1

Frequency Response and Bode Plots

1.1 Preliminaries

The steady-state sinusoidal frequency-response of a circuit is described by the phasor transfer function $H(j\omega)$. A *Bode plot* is a graph of the magnitude (in dB) or phase of the transfer function versus frequency. Of course we can easily program the transfer function into a computer to make such plots, and for very complicated transfer functions this may be our only recourse. But in many cases the key features of the plot can be quickly sketched by hand using some simple rules that identify the impact of the poles and zeroes in shaping the frequency response. The advantage of this approach is the insight it provides on how the circuit elements influence the frequency response. This is especially important in the *design* of frequency-selective circuits. We will first consider how to generate Bode plots for simple poles, and then discuss how to handle the general second-order response. Before doing this, however, it may be helpful to review some properties of transfer functions, the decibel scale, and properties of the log function.

Poles, Zeroes, and Stability

The s-domain transfer function is always a rational polynomial function of the form

$$H(s) = K\frac{N(s)}{D(s)} = K\frac{s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \cdots a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots b_1 s + b_0} \tag{1.1}$$

As we have seen already, the polynomials in the numerator and denominator are factored to find the poles and zeroes; these are the values of s that make the numerator or denominator zero. If we write the zeroes as $z_1, z_2, z_3 \cdots$ etc., and similarly write the poles as $p_1, p_2, p_3 \cdots$, then $H(s)$ can be written in factored form as

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{1.2}$$

The pole and zero locations can be real or complex. When the roots are real they are called *simple poles* or *simple zeros*. When the roots are complex they always occur in pairs that are complex conjugates of each other.

Another important observation is that stable networks must always have poles and zeroes in the left-half of the complex s-plane, such that the real parts of the poles/zeroes will be *negative*. As an example, lets assume a stable network with simple poles at $p_1 = -1$ and $p_2 = -10$. The transfer function would then be

$$H(s) = \frac{1}{(s-p_1)(s-p_2)} = \frac{1}{(s+1)(s+10)} \tag{1.3}$$

Thus for stable networks we *always* will find terms of the form $(s+a)$ in the denominator, where a is a *positive* number. Students sometimes get confused by the use of $(s-p)$ or $(s+a)$ to represent the same pole location; just remember that the poles are the values of s that make the denominator zero, *i.e.* $s = p$ or $s = -a$ in this example; clearly these will represent the same pole if $p = -a$, and will represent a *stable* pole if $\text{Re}\{a\} > 0$ or $\text{Re}\{p\} < 0$.

When there are multiple roots at the same location the denominator will contain factors of the form $(s+a)^r$, where r is an integer that tells us how many times the root is repeated. For example, a critically-damped second-order response would have $r = 2$.

When the stable network includes a complex-conjugate pole pair, we can represent the pole locations as $s = -\alpha \pm j\beta$ where α and β are both positive real numbers. The transfer function will then have a factor of the form

$$H(s) = \frac{1}{\left[s-(-\alpha+j\beta)\right]\left[s-(-\alpha-j\beta)\right]} = \frac{1}{s^2+2\alpha s+\alpha^2+\beta^2} = \frac{1}{(s+\alpha)^2+\beta^2} \tag{1.4}$$

and thus all the coefficients in the denominator are *positive*, even though the roots in fact have negative real parts. For reasons which will become clear later it is more convenient to write the second-order polynomial in the "standard form"

$$s^2 + 2\xi\omega_n s + \omega_n^2 \tag{1.5}$$

where ω_n is called the *corner frequency* or break point, and ξ is called the *damping factor*. Comparing (1.4) and (1.5) we can relate the corner frequency and damping factor to the poles using

$$\omega_n = \sqrt{\alpha^2+\beta^2} \qquad \xi = \alpha / \omega_n = \alpha / \sqrt{\alpha^2+\beta^2} \tag{1.6}$$

Decibel Scale and Log Functions

Logarithmic scales are useful when plotting functions that vary over many orders of magnitude. This is certainly the case with electrical signals; for example, the signal received by your cell phone is often more than 12 orders of magnitude lower in power than the signal transmitted from the base station! In a filter circuit, the magnitude of the transfer function in the passband may be several orders of magnitude larger than it is in the stop band. We are also interested in the frequency response of circuits over a wide range of frequencies, so it makes sense to use a logarithmic scale for frequencies as well as signal intensity. Electrical engineers use the base-ten logarithm function and denote that as "log", reserving "ln" for the natural log function (base e), such that

$$\log x \equiv \log_{10} x \qquad \ln x \equiv \log_e x \tag{1.7}$$

This notation is not universal; some computer math programs (such as *Mathematica*) use Log[x] for the natural log. In order to compute the base-ten log in *Mathematica*, you have to specify the base by writing Log[10, x]. Fortunately all log functions share the following useful properties regardless of base

$$\begin{aligned} \log AB &= \log A + \log B \\ \log A / B &= \log A - \log B \\ \log y^x &= x \log y \end{aligned} \tag{1.8}$$

The "bel" scale (after inventor Alexander Graham Bell) is defined as the log-base-ten of the ratio of two signal "intensities" (quantities relating to the power or energy associated with the signal). In circuits work we are often interested in the output-to-input power ratio, P_{out} / P_{in}, but the bel scale can be used to compare any two like quantities (for example, the ratio of signal power to carrier in an AM signal, or the ratio of signal power to noise power in a certain bandwidth). Since there are 10 "decibels" per bel the power ratio in dB is defined as

$$10\log_{10}\frac{P_{out}}{P_{in}} \qquad \text{(power ratio in dB)} \tag{1.9}$$

Each time the power increases by a factor of ten, the power ratio in dB increases linearly by 10dB. Since power is related to the square of voltage or current, the dB scale for those quantities becomes (assuming identical source and load impedances[1])

$$10\log_{10}\frac{V_{out}^2}{V_{in}^2} = 20\log_{10}\frac{V_{out}}{V_{in}} \qquad \text{(voltage ratio in dB)} \tag{1.10}$$

In most cases our transfer function is a voltage or current ratio, so we will use $20\log\left|H(j\omega)\right|$ to compute the magnitude in dB. Some important dB conversions to remember are summarized below:

|H|	**$\|H\|_{dB}$**
1	$20\log(1) = 0$ dB
$\sqrt{2}$	$20\log(\sqrt{2}) = 10\log 2 = 3$ dB
2	$20\log(2) = 6$ dB
4	$20\log(4) = 12$ dB
5	$20\log(5) = 14$ dB
10	$20\log(10) = 20$ dB

A logarithmic scale like the dB scale prove to be a great advantage when dealing with circuit transfer functions, which are always of the form of a rational polynomial function as in (1.2).

