

Saffman-Taylor Instability

**Math-451H
Dr. Kondic**

Motivation:

Saffman-Taylor instability has real world applications in understanding and improving the recovery of gases and oil from the earth. I

Experimental:

Two large plates of plastic were used as a rudimentary Helle-Shaw cell. The plates were separated using small pieces of transparency and a hole was drilled in the middle for injection purposes. The cell was filled with glycerol and then colored water or air was injected resulting in fingering.

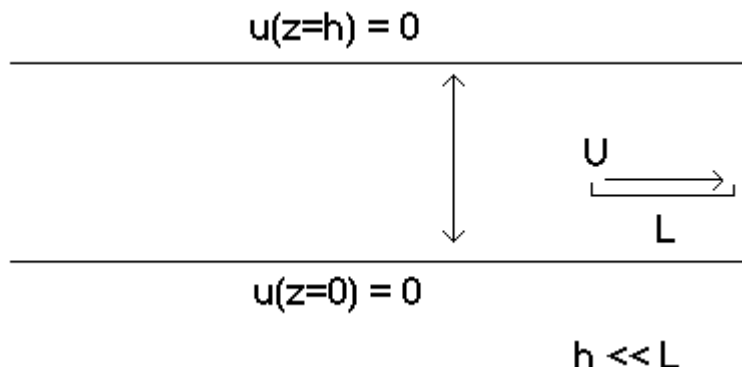


Derivation of Darcy's Law

Start with Navier-Stokes equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \frac{-1}{\rho} \nabla P + \nu \cdot \nabla^2 u + g$$

We are looking at flow in a Helle-Shaw cell and assume no-slip condition and uniform flow.



Because it is a uniform flow we can reduce the equation to:

$$(u \cdot \nabla)u = \frac{-1}{\rho} \nabla P + \nu \cdot \nabla^2 u + g$$

Because we are in a Helle-Shaw cell and h is so small we look at it as a 2D problem we can neglect gravity

$$(u \cdot \nabla)u = -\frac{1}{\rho} \nabla P + \nu \cdot \nabla^2 u$$

Due to no-slip condition

$$\nu \cdot \nabla^2 u = \nu \cdot \left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} \sim O\left(\frac{U}{L^2}\right)$$

$$\frac{\partial^2 u}{\partial y^2} \sim O\left(\frac{U}{L^2}\right)$$

$$\frac{\partial^2 u}{\partial z^2} \sim O\left(\frac{U}{h^2}\right)$$

since $h \ll L$ we can throw out the x-direction and y-direction components

$$\nu \cdot \nabla^2 u = \nu \cdot \frac{\partial^2 u}{\partial z^2}$$

the equation then becomes:

$$(u \cdot \nabla)u = -\frac{1}{\rho} \nabla P + \nu \cdot \frac{\partial^2 u}{\partial z^2}$$

then the left hand side can be reduced

$$u = \langle u, v, w \rangle$$

$$(u \cdot \nabla)u = u \frac{\partial P}{\partial x} + v \frac{\partial Q}{\partial y} + w \frac{\partial R}{\partial z}$$

$$(u \cdot \nabla)u \sim O\left(\frac{U^2}{L}\right)$$

$$\nu \cdot \frac{\partial^2 u}{\partial z^2} \sim O\left(\nu \frac{U}{h^2}\right)$$

$$\frac{\frac{U^2}{L}}{\frac{\nu \cdot U}{h^2}} = \frac{U^2}{L} \cdot \frac{h^2}{\nu \cdot U} = \frac{U^2}{L} \cdot \frac{h^2}{\nu \cdot U} \cdot \left(\frac{L}{L}\right) = \frac{UL}{\nu} \cdot \left(\frac{h}{L}\right)^2$$

Experimentally this is approximately

$$\frac{UL}{\nu} \cdot \left(\frac{h}{L}\right)^2 = \frac{5cm/s \cdot 1cm}{13} \left(\frac{0.05cm}{0.1cm}\right)^2 = 0.09615$$

0.001717 << 1 so it would appear this is a reasonable step

Note: the Reynolds number of the glycerol is R = 0.38461

If $\frac{UL}{\nu} \cdot \left(\frac{h}{L}\right) \ll 1$ then we can neglect $(u \cdot \nabla)u$

