R_1(1+\epsilon\eta)} = -\gamma \left(\frac{1}{R_1} - \epsilon \frac{\eta + \eta_{\theta\theta}}{R_1} \right) \tag{10}$$

Next, use $\ln(1+\epsilon) \approx \epsilon$ to calculate P_0 from (4), and also $\eta_{\theta\theta} = -m^2\eta$. The result for $P_1(r,\theta)$ is

$$P_1(r, \theta) = \frac{\gamma}{R_1} \left(\frac{1}{\ln R_2/R_1} + (1 - m^2) \right) r^{-m} \eta \tag{11}$$

correct to $O(1)$; this is all what is needed since P_1 is multiplied by ϵ in (10). From the Darcy's law, the result for the radial component of velocity, to $O(\epsilon)$, calculated at $r = R_1(1+\epsilon\eta)$ is then

$$u_r|_{R_1(1+\epsilon\eta)} = \frac{\gamma}{R_1} \frac{1}{\ln R_2/R_1} + \epsilon\gamma \frac{m}{R_1} \left(\frac{1}{\ln R_2/R_1} + (1 - m^2) \right) \eta \tag{12}$$

where (5) was used.

Knowing u_r, we can now analyze the time evolution of the interface. One easy way of doing that is to use the following trick:

$$\frac{d}{dt}\left(\frac{1}{2}\mathbf{R}_1 \cdot \mathbf{R}_1\right) = \mathbf{R}_1 \cdot \frac{d\mathbf{R}_1}{dt} \tag{13}$$

where $\mathbf{R}_1$ is given by (3), and to realize that

$$\mathbf{R}_1 \cdot \mathbf{R}_1 = R_1^2(1 + 2\epsilon\eta) \tag{14}$$

and that $d\mathbf{R}/dt = u_r\hat{\mathbf{r}}$. Next, calculate the time derivative:

$$\frac{d}{dt}\frac{R_1^2}{2}(1 + 2\epsilon\eta) = R_1\dot{R}_1(1 + 2\epsilon\eta) + R_1^2\epsilon\dot{\eta} \tag{15}$$

where $\dot{R}_1 = dR_1/dt = u_{0r}(r = R_1)$. For simplicity, put now $R_1 = 1$, so that $\dot{R}_1 = \gamma/\ln R_2$ to reduce (13) to this equation:

$$\dot{\eta} = \dot{R}_1\left[-1 + m\left(1 + \frac{\gamma}{\dot{R}_1}(1 - m^2)\right)\right]\eta \tag{16}$$

Assume now that the quantity in the square brackets is a constant, call it σ. The solution of the last equation is then

$$\eta(m, t) = \eta(m, t = 0)e^{\sigma t} \tag{17}$$

The quantity σ is often called growth rate, since it determines the "growth" of the perturbation η. If $\sigma > 0$, η grows, and the perturbation is unstable; otherwise, it is stable. Stability obviously depends on the wave number of the perturbation, with the perturbations characterized by large m being stable (due to the term proportional to $-m^3$). This is what we expect due to known stabilizing effect of surface tension.

This result expresses the competition between the destabilizing effect of viscosity contrast (destabilizing), and the surface tension (stabilizing). [Recall that dimensional form of Darcy's law includes viscosity, as well as the gap separation].