Two related terms we will use in our discussion of frequency response plots are "decade" and "octave". A decade change in frequency is a factor of ten. So, for example, 1 kHz is a decade *above* 100 Hz and a decade *below* 10 kHz. An "octave" is a factor of two, so similarly 1 kHz is an octave above 500 Hz and an octave below 2 kHz.

[1] If the source and load impedances are not the same this shows up as an additive constant in (1.10), not especially critical for the discussion of this chapter.

1.2 Bode Amplitude Plots

Simple Poles and Zeroes

Consider the transfer function of a first-order circuit with a simple pole at $s=-1$. The AC steady-state frequency-response is determined by letting $s \to j\omega$

$$H(s)=\frac{1}{s+1} \quad \Rightarrow \quad H(j\omega)=\frac{1}{j\omega+1} \tag{1.11}$$

The *magnitude* of the transfer function is then given by

$$\left|H(j\omega)\right|=\left[\omega^2+1\right]^{-1/2} \tag{1.12}$$

This function is plotted in Figure 1-1 below for frequencies that are two orders of magnitude above and below $\omega=1$; clearly the response is quite different on either side of this point. The asymptotic behavior for $\omega \ll 1$ and $\omega \gg 1$ can be found from (1.12) as

$$\left|H(j\omega)\right|_{\text{dB}}=\begin{cases} 0 \text{ dB} & \omega \ll 1 \\ -20\log\omega \text{ dB} & \omega \gg 1 \end{cases} \tag{1.13}$$

These asymptotes are just straight lines on the dB vs. $\log\omega$ plot. For $\omega \ll 1$ the function is a constant, $|H|=1$, or 0 dB. At the other extreme where $\omega \gg 1$, the transfer function decreases as $-20\log\omega$ in dB; on a log-frequency scale this is a straight line with a slope of -20 dB/decade; that is, the transfer function *decreases* by 20dB for every factor of ten increase in frequency. This slope is equivalent to -6dB/octave, a helpful thing to remember.

The two straight-line asymptotes capture the essential features of the plot, meeting at a frequency corresponding to the pole location. This is the "break point". At this point the transfer function has a magnitude

$$\left|H(j1)\right|=\frac{1}{\sqrt{2}}, \quad \text{or} \quad \text{-3 dB}$$

A transfer function with a simple zero behaves similarly, as shown in Figure 1-2, except that

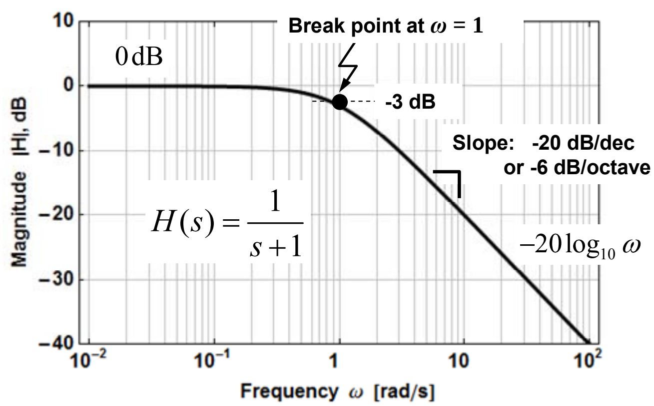


Figure 1-1 – Frequency response for a simple pole at $s=-1$

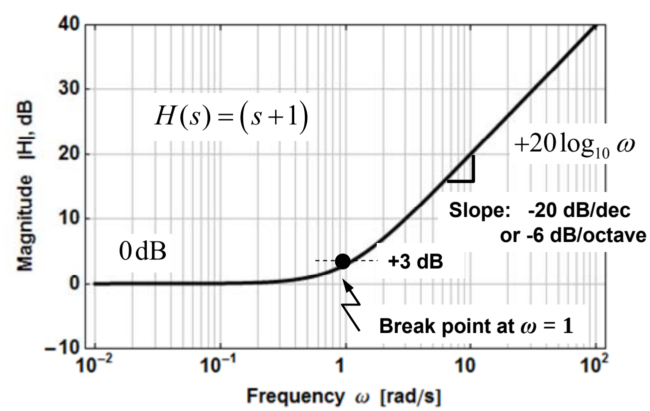


Figure 1-2 – Frequency response for a simple zero at $s=-1$

the function turns *up* at the break point instead of down. Otherwise the rate of change is the same (20 dB per decade above the breakpoint).