This is because if we divide the order of the term in question by the order of a significant term if we get something small like $\frac{UL}{\nu} \cdot \left(\frac{h}{L}\right) \ll 1$ then that term can be neglected. The equation reduces to:

$$0 = \frac{-1}{\rho} \nabla P + \nu \cdot \frac{\partial^2 u}{\partial z^2}$$

Then add the first term to the other side and then multiply both sides by ρ , then $\mu = \rho \nu$

$$\nabla P = \mu \cdot \frac{\partial^2 u}{\partial z^2}$$

Grad P can and u can now be separated into its components to form the following equations

$$\frac{\partial P}{\partial x} = \mu \cdot \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial P}{\partial y} = \mu \cdot \frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial P}{\partial z} = \mu \cdot \frac{\partial^2 w}{\partial z^2}$$

Since change in pressure in the z direction is significantly small than in the x and y direction so we can approximate P with just the x and y equations. We can then solve them by just integrating twice with respect to z to get:

$$u = \frac{-1}{2\mu} \cdot \frac{\partial P}{\partial x} \cdot z(h-z)$$

$$v = \frac{-1}{2\mu} \cdot \frac{\partial P}{\partial y} \cdot z(h-z)$$

Then put u and Ps components back together

$$u = \frac{-\nabla P}{2\mu} \cdot z(h-z)$$

Now take the average value

$$u = \frac{1}{h} \int_0^h \frac{-\nabla P}{2\mu} \cdot z(h-z) dz$$

$$u = \frac{-\nabla P}{2h\mu} \int_0^h z(h-z) dz$$

$$u = \frac{-\nabla P}{2h\mu} \int_0^h hz - z^2 dz$$

$$u = \frac{-\nabla P}{2h\mu} \left(\frac{hz^2}{2} - \frac{z^3}{3} \right)_0^h$$

$$u = \frac{-\nabla P}{2\mu} \left(\frac{z^2}{2} - \frac{z^3}{3h} \right)_0^h$$

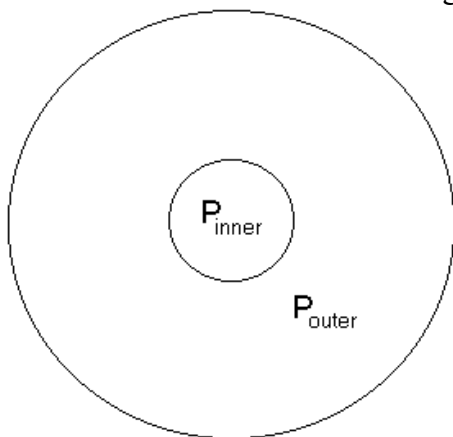
$$u = \frac{-\nabla P}{2\mu} \left(\frac{3h^2 - 2h^2}{6} \right) = \frac{-\nabla P}{2\mu} \cdot \frac{h^2}{6}$$

Darcy's Law

$$u = \frac{-h^2}{12\mu} \cdot \nabla P$$

Laplace-Young Boundary Conditions:

Here we assume that the inner pressure being greater than the outer pressure. If this were not the case the interface would not grow.



Laplace-Young boundary conditions are that $P_{\text{inner}} - P_{\text{outer}} = \gamma\kappa$

Where,

γ is surface tension

κ is curvature

Saffman-Taylor Instability:

Saffman-Taylor instability is what we call the instability that arises when you displace a more viscous fluid by a less viscous fluid. Experimentally we injected water into glycerol and air into glycerol in order to see the effects of Saffman-Taylor instability on the interface between the two fluids which resulted in interesting fingering patterns as seen below:



Nondimensionalize Equations:

Darcy's Law

$$u = \frac{-h^2}{12\mu} \cdot \nabla P$$

$$u' = \frac{u}{U}$$

$$\nabla P' = \frac{\nabla P}{\frac{12\mu U}{h^2}}$$

$$u' = -\nabla P'$$

where U is the typical flow velocity

Capillary Number:

Capillary number describes the ratio of the destabilizing viscous force to the surface tension and is calculated by using the following formula:

$$Ca = \frac{12\mu_0 \dot{R}_0 R_0^2}{(\gamma h^2)}$$

where

μ_0 - density times viscosity

\dot{R}_0 - velocity of inner boundary

R_0 - radius of inner boundary

γ - surface tension

h - gap width between plates

Find units of Capillary number:

$$\frac{g^2 \cdot \text{sec} \cdot \frac{\text{cm}}{\text{sec}} \cdot \text{cm}^2}{\frac{\text{dynes}}{\text{cm}} \cdot \text{cm}^2} = \frac{g^2 \cdot \text{cm}^2}{\text{dynes}} = \frac{g^2 \cdot \text{cm}^2}{\text{cm} \cdot g \cdot \text{sec}} = \frac{g \cdot \text{cm}}{\text{sec}}$$

