Saffman-Taylor Instability

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What is it?

- When a more viscous fluid is displaced by a less viscous fluid Saffman-Taylor instability occurs along the interface of the two fluids.
- If you do it the other way around nothing interesting happens because it is stable.

Basic Idea

- Inject air or dyed water into a makeshift Hele-Shaw cell which contains glycerol at a constant pressure.
- Due to ST instability fingering will occur

- Look at this theoretically and try to simulate numerically

Theory

- If everything was perfect even with the instability the fingering would not arise
- Fingering occurs due to the instability and the introduction of small perturbations along the interface of the two fluids.
- In stable situations these perturbations would heal back into the interface rather than finger

Fingering

- Initially the interface is circle centered at the injection point.
- As the less viscous fluid is injected a circle grows at first because surface tension prevents fingering.
- While circumference of the interface is less than the critical wavelength then fingering does not occur.

- After the circumference grows beyond the critical wavelength, fingers start to form due to perturbations which occur from the fluid not being homogeneous throughout.
- We can look at this instability and perturbations using linear stability model, but first we need Darcy’s Law.

Darcy's Law

- Darcy's Law can be derived from Navier-Stokes equation

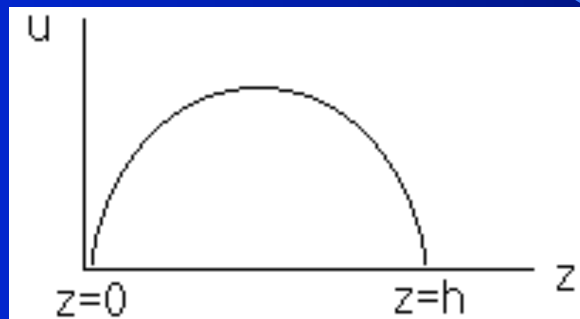
$$\frac{\partial u}{\partial t} + (u \bullet \nabla)u = \frac{-1}{\rho} \cdot \nabla P + \nu \nabla^2 + g$$

Assume uniform flow

Gravity neglected in Hele-Shaw Cell

$$(u \bullet \nabla)u = \frac{-1}{\rho} \cdot \nabla P + \nu \nabla^2$$

- Assume no-slip condition
  - U(z=0) = 0   U(z=h) = 0



Then greatest change in velocity is with respect to the Z direction.

[illegible] [illegible]^{2}u [illegible] [illegible] d^2u/dz^2

After additional simplifying assumptions

$$0 = \frac{-1}{\rho} \cdot \nabla P + \nu \cdot \frac{\partial^2 u}{\partial z^2}$$

- We then solve the resulting equations by direct integration and get
- $$u = \frac{-1}{2\mu} \cdot \frac{\partial P}{\partial x} Z(h - Z)$$
$$v = \frac{-1}{2\mu} \cdot \frac{\partial P}{\partial y} Z(h - Z)$$

- Then after taking the average value of the function we get Darcy's Law

$$U = \frac{-\nabla P}{2\mu h}\int_0^h hz - z^2 \, dz$$

$$U = \frac{-h^2}{12\mu}\nabla P$$

Linear Stability

- □ is surface tension and □ is curvature
- Laplace-Young BC are
  - The inner domain pressure is greater than the outer domain pressure
  - Pressure jump from Inner Boundary to outer is

    $P_{inner} - P_{outer} = □□$

The inner boundary is defined by the following function

$R_1(□,t) = R_1(t)\ (1 - □□(□,t)\ )$

N(t)   is the amplitude

m   is the wave number

$□(□,t) = N(t)\cos(m\ □)$

- As we increase the value of [illegible] the perturbations more significant.
- Notice if we set [illegible] = 0 the inner boundary reduces to a circle and which would just stably expand. So when [illegible][illegible]0 we have instability.
- We are interesting in determining if [illegible]([illegible],t) will grow into finger or heal back into the interface.