This general behavior can be demonstrated for any simple pole or zero, including repeated roots. For example, let's take a repeated pole at $s=-a$

$$H(s)=\frac{1}{(s+a)^r} \quad \Rightarrow \quad H(j\omega)=\frac{1}{(j\omega+a)^r} \tag{1.14}$$

where r is an integer representing the number of times the pole is repeated. The magnitude of the frequency response is now

$$|H(j\omega)|=\left[\omega^2+a^2\right]^{-r/2} \tag{1.15}$$

In this case the asymptotic behavior for $\omega \ll a$ and $\omega \gg a$ can be found from (1.15)

$$|H(j\omega)|_{\text{dB}}=\begin{cases}-20r\log a & \omega \ll a \\ -20r\log\omega & \omega \gg a\end{cases} \qquad \text{[dB]} \tag{1.16}$$

Once again the asymptotes are just straight lines meeting at $\omega=a$, shown in Figure 1-3 as the dashed lines. In this case the slope breaks downward by $20r$ dB/decade, or 20dB/decade for each time the pole is repeated. The dashed lines are called the uncorrected or "straight-line" Bode plot for the transfer function. Clearly the uncorrected plot captures the essential behavior of the frequency response with a minimum of effort. We can always improve the accuracy of the sketch by drawing in a smoothed or "corrected" version that meets the straight-line asymptotes away from the break-point and passes through the true value of the transfer function immediately at the break point, which in this case is given by

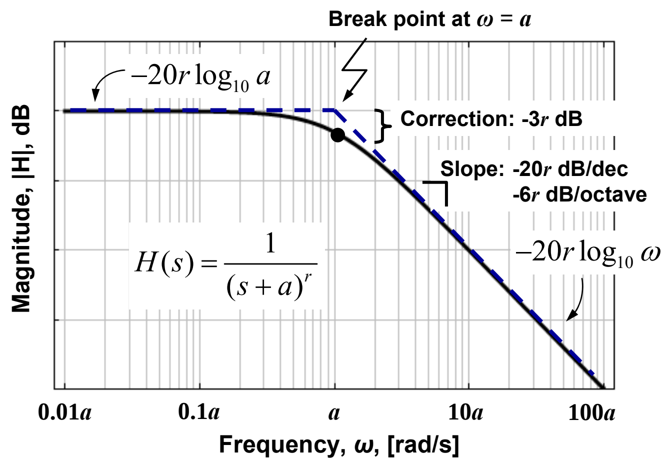


Figure 1-3 – Bode plot for a repeated pole at $s=-a$. The dashed line is a quick estimate called the "uncorrected" Bode plot. The solid line is the "corrected" Bode plot, passing through the correct location at the break point.

$$|H(ja)|_{\text{dB}}=-20r\log\left(a\sqrt{2}\right)\approx(20r\log a-3r)\,\text{dB} \tag{1.17}$$

This shows that the corrected plot should passes through a point that is $3r$ dB below the uncorrected curve at the break point, or 3dB for each time the pole is repeated. The corrected Bode plot is shown as the solid line in Figure 1-3.

Transfer Functions with Multiple Simple Poles and Zeroes

Suppose we have a transfer function with more than one pole or zero, or a combination of simple poles and zeroes. For example:

$$H(s)=A\frac{(s+z)}{(s+p)} \tag{1.18}$$

An interesting thing happens when we express the magnitude of this transfer function in dB: using the properties of the log function (1.8), we get

$$\left|H(s)\right|\text{dB} = 20\log\left|A\right| + 20\log\left|s+z\right| + 20\log\frac{1}{\left|s+p\right|} \tag{1.19}$$

Thus converting to dB breaks the transfer function into a simple sum of the individual factors that we have already considered. The composite response is then just a simple sum of the individual responses. Let's look at a specific example:

$$H(s) = \frac{10(s+100)}{(s+1)} \tag{1.20}$$

This is plotted in Figure 1-4. In the composite response the transfer function breaks downward at the pole location ($\omega = 1$), and then flattens out again when the zero location is reached ($\omega = 100$). Can you see why? When the zero is reached, the downward break of the first pole is canceled out by the upward break of the zero. At low frequencies ($\omega \to 0$) the magnitude of the transfer function is a constant representing a sum of the values (in dB) of the low-frequency asymptotes of each individual term: 20dB + 0dB + 40dB = 60dB. At the high frequencies ($s \to \infty$) the transfer function in (1.20) approaches the limiting value of 10 (20 dB).

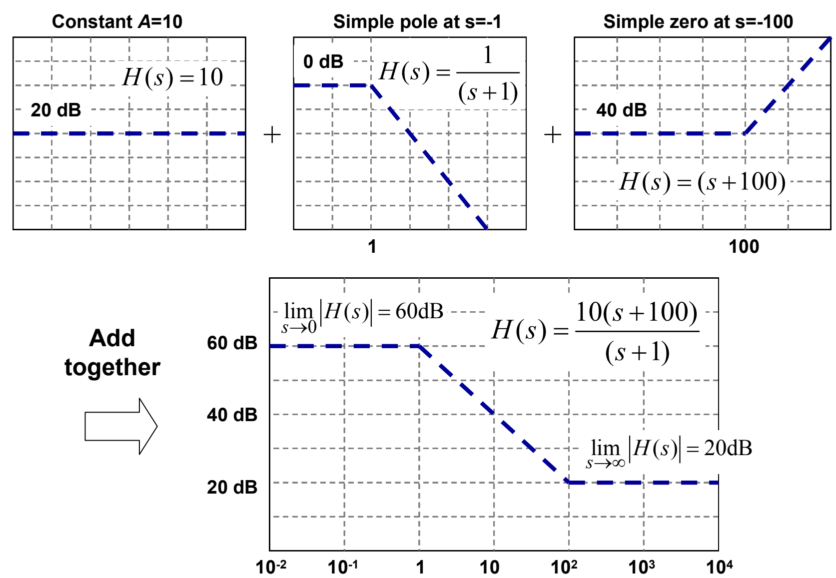


Figure 1-4 – Illustration of how the composite Bode plot of the transfer function in (1.20) is a superposition of the individual terms.

From this example some simple rules for generating uncorrected Bode plots begin to emerge: when poles are encountered the slope always *decreases* by 20 dB/decade. When zeroes are encountered the slope always *increases* by +20 dB/decade. All we need to do is choose a suitable starting point and then start drawing straight lines, changing the slope up or down depending on whether we encounter a pole or a zero.