The following is for an air and water scenario:

$$\mu_0 = 1 \text{ g/cm} \cdot 1 \text{ cm-g-sec} = 1 \text{ g}^2\text{-sec}$$

$$\dot{R}_0 = 5 \text{ cm/sec}$$

$$R_0 = 6.5 \text{ cm}$$

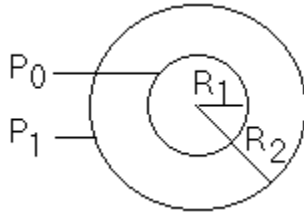
$$\gamma = 72.8 \text{ dynes/cm}$$

$$h = 0.05 \text{ cm}$$

$$Ca = \frac{12 \cdot 1 \cdot 5 \cdot 6.5^2}{(72.8 \cdot 0.05^2)} = 2,142.86 \frac{\text{g} \cdot \text{cm}}{\text{sec}}$$

Circular Boundary Problem:

The following solves for pressure in a circular boundary problem as shown:



$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial P}{\partial r} = 0 \quad \text{B.C.'s} \quad P(R_0) = P_0 \quad P(R_1) = P_1$$

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \cdot \frac{\partial P}{\partial r} \right) = 0$$

$$\int \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \cdot \frac{\partial P}{\partial r} \right) = \int 0$$

$$r \cdot \frac{\partial P}{\partial r} = c$$

$$\frac{\partial P}{\partial r} = \frac{c}{r}$$

$$\int \frac{\partial P}{\partial r} = \int \frac{c}{r}$$

$$P = c \cdot \ln r + c_1$$

$$P(R_0) = c \cdot \ln R_0 + c_1 = P_0$$

$$P(R_1) = c \cdot \ln R_1 + c_1 = 0$$

$$c = \frac{P_0}{\ln R_1 / R_0}$$

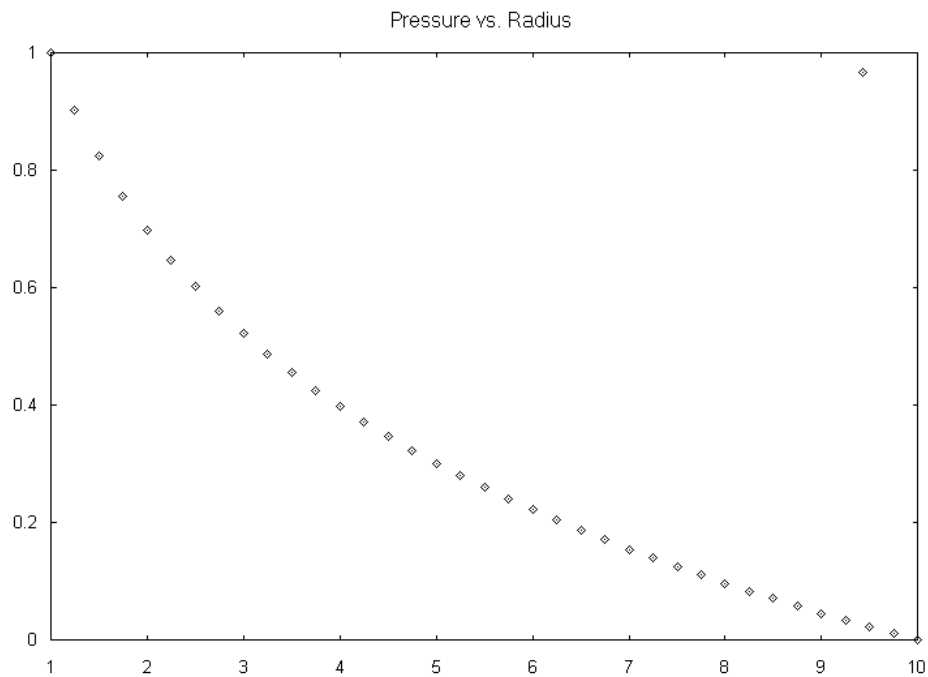
$$c_1 = P_0$$

$$P(r) = P_0 \left(1 - \frac{\ln r / R_0}{\ln R_1 / R_0} \right)$$

The below graph uses the above solution with:

$$P_0 = 1 \quad R_0 = 1$$

$$R_1 = 10$$



Linear Stability:

If everything were perfect then the interface between a more viscous fluid being displaced by a less viscous fluid would grow as a circle. However, in the real world things are not perfect. Experimentally as the interface grows small disturbances occur due to dust, inhomogeneities in the fluid, imperfections in the plates, etc. and because the interface is unstable these perturbations grow and form the fingering patterns that are seen. In order to theoretically look at this we have to mathematically introduce perturbations. We can do this by perturbing the initial inner boundary. Then we want to find out how these perturbations evolve; when do they grow and when do they heal. All of this though is only valid near the initial time. At the very beginning the interface grows as a circle because its circumference is smaller than the critical wavelength for fingering to occur; this will later be derived

$$\nabla^2 P = 0$$

$$P(R_1) = -\gamma\kappa$$

$$P(R_2) = 0$$

Let the inner boundary be

$$R_1(\theta, t) = R_1(t) (1 - \epsilon \eta(\theta, t))$$

where

$$\eta(\theta, t) = N(t) \cos(m \theta)$$

$N(t)$ is amplitude

m is wave number

and $\epsilon \ll 1$

Base Problem:

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial P}{\partial r} = 0$$

$$P(R_1) = -\gamma\kappa$$

$$P(R_2) = 0 \quad \text{Here } \kappa \text{ or curvature is } 1/R_1$$

Let $\varepsilon=0$

$$R_1(\theta, t) = R_1(t)$$

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \cdot \frac{\partial P}{\partial r} \right) = 0$$

$$r \cdot \frac{\partial P}{\partial r} = c$$

$$\frac{\partial P}{\partial r} = \frac{c}{r}$$

$$P = c \cdot \ln r + c_1$$

$$P(R_1) = c \cdot \ln R_1 + c_1 = -\gamma\kappa$$

$$P(R_2) = c \cdot \ln R_2 + c_1 = 0$$

$$c = \frac{-\gamma}{R_1 \ln R_1 / R_0}$$

$$c_1 = \frac{-\gamma}{R_1}$$

$$P(r) = \frac{-\gamma}{R_1} \left(1 - \frac{\ln r / R_1}{\ln R_2 / R_1} \right)$$

Now apply an approximate form of Darcy's Law to get the radial velocity

$$u = -\nabla P$$

Find grad P

$$u_{0r} = \frac{\partial}{\partial r} \left(\frac{-\gamma}{R_1} \left(1 - \frac{\ln r / R_1}{\ln R_2 / R_1} \right) \right) = \frac{-\gamma}{R_1 \ln R_2 / R_1} \cdot \frac{R_1}{r} = \frac{-\gamma}{\ln R_2 / R_1} \cdot \frac{1}{r}$$

Linear Stability Analysis:

Now let $\varepsilon \neq 0$

We look at the terms where powers of ε are not greater than 1.

$$P = P_0 + \varepsilon P_1(r, \theta)$$

$$u = u_0 + \varepsilon u_1(r, \theta)$$

$$\kappa = \kappa_0 + \varepsilon \kappa_1$$

$$\kappa_0 = \frac{1}{R_1}$$

$$\kappa_0 = \frac{\eta + \eta_{\theta\theta}}{R_1}$$

Assume solution form from Fourier series for P_1 , which is a good assumption since it satisfies Laplace equation.

$$P_1 = F_m \cos(m\theta) r^{-m}$$

B.C. become

$$(P_0 + \varepsilon P_1) \big|_{R_1} (1 + \varepsilon \eta) = -\gamma \kappa \big|_{R_1} (1 + \varepsilon \eta) = -\gamma \left(\frac{1}{R_1} + \varepsilon \frac{\eta + \eta_{\theta\theta}}{R_1} \right)$$

Applying B.C's to the assumed solution form of P_1 since $\ln(1+\varepsilon) \approx \varepsilon$ and $\eta_{\theta\theta} = -m^2 \eta$ and results in

$$P_1(r, \theta) = \frac{\gamma}{R_1} \left(\frac{1}{\ln(R_1 / R_2)} + (1 - m^2) \right) r^{-m} \eta$$

Apply Darcy's law and evaluate at $R_1(1+\epsilon\eta)$

$$u_r = \frac{-\gamma}{R_1 \ln R_2 / R_1} + \epsilon \gamma \frac{m}{R_1} \left(\frac{1}{\ln R_2 / R_1} + (1 - m^2) \right) \eta$$

We want to know how the interface evolves and we know u_r , we can use this derivative of products trick that can be used

$$\frac{\partial}{\partial t} \left(\frac{1}{2} R_1 \cdot R_1 \right) = R_1 \cdot \frac{\partial R_1}{\partial t}$$