We need to know how $\square(\square,t)$ evolves, so after some work we arrive at $\square'(\square,t)$

$$\square'(\square,t) = R'_1 \, [\, -1 + m\,(1 + (\square / R'_1)\,(1 - m^2)\,)\,]\, \square$$

$$\square = [\, -1 + m\,(1 + (\square / R'_1)\,(1 - m^2)\,)\,]$$

and is assumed to be constant

$$\square(m,t) = \square(m,0)\, e^{\square t}$$

$$(m,t) = \quad (m,0)\; e^{\quad t}$$

This tells us what will happen to the perturbation

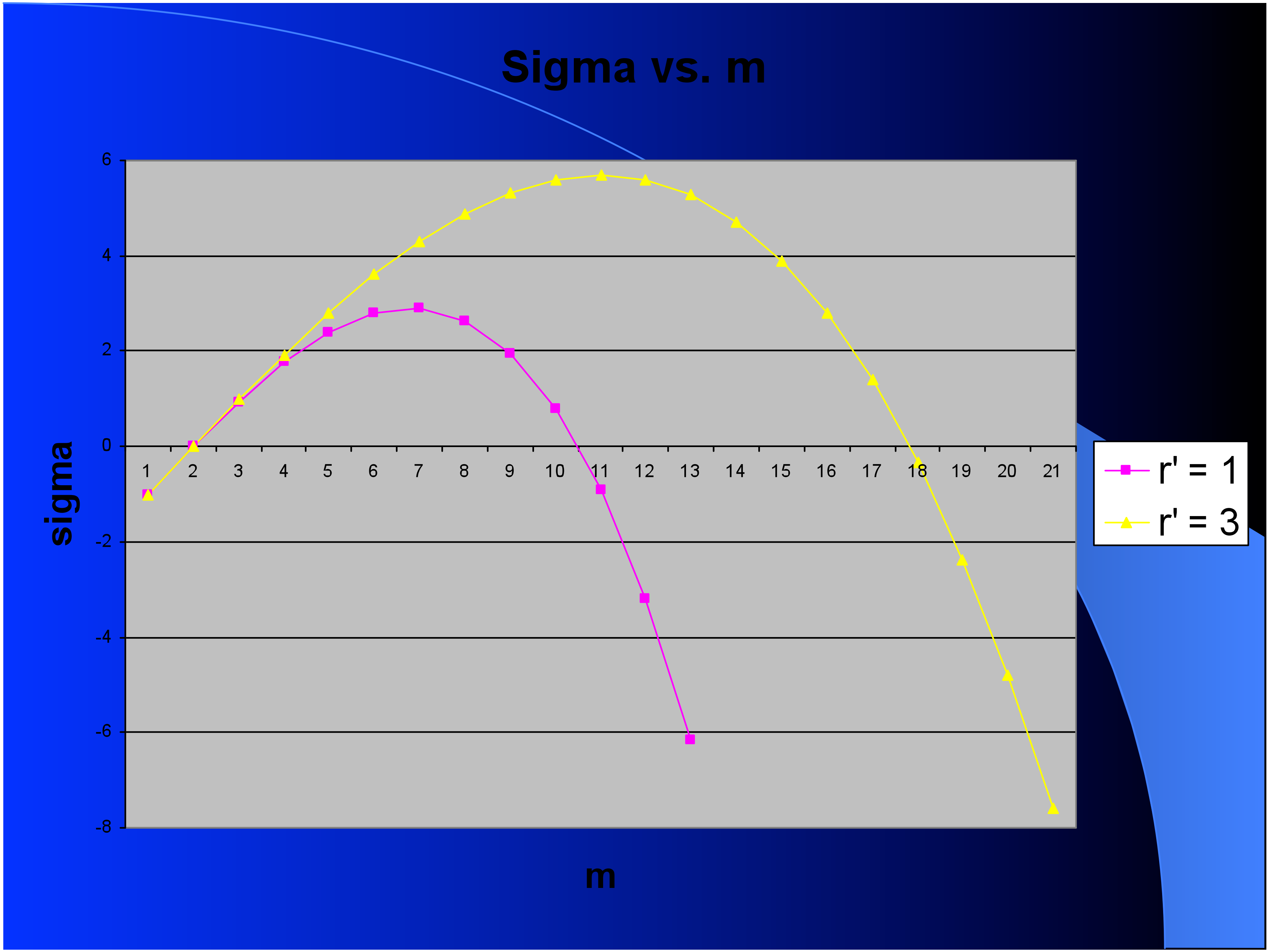
< 0 Heals

> 0 Grows

$$= [\,-1 + m\,(1 + (\quad / R'_1)\,(1 - m^2)\,)\,]$$

- Looking at tells us what stability is dependent on

  1. m – wave number
  2. - surface tension
  3. R'_1 – velocity of the inner boundary



Sigma vs. m
sigma
m
6
4
2
0
-2
-4
-6
-8
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
r' = 1
r' = 3

Fingering related to Injection P

$$\lambda_m = \frac{2\sqrt{3}\,\pi R}{\left(\dfrac{QR}{2\pi M_2 \sigma}+1\right)^{1/2}}$$

Q is rate at which water is injected and is inversely proportional to finger width