Often the most difficult part is figuring out where to start the plot, or how to position the asymptotes on the vertical scale. In the previous example the transfer function begins with a constant value at low frequencies which makes things easy; we just let $\omega \to 0$ in the transfer function and take the magnitude of what is left over,

$$\lim_{s\to 0}\left|H(s)\right| = \frac{10(100)}{1} = 1000 \Rightarrow 60\text{dB} \tag{1.21}$$

Here is a slightly more challenging example:

$$H(s) = \frac{10s}{(s+1)} \tag{1.22}$$

The first thing to notice is that the frequency response will begin on an upward trajectory because of the zero at $s=0$; can you see why? We've already found that the slope *increases* after each zero, and since we are always plotting frequency on a *log scale* we can never include the point $\omega = 0$ on the plots. No matter how we choose the limits the plot must always start at a frequency *above* the first zero, and thus the plot will begin with an upward slope of +20 dB/decade. When we reach the next break-point (associated with the pole at $s=-1$) the slope will decrease by 20 dB/dec, flattenening the response. The result is a high-pass filter response. The only question that remains is where to position the asymptotes on the vertical scale; that is, where to start drawing lines? For this we need some convenient reference point to begin the plot. In this case it seems best to start at high-frequencies and work backwards, since for $\omega \gg 10$ the magnitude approaches a limiting value of

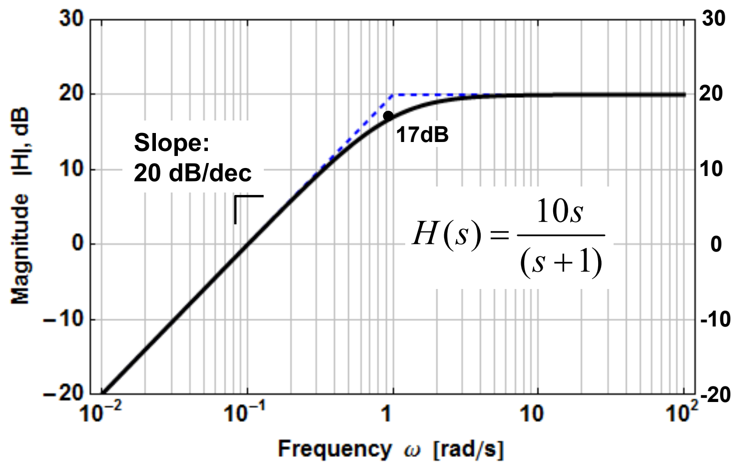


Figure 1-5 – Bode plot for the example of (1.22). The plot begins with an upward slope of 20 dB/decade because of the zero at s=0.

$$\lim_{s\to\infty}\left|H(s)\right| = 10, \quad \text{or } 20 \text{ dB} \tag{1.23}$$

So now we can draw the Bode plot as shown in Figure 1-5: the curve starts up at a slope of +20 dB/decade due to the zero at s=0, and flattens out at a level of 20 dB at the break point. These two asymptotes are shown as the dashed lines. We then sketch in the "corrected" plot which passes through a point 3dB below the uncorrected plot at the break point, or 17dB.

Let's do one more example of multiple poles and zeroes:

$$H(s) = \frac{10s}{(s+10)(s+100)^2} \tag{1.24}$$

First think abut this qualitatively: there is a simple zero at $s=0$, a simple pole at $s=-10$, and a double pole at $s=-100$. Can you start to visualize the shape of the Bode plot? The first part of the plot (for $\omega < 100$) the shape should be similar to the previous example, starting on a positive slope of +20 dB/decade and flattening out above $\omega = 10$, but the double pole at $\omega = 100$ will cause the slope to break downward again by -40 dB/decade. So the function has a band-pass response shape. The only difficult part here is how to position the asymptotes on the vertical scale. In this example the techniques we used previously don't work; if we test the low- and high-frequency limits by letting $s \to 0$ or $s \to \infty$, the transfer function goes to zero, which is negative infinity on a dB scale! In this situation there are two common methods of attack. The first (and most straightforward) method is just to choose a specific frequency, preferably far from all the other poles and zeroes, and simply evaluate the function numerically. The function will have to pass through that point, correct? Usually it is best to choose the lowest or highest frequency on the plot for this purpose, assuming it is a factor of ten below the nearest pole or zero. For example, at $\omega = 1$ we have

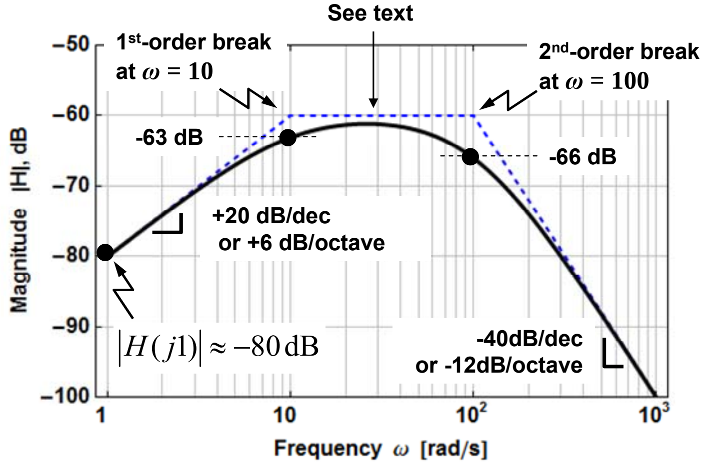


Figure 1-6 – Bode plot for the function given in (1.24)

$$|H(j1)| = \left| \frac{10(j1)}{(j1+10)(j1+100)^2} \right| \tag{1.25}$$

This looks nasty, but remember that we can find the magnitude of a complex expression like this by evaluating the magnitude of each complex term individually:

$$|H(j1)| = \frac{10|j1|}{|j1+10||j1+100|^2} = \frac{10}{\sqrt{1+10^2}\left(\sqrt{1+100^2}\right)^2} \approx 10^{-4} \Rightarrow -80\,\text{dB} \tag{1.26}$$

(the exact value computes to -80.04 dB). As shown in Figure 1-6, we position the first dashed-line asymptote at -80dB for $\omega = 1$, sloping up at +20dB/decade, and from there we just follow the basic rules of changing slope for each pole and zero that is encountered.