Now manipulate actual equation for R_1 so we can use the above

$$R_1 \cdot R_1 = R_1(t)(1 - \epsilon\eta) \cdot R_1(t)(1 - \epsilon\eta) = (R_1(t) - R_1(t)\epsilon\eta)(R_1(t) - R_1(t)\epsilon\eta)$$

$$R_1^2 - R_1^2 \epsilon\eta - R_1^2 \epsilon\eta + R_1^2 \epsilon^2 \eta^2$$

We can throw away the last term cause its ϵ of order higher than 1 and we get

$$R_1 \cdot R_1 = R_1^2 (1 - 2\epsilon\eta)$$

$$\frac{\partial R}{\partial t} = u_r \hat{r}$$

Now put it all together

$$\frac{\partial}{\partial t} \left(\frac{R_1^2}{2} (1 + 2\epsilon\eta) \right) = R_1 \cdot \dot{R}_1 (1 + 2\epsilon\eta) + R_1^2 \epsilon \dot{\eta}$$

$$\dot{R}_1 = \frac{\partial R}{\partial t} = u_r$$

To make things simpler let $R_1 = 1$ and so $\dot{R}_1 = \frac{\gamma}{\ln(R_2)}$

We then can get

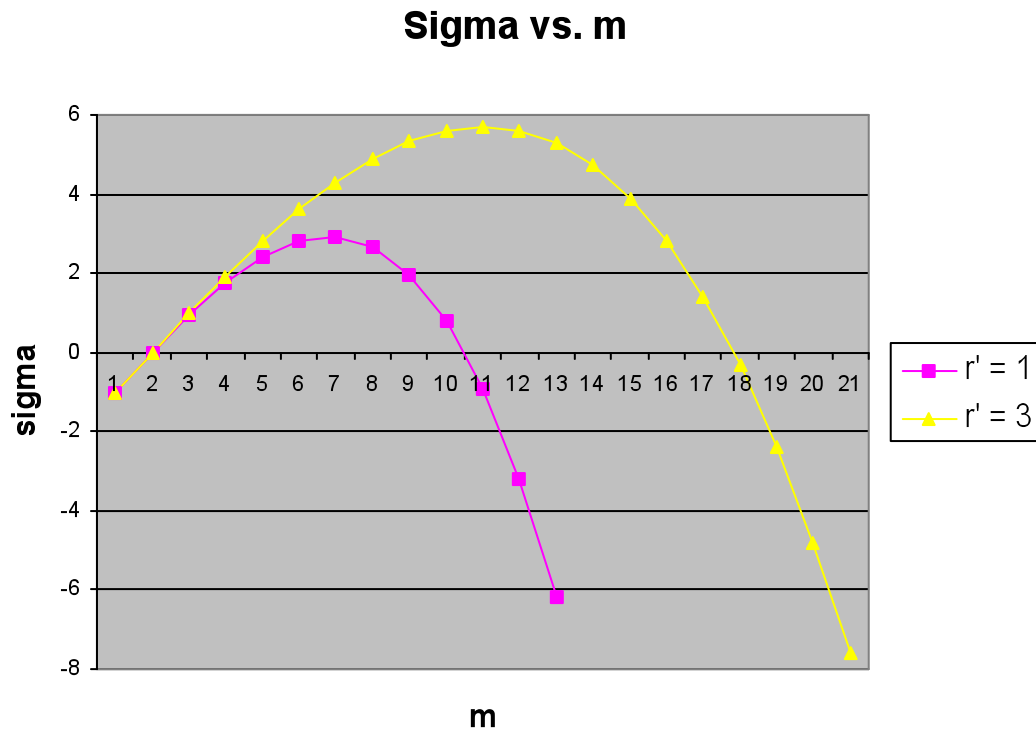
$$\dot{\eta} = \dot{R}_1 \left[-1 + m \left(1 + \frac{\gamma}{\dot{R}_1} (1 - m^2) \right) \right] \eta$$

Assume the quantity in the square brackets is constant σ

$$\eta(m, t) = \eta(m, t = 0) e^{\sigma t}$$

We can see that if $\sigma > 0$ then the perturbation will grow exponentially. However if $\sigma < 0$ then the perturbation will heal back into the interface since it would then exponentially decay. By looking at σ we can see what quantities stability is dependent on: surface tension, wave number, and velocity of inner boundary. σ is known as the growth rate. There is a critical wave number where the growth is maximal.

Lets look at how σ and the critical wave number depend on velocity



This graph makes sense when we look at the experiments. If we inject the water quickly the fingers are smaller, that's because it has a larger wave number and the finger width is inversely proportional to wave number. If we inject the water slowly the fingers are thicker because they have a smaller critical wave number.