Fast injection – small finger width

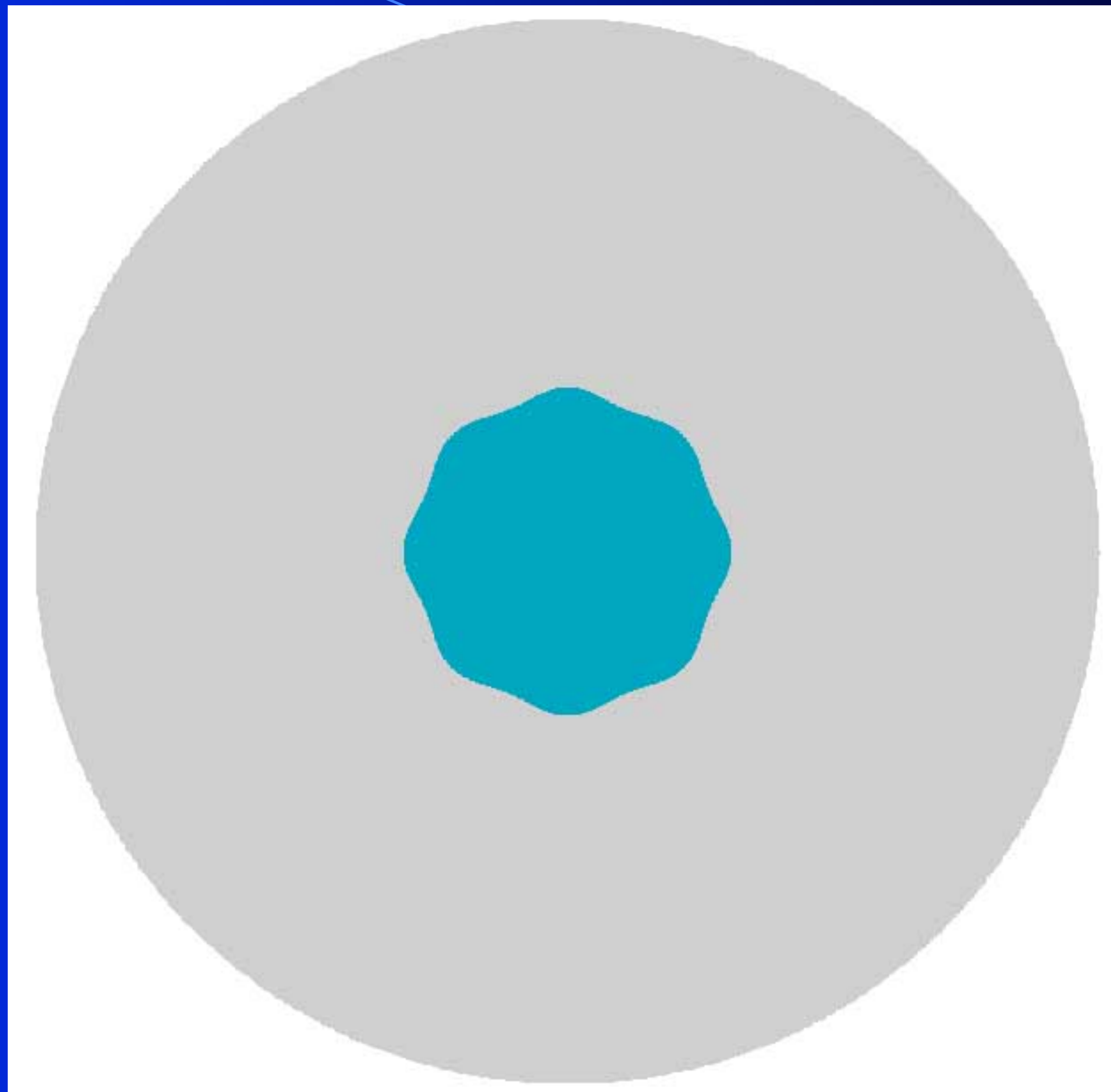
Slow injection – larger finger with

Numerical Simulation

Numerical Simuation

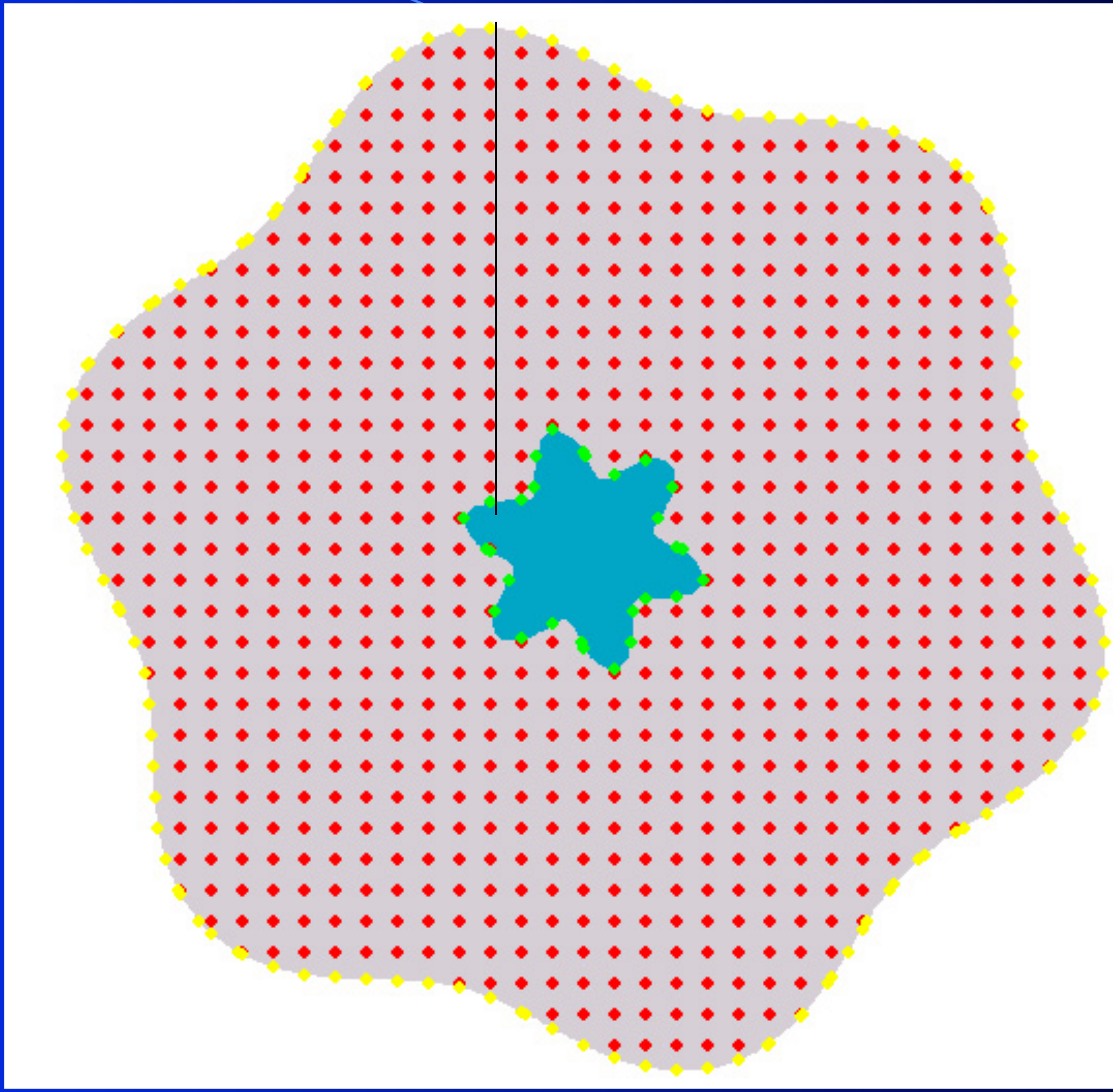
- Procedure for Simulating Bubble Expanding Into a viscous incompressible fluid
- Step 1.
  - Define Initial Boundaries:
    - Inner boundary
    - Outer boundary