The second method for positioning the curve vertically is similar in that we try to evaluate the function at some point numerically, but focusing on the asymptotic behavior of each term in the transfer function. At any particular frequency we can split up the transfer function into two parts, grouping all the poles and zeros that lie at or below this frequency, and grouping all the terms with poles and zeroes that lie above this frequency. For example, at a frequency just above the first pole location $\omega = 10$ we could write

$$|H(j10)| \approx \overbrace{\underbrace{\left[\frac{10s}{s+10}\right]}_{\approx 10 \text{ for } \omega \gg 10}}^{\substack{\text{terms with poles}\\\text{or zeros at or}\\\text{below } \omega=10}} \times \overbrace{\underbrace{\left[\frac{1}{(s+100)^2}\right]}_{\approx 10^{-4} \text{ for } \omega \ll 100}}^{\substack{\text{terms with poles}\\\text{or zeros at or}\\\text{above } \omega=10}} \approx 10^{-3} \Rightarrow -60\,\text{dB} \tag{1.27}$$

This is the level the *uncorrected* Bode plot should pass through at the first pole location $\omega = 10$. The terms in the first bracket have poles and zeroes at or below $\omega = 10$ so they each contribute their *high-frequency* asymptotic behavior to the uncorrected Bode plot. The terms in the second brackets contribute their *low-frequency* asymptotic behavior to the plot. We can see in Figure 1-6 that the uncorrected bode plot does indeed pass through -60dB at

$\omega = 10$. This latter method is how certain software packages generate uncorrected Bode plots from complicated transfer functions.

We can now summarize our findings as a set of rules or guidelines for drawing Bode plots involving simple/repeated poles and zeroes:

Rules for Drawing Bode Magnitude Plots with Simple Poles and Zeroes

- First determine all the break points (pole and zero locations) and arrange in order of increasing frequency. Choose a frequency range for the plot that encompasses all these points, adding an extra decade of frequency above and below this range.
- Based on the poels and zeroes, make a quick sketch of the expected shape of the Bode plot on a piece of scrap paper. This will help you find the appropriate vertical scales. For a simple pole or zero of the form $(s+a)$ the slope of the uncorrected Bode plot changes at the break point $\omega = a$, increasing by 20 dB/decade for a zero, and decreasing by 20dB/decade for a pole. For a *repeated* pole or zero $(s+a)^r$ the slope changes by $20r$ dB/decade, or 20 dB for each time the pole or zero is repeated.
- To find a reference level we first consider the behavior of the function for low-frequencies ($\omega \to 0$) or high frequencies ($\omega \to \infty$). If the limiting behavior approaches a constant value at these extremes that is a good starting point. Otherwise, we must evaluate the function numerically at some particular frequency, preferably in a region with a constant-value "plateau".
- Once the uncorrected Bode plot is finished, a corrected version can be drawn. For simple/repeated roots the true response passes through a point that is $3r$ dB below the uncorrected curve at the break point, or 3dB for each time the pole is repeated

These rules work well for transfer functions that have poles and zeroes that are well separated in frequency (by a factor of 10 or more). If the poles and zeroes are very close together the rules break down and we must evaluate the function numerically.

Normalized Functions and Time Constants

Our approach thus far has been to work with transfer function in the pole-zero form (1.2). Many books recommend re-normalizing the transfer function first by dividing the numerator by all the zeroes, and dividing the denominator by all the poles. For example, if we had a transfer function given by

$$H(s) = \frac{5(s+2)}{(s+10)(s+100)} \tag{1.28}$$

we could factor out the zeroes from the numerator and the poles in the denominator to give

$$H(s) = \underbrace{\frac{5(2)}{10(100)}}_{0.01 \text{ (or } -40\text{dB)}} \frac{(1+s/2)}{(1+s/10)(1+s/100)} \tag{1.29}$$

In this procedure all the poles and zeroes have the form $(1+s/a)^r$, from which you can see that the low-frequency asymptote for each term is now always 1, or 0 dB. The break point is still at $\omega = a$, and the same rules apply: the slope goes up by +20dB/decade for each zero, and down by 20dB/decade for each pole.

Is it an advantage to renormalize the function in this way? Probably not, at least in terms of the effort that goes into making a Bode plot by hand. Factoring out the terms as in (1.29) does tell us that the starting value of the plot will be -40dB at low frequencies, but we could get this information just as easily from (1.28) by letting $s \to 0$. And when there are zeroes at $s = 0$, any potential advantage of renormalizing disappears, because we still have to invest the same amount of effort (or more) in figuring out where to position the lines vertically.

However, there are times in circuit analysis when this normalized form does appear naturally, so it is important to be familiar with generating Bode plots from both forms. When it appears it us usually written in terms of time constants, like this

$$H(s) = \frac{s(1+s\tau_1)}{(1+s\tau_2)(1+s\tau_3)} \tag{1.30}$$

The break points in the Bode plot are now at $1/\tau_1$, $1/\tau_2$, and $1/\tau_3$. Recall that a simple pole of the form $1/(s-p)$ is associated with an exponential in the time-domain, e^{pt}, thus simple poles are always related to time constants as $\tau = -1/p$. Some advanced circuit analysis techniques focus specifically on the rapid estimation of time-constants in complex circuits, in which case it is often easier to work with the time constant form as in (1.30). It should be clear by now that generating a Bode plot for this case is a simple extension of the techniques we have developed earlier.