Lets find the critical wavelength from σ . First take its derivative with respect to m and then set it to zero. Then replace m with $1/\lambda_c$ where λ_c is wavelength.

$$\frac{\partial}{\partial m} \left[-1 + m \left(1 + \frac{\gamma}{\dot{R}_1} (1 - m^2) \right) \right] = 1 + \frac{\gamma}{\dot{R}_1} - 3m^2 = 0$$

$$m = \frac{1}{\lambda_c}$$

$$1 + \frac{\gamma}{\dot{R}_1} = 3 \left(\frac{1}{\lambda_c} \right)^2$$

$$\sqrt{\frac{1 + \frac{\gamma}{\dot{R}_1}}{3}} = \frac{1}{\lambda_c}$$

$$\lambda_c = \frac{\sqrt{3}}{\sqrt{\frac{\gamma}{\dot{R}_1} + 1}}$$

Experimentally this was approximately

$$\lambda_c = \frac{\sqrt{3}}{\sqrt{\frac{72.8}{5} + 1}} = 4.4mm \quad \text{which seems resonable}$$

Numerical Simulation

We use the following procedure to simulate a bubble expanding into a viscous incompressible fluid:

1. Define the two initial boundaries
2. Discretize: Find all the points inside the domain (which is formed between the two boundaries) that we want to solve for.
3. Solve Laplace's equation for pressure inside the domain with Laplace-Young BCs using finite-differences method
 - a. Use two point differencing to calculate curvature in the boundary condition
 - b. Use Gauss-Seidel to iteratively solve the linear system.
4. Update boundaries according to Darcy's law
 - a. Calculate gradient of pressure at points inside the domain using two-sided differences
 - b. Extrapolate at inner and outer boundaries to find gradient of pressure at the two boundaries
 - c. Use Runge-Kutta to calculate new coordinates
5. Continue time evolution by going back to step 2.