$$R(\theta) = a + \varepsilon \cos(m\theta)$$



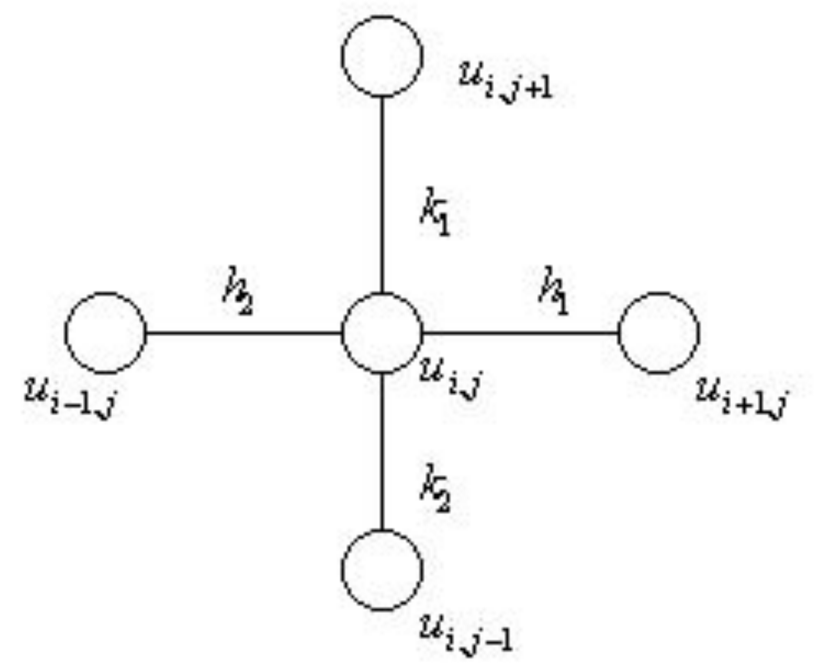
Step 2

- Discretenize
  - Find all the points inside the domain (which is formed between two boundaries) that we want to solve* for
    - *Find pressures



Step 3

- Solve Laplaces Equation for pressure at the grid points using finite-difference method with Laplace-Young BCs.
  - Use two point differencing to calculate curvature in the boundary condition
  - Use Gauss-Seidel to iteratively solve that linear system



$$u_{ij} = \frac{k_1 k_2 (k_1 + k_2)(h_2 u_{i+1,j} + h_1 u_{i-1,j}) + h_1 h_2 (h_1 + h_2)(k_2 u_{i,j+1} + k_1 u_{i,j-1})}{(k_1 k_2 + h_1 h_2)(h_1 + h_2)(k_1 + k_2)}$$