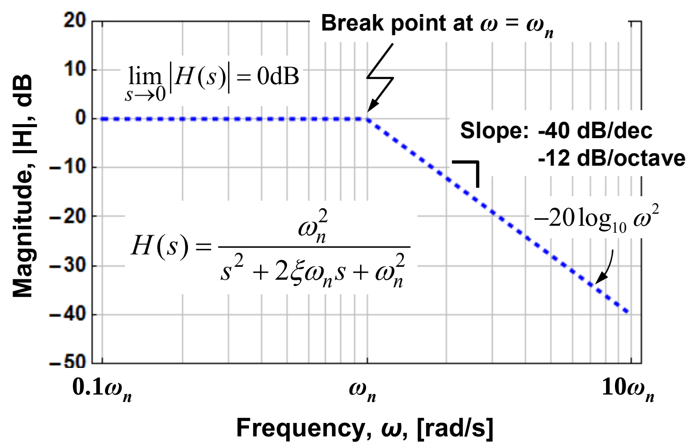


Figure 1-7 – Uncorrected Bode amplitude plot for a second-order response with $0 \le \xi \le 1$.

Second-Order Response with Complex Roots

An important remaining issue is the case of complex-conjugate pole pairs, as in (1.4). For this purpose it proves helpful to write the second-order polynomial in the form (1.5), *e.g.*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{1.31}$$

In this form the quadratic formula gives the pole locations as

$$s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \tag{1.32}$$

For $\xi > 1$ the 2nd-order response involves two simple (real) poles, and we already know how to deal with that situation. Stable *complex-conjugate pole pairs* occur when $0 < \xi < 1$, and this is case we are most interested in here. The amplitude-frequency response is given by

$$|H(j\omega)| = \frac{\omega_n^2}{\left|\omega_n^2 - \omega^2 + j2\xi\omega_n\omega\right|} = \omega_n^2\left[\left(\omega_n^2 - \omega^2\right)^2 + 4\xi^2\omega_n^2\omega^2\right]^{-1/2} \tag{1.33}$$

Figure 1-7 shows the asymptotic behavior of (1.33), well above and well below ω_n, i.e. the uncorrected Bode plot. These asymptotes are given by

$$|H(j\omega)|_{\text{dB}} = \begin{cases} 0\,\text{dB} & \omega \ll \omega_n \\ -20\log\omega^2 & \omega \gg \omega_n \end{cases} \qquad \text{[dB]} \tag{1.34}$$

Interestingly the asymptotic behavior is the same as it would be for a *repeated simple pole* at $\omega = \omega_n$; the slope decreases by 40dB/decade at this location. So the uncorrected Bode plot for the complex conjugate poles is the same as it would be for a simple repeated pole, with $\omega = \omega_n$ behaving as the break point in this case. For this reason ω_n is called the *corner frequency* for the complex second-order response.

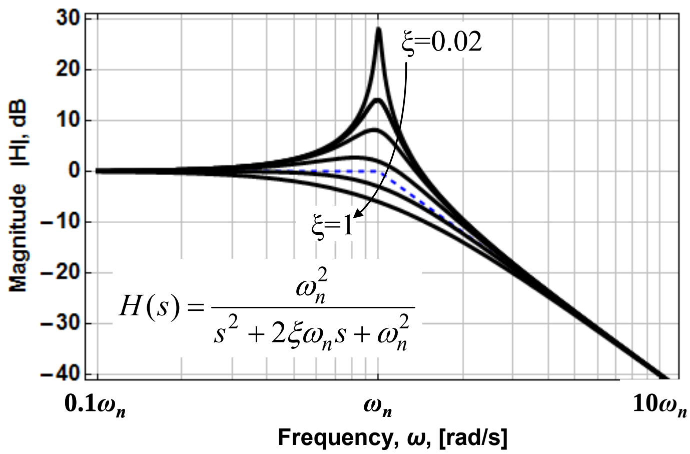
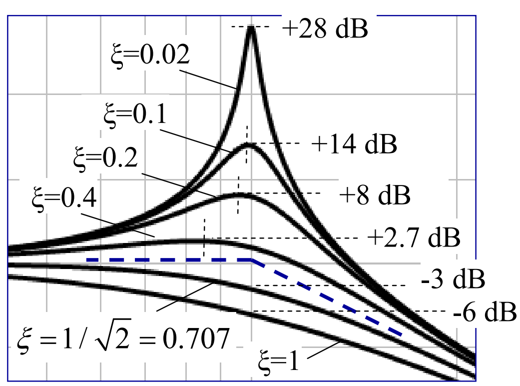


Figure 1-8 – Bode amplitude plot for the general 2nd-order low-pass response for various values of ξ.

It is when considering the *corrected* Bode plot for the complex second-order response that things start to get interesting. As shown in Figure 1-8, the behavior near the break point is a strong function of the parameter ξ, which we call the *damping factor*. For small ξ the curves are peaked sharply near the corner frequency. Exactly at the corner frequency the curve must pass through the point

$$|H(j\omega_n)| = \frac{1}{2\xi} \quad \Rightarrow \quad -20\log 2\xi \;\; \text{[dB]} \tag{1.35}$$

Note that this correction may be above the asymptote (positive) or below (negative) depending on the value of the damping factor ξ.



ξ	$\lvert H(j\omega_n)\rvert$	ω_p/ω_n	$\lvert H(j\omega_p)\rvert$
0.02	+28 dB	~1	+28 dB
0.05	+20 dB	0.997	+20 dB
0.1	+14 dB	0.990	+14dB
0.2	+8 dB	0.959	+8.1 dB
0.4	+1.9 dB	0.825	+2.7dB
0.5	0 dB	0.707	+1.3 dB
0.707	-3 dB	0	0 dB
1	-6 dB	—	—

Figure 1-9 – Behavior near the corner frequency for various values of the damping factor ξ.