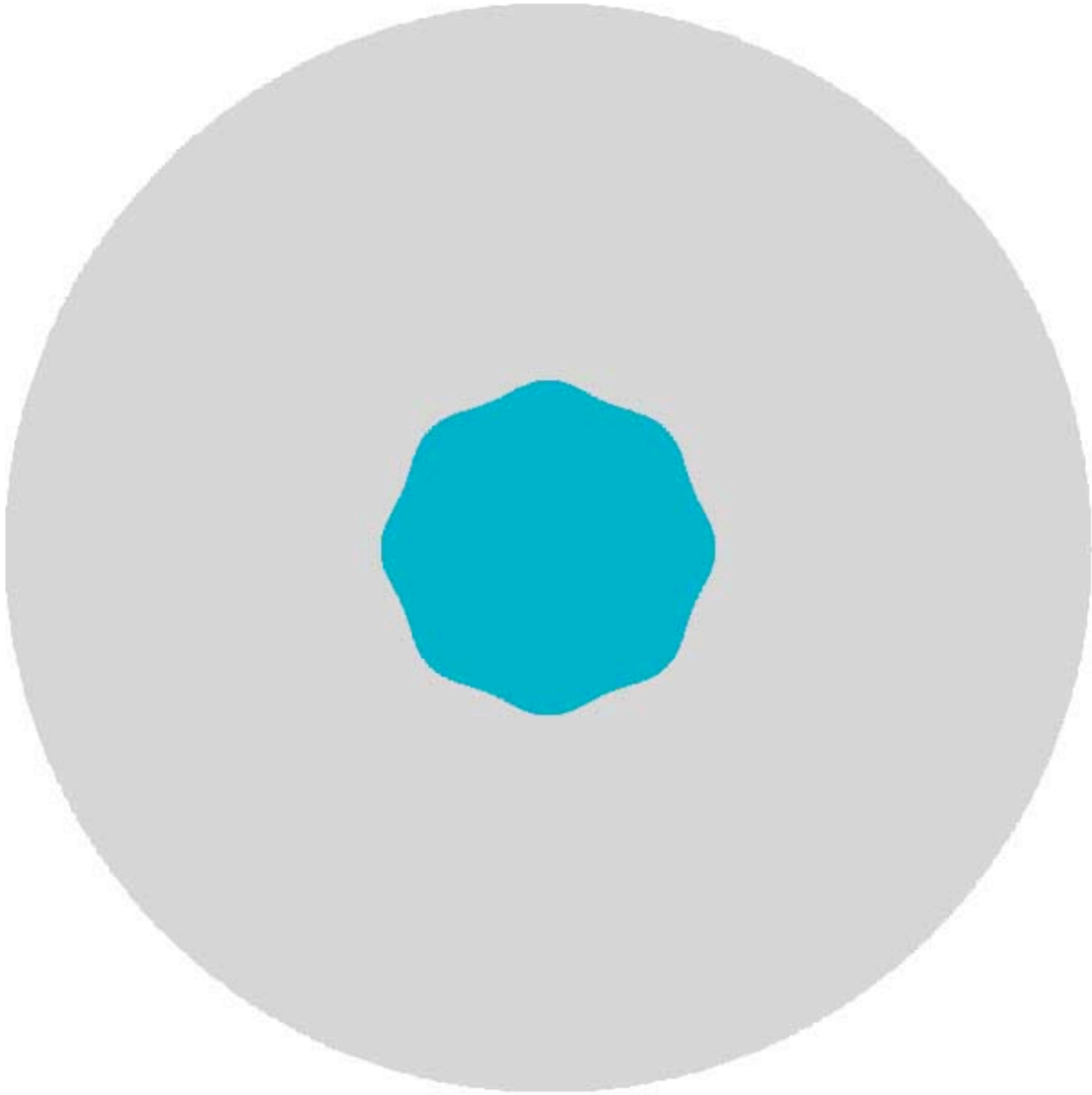
Initial Boundaries

The initial inner and outer boundaries are defined by the following polar equation:

$$R(\theta) = a + \varepsilon \cos(m\theta)$$

with different parameters a , ε , and m . Note that we could have used different shapes for boundaries than ones described by the above as long as the domain formed is closed and the two boundaries do not cross. The above was also chosen to compare results with linear stability analysis.

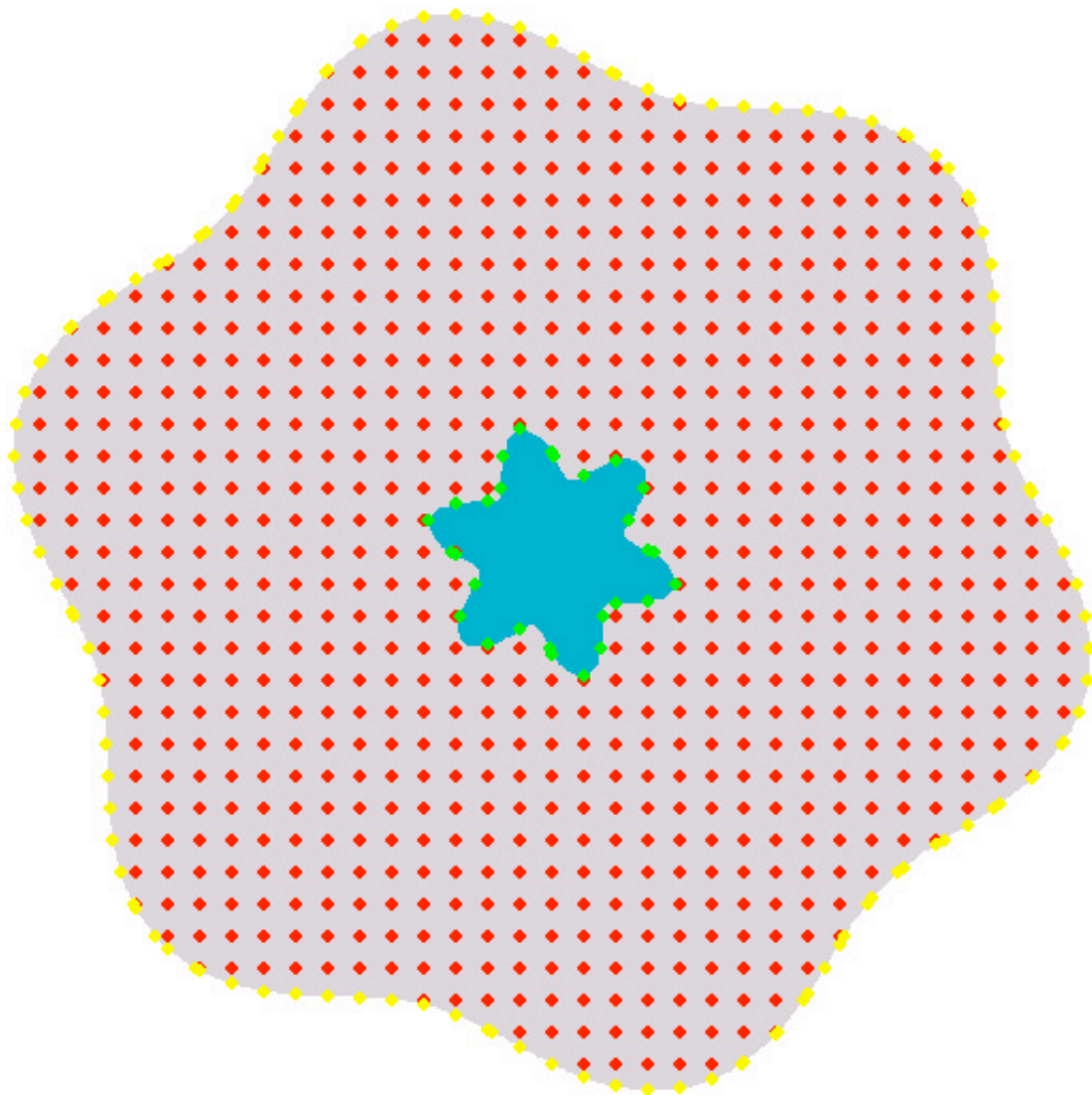
Figure 1: Sample Boundary. Outer boundary has $a = 10$ and $\varepsilon = 0$. Inner boundary has $a=3$, $\varepsilon = 0.1$, $m=8$