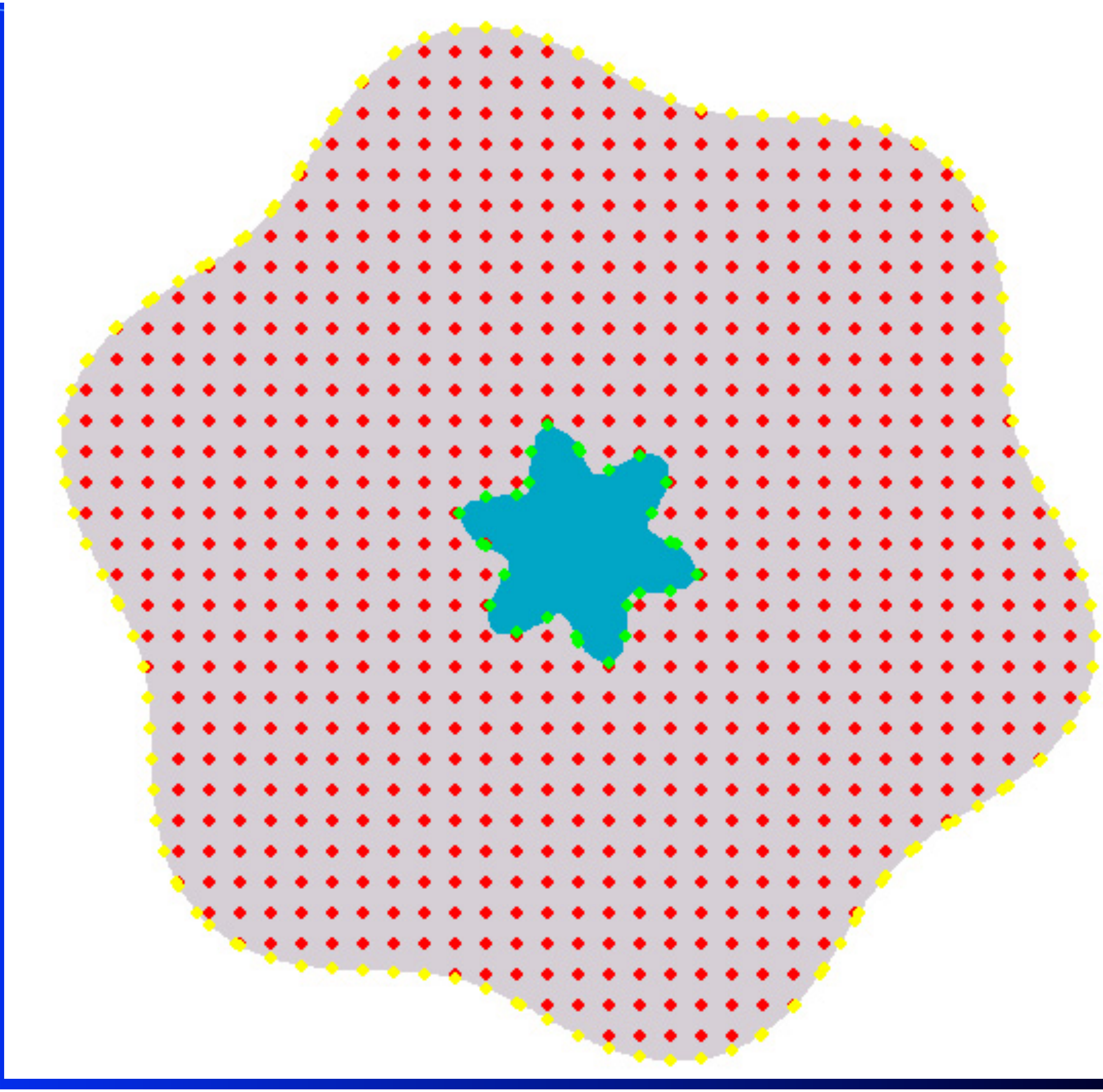
Step 4: Update boundaries according to Darcy's Law