Also note that the peak value is not necessarily centered exactly at the corner frequency; to find the peak location we set the first derivative equal to zero, giving

$$\frac{\partial}{\partial\omega}|H(j\omega)| = 0 \quad \Rightarrow \quad \omega = \omega_p = \omega_n\sqrt{1-2\xi^2} \;\; \text{(low-pass)} \tag{1.36}$$

This result tells us that there is a peak or maximum in the response only when $1-2\xi^2>0$, or equivalently for $0\le\xi\le 1/\sqrt{2}$. In this range the peak amplitude is given by

$$\left|H(j\omega_p)\right| = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \Rightarrow \quad -20\log\left(2\xi\sqrt{1-\xi^2}\right) \text{ [dB]} \tag{1.37}$$

Figure 1-9 gives a close-up of the region near the corner frequency for various values of damping factor. Physically this behavior near the break point is associated with a resonance condition in the circuit; we will discuss this later.

If we instead have a complex pair of *zeroes* in the transfer function, *e.g.*

$$H(s) = \frac{s^2+2\xi\omega_n s+\omega_n^2}{\omega_n^2} \tag{1.38}$$

Then we get the response shown in Figure 1-10. As you might expect it is just the mirror image of the complex pole pair, so all of the main conclusions are the same. Only the sign of the correction near the break point changes. We can still use the table in Figure 1-9 if we remember to reverse the sign (so, for example, the curve lies 28dB *below* the asymptotes at the break point for $\xi=0.02$).

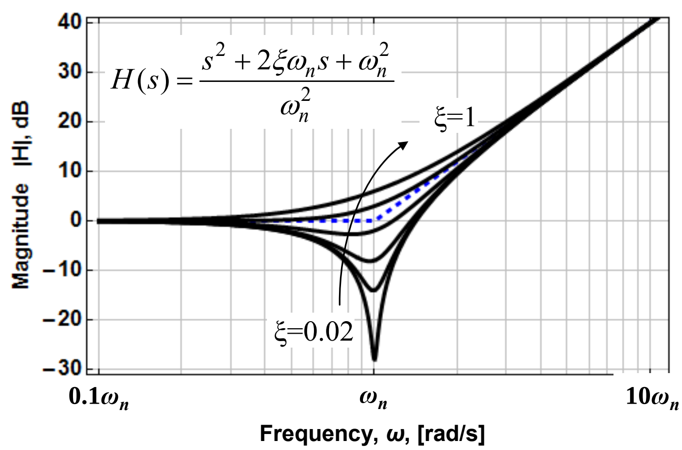


Figure 1-10 – Amplitude response with a complex zero pair.

Similarly if we add a repeated zero at $s=0$ to the general 2nd-order low-pass response of Figure 1-8 we get a high-pass shape as shown in Figure 1-11. Again the general conclusions are unchanged. The only thing to note here is that the peak location shifts *above* the break point in this case. We can still use the results in the table of Figure 1-9, but the data in the column for the peak frequency should be interpreted as ω_n/ω_p instead, since the peak location is now given by

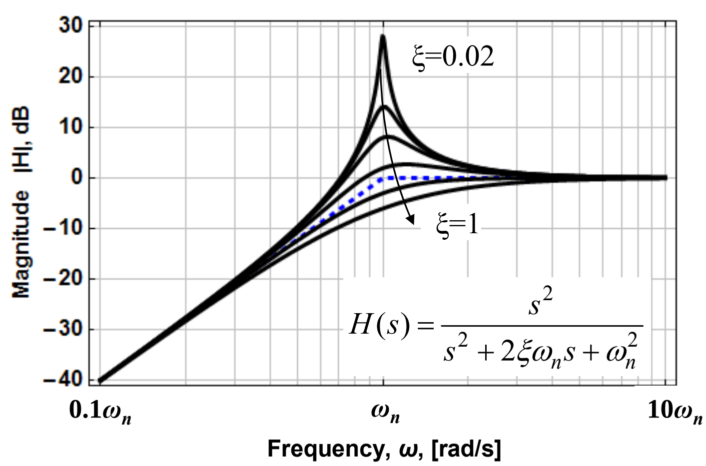


Figure 1-11 – Amplitude response for the general 2nd-order high-pass function for various values of ξ.

$$\frac{\partial}{\partial\omega}\left|H(j\omega)\right| = 0 \quad \Rightarrow \quad \omega=\omega_p=\frac{\omega_n}{\sqrt{1-2\xi^2}} \quad \text{(high-pass)} \tag{1.39}$$

For example, the peak location for $\xi = 0.4$ in Figure 1-11 would shift to $\omega_p = \omega_n / 0.825 = 1.212\omega_n$.

What if we had a *repeated* complex pole pair? For example, a transfer function that involves the square of a term like that in (1.31). You should be able to convince yourself by the stage that the asymptotic slope should now change by -80 dB/decade, and that the correction factors near the break point are all doubled.

In terms of generating Bode amplitude plots in the presence of complex pole pairs, our procedure is the same as before but can be amended as follows:

Additional Rules for <u>Amplitude</u> Plots with Complex Pole and Zero Pairs

- First write the relevant 2nd-order polynomials in the standard form $s^2 + 2\xi\omega_n s + \omega_n^2$, and extract the corner frequency ω_n and the damping factor ξ. Complex roots are associated with $0 \le \xi \le 1$. For $\xi > 1$ the roots are real and hence correspond to simple poles or zeroes that we already considered.
- Sketch the uncorrected Bode plot, which is equivalent to the case of a repeated simple pole with a break point at $\omega = \omega_n$. If the complex roots are in the numerator the slope increases by 40dB/decade at the break point. If the complex roots are in the denominator the slope decreases by 40 dB/decade.
- Sketch in the corrected Bode plot. The peak value, peak location, and value of the function exactly at the break point can be determined from the table in Figure 1-9 or the equations in (1.35)-(1.39).