Finding Grid Points

Figure 1 shows points used in discretization. The red colored dots represent points inside the domain. We'll solve pressure at each of these points. The green colored dots are points on the inner boundary that are needed when using finite differences. Similarly, the yellow colored dots are points on the outer boundary that are used when solving Laplace's equation. Note that those points are not (the only) ones used for advancing the boundary.

Figure 3: Grid Points



Finite-Differences

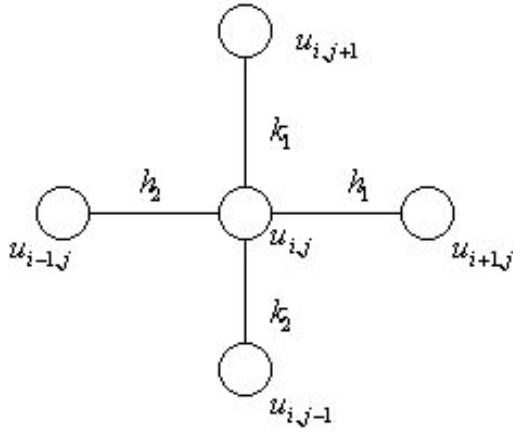
Laplace's equation was solved in Cartesian coordinates. Since the spacing can be unequal near the curved boundaries, we need to modify our approximations to the derivatives.

Alternating notation and plugging it to a two-sided difference formula we get the following approximation to the second derivative.

$$u_{xx}(x, y) \cong \frac{2(-u_{i,j}(h_2 + h_1) + h_2 u_{i+1,j} + h_1 u_{i-1,j})}{h_1 h_2 (h_1 + h_2)}$$

where point $u_{i,j}$ is located as in the following figure:

Figure 2: Point Orientation



Formula for the second partial with respect to y is analogous.

Plugging the approximations to the second derivative with respect to x and y and solving for $u_{i,j}$ we get the recursive formula for pressure at point $u_{i,j}$:

$$u_{ij} = \frac{k_1 k_2 (k_1 + k_2) (h_2 u_{i+1,j} + h_1 u_{i-1,j}) + h_1 h_2 (h_1 + h_2) (k_2 u_{i,j+1} + k_1 u_{i,j-1})}{(k_1 k_2 + h_1 h_2) (h_1 + h_2) (k_1 + k_2)}$$

Calculating Velocity of Boundary points

In order to find the velocity at the two boundaries we first calculated the velocities at the grid points (those are easy to calculate since we always know the pressure to the left, right, up, and bottom of that point). We then extrapolated in two dimensions for each point on the two boundaries using Lagrange polynomial. Only local points were used in extrapolation since using all the grid points can lead to large errors.

Results:

To verify our calculations we initially made the domain annular. Our numerical calculations for pressures, boundary condition, the partial derivatives inside the domain as well as on the boundaries were very close to the values obtained with the analytical

solutions. We thus assumed our methods in steps 2 through 4 were correct. However for larger time the results of the program runs do not create real fingers. The reason we think that only the initial evolution is correct, is since more points are needed in between the boundaries to simulate an actual growth further on. But such a conduct is impossible when tried on a computer, since it would take too long to run with a small enough time step. Because of that we think calculations of the velocities at the boundaries using extrapolation of the gradient of pressure is incorrect.