Calculate gradient of pressure at points inside the domain using two-sided differences

– Extrapolate in two dimensions using Lagrange Polynomial at inner and outer boundaries to find gradient of pressure at the two boundaries (Only local points are used)

– Use Forward Euler to calculate new coordinates

$$X_n = X_{n-1} + (-M) * P_x * \text{timestep}$$



Step 5

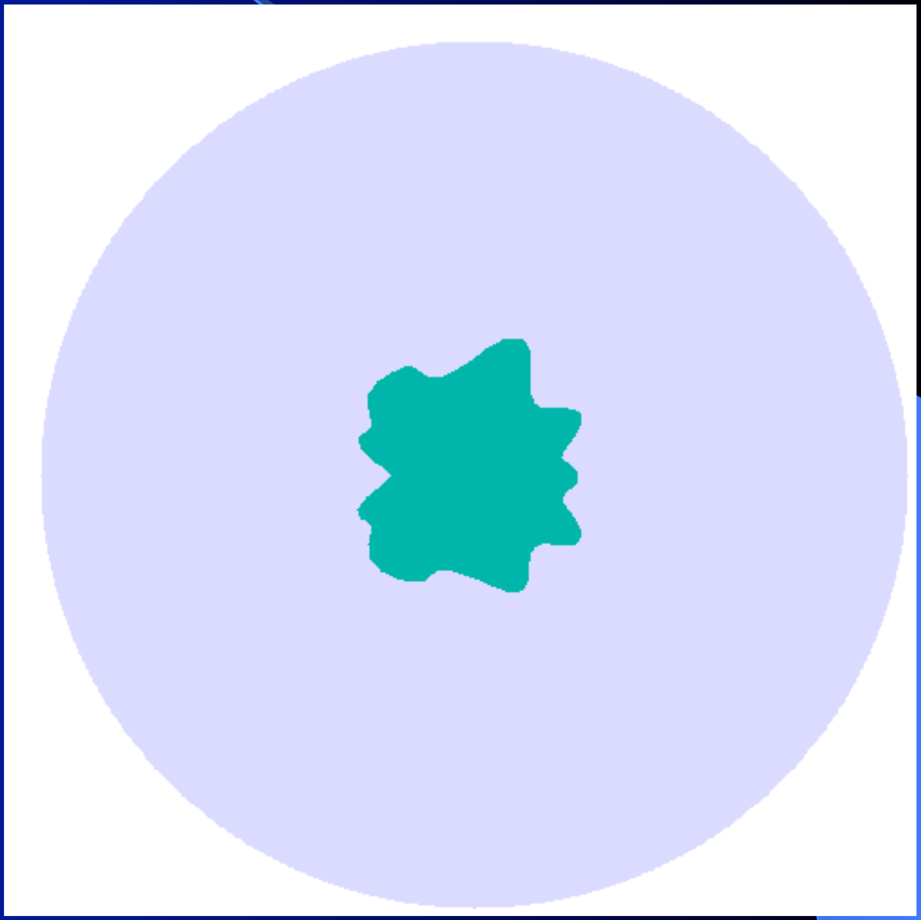
- Continue Time evolution by going back to step 2

Analysis of Results

Analysis

- The results of the program runs do not create real fingers
- Although our numerical calculations were very close to the analytical solution for the initial time step and circular boundaries, at later times it did not seem that the boundaries grew as they should

Comparison



Comparison Cont.

- The reason we think that only the initial evolution is correct, is since more points are needed in between the boundaries to simulate an actual growth further on
- But such a conduct is impossible when tried on a computer, since it would take ages to run with a small timestep