Let's conclude with an example that combines many of the details we've considered:

$$H(s) = \frac{10^5 s(s+100)}{(s+10)^2(s^2+400s+10^6)} \tag{1.40}$$

This function has a zero at $s = 0$ and a zero at $s = -100$, a repeated simple pole at $s = -10$, and two other poles coming from a second-order polynomial. By comparing the quadratic to the standard form we find

$$\omega_n^2 = 10^6 \Rightarrow \omega_n = 10^3$$
$$2\xi\omega_n = 400 \Rightarrow \xi = 0.2$$

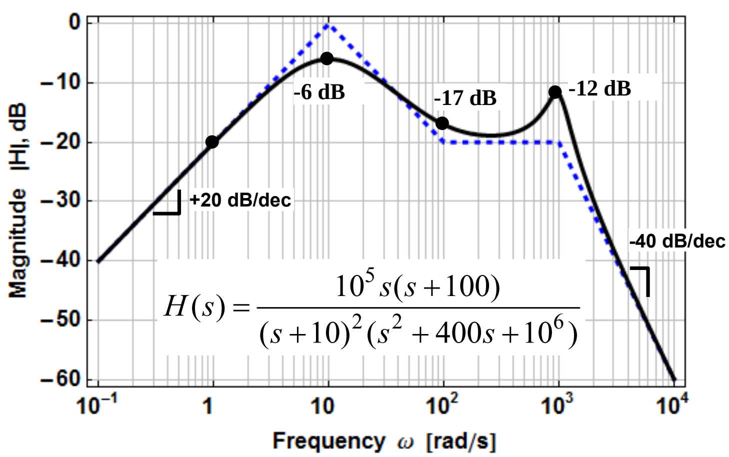


Figure 1-12 – Example problem integrating many common features of Bode plots.

From the table in Figure 1-9 we find a correction of +8dB at $\omega = \omega_n$. Where do we start the plot? Let's evaluate the function at $\omega = 1$, which is a decade below the lowest break point. This gives:

$$\left|H(j1)\right| \approx \frac{10^5(1)(100)}{(10)^2(10^6)} = 0.1 \Rightarrow -20\,\text{dB}$$

This completed plot is shown in Figure 1-12, with dots marking the starting point and the values of the functions at the break points using our basic rules: down 6dB for a repeated pole, up 3 dB for a simple zero, and up $-20\log 2\xi$ for a complex pole pair.

Root Locus Plot

Please remember that a second-order polynomial doesn't always have complex-conjugate roots, but we CAN always put it in the standard form $s^2 + 2\xi\omega_n s + \omega_n^2$! You should make sure that the damping factor is in the range of $0 \le \xi \le 1$ before you start plugging it into the formulas in (1.35)-(1.39).

A helpful way to visualize solutions of the second-order polynomial for <u>all</u> possible values of the damping factor is shown in Figure 1-13. This is called a *root-locus plot*. The arrows show the path of the roots in the complex plane, beginning with $\xi = 0$. At this starting point the complex conjugate roots have no real part, lying on the imaginary axis. As the damping factor is increased the roots travel on a circular arc towards the negative real axis. When $\xi = 1$ the roots converge to a common point on the negative real axis, corresponding to a repeated simple pole. As the damping factor is increased beyond that point the roots split along the real axis, one growing and one shrinking.

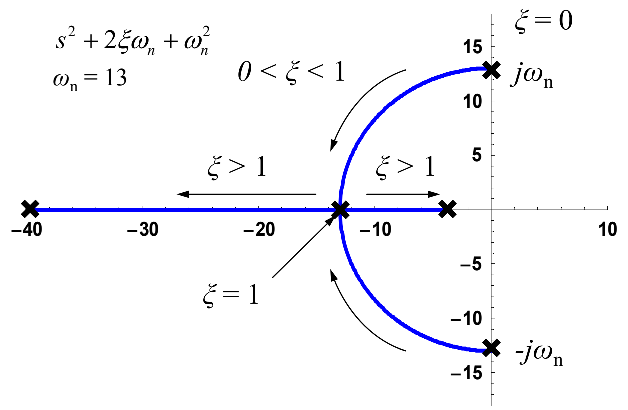


Figure 1-13 – Root-locus in the complex plan for the second-order polynomial as a function of damping factor.

1.3 Bode Phase Plots

The transfer function $H(j\omega)$ is a phasor. Thus far we have concentrated on the magnitude of the transfer function, but the phase response is important as well; it tells us how the phase of a sinusoidal signal changes as it passes through the network. As you will see in later coursework, phase vs. frequency plots are important for investigating potential instabilities in feedback systems. So it is important to be just as familiar with making Bode phase plots as with Bode magnitude plots.

Before we proceed it may be helpful to summarize some important points for finding the phase of complex rational functions. The key is to remember that any complex number z can be written as $z = |z| e^{j\angle z}$, and when exponentials are multiplied the exponents add. So for a product of two phasor functions the net phase is the sum of individual phases:

$$H(j\omega) = F(j\omega)G(j\omega) \quad \Rightarrow \quad \angle H(j\omega) = \angle F(j\omega) + \angle G(j\omega) \tag{1.41}$$

Similarly for a rational function we subtract the phases

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} \quad \Rightarrow \quad \angle H(j\omega) = \angle N(j\omega) - \angle D(j\omega) \tag{1.42}$$

Here is a specific example that will encompass many of the situations to be encountered,

$$H(j\omega) = \frac{(j\omega + z)^m}{(j\omega + p)^n} \Rightarrow \quad \angle H(j\omega) = m\tan^{-1}\left(\frac{\omega}{z}\right) - n\tan^{-1}\left(\frac{\omega}{p}\right) \tag{1.43}$$

Phase Plots for a Simple Poles and Zeroes

Let's start as before by considering a first-order circuit with a simple pole at $s=-1$,

$$H(s)=\frac{1}{s+1} \quad\Rightarrow\quad H(j\omega)=\frac{1}{j\omega+1} \tag{1.44}$$

The phase of the transfer function is given by

$$\angle H(j\omega)=-\tan^{-1}(\omega) \tag{1.45}$$

The asymptotic behavior for $\omega \ll 1$ and $\omega \gg 1$ can be easily found

$$\angle H(j\omega)=\begin{cases} 0^\circ & \omega \ll 1 \\ -90^\circ & \omega \gg 1 \end{cases} \tag{1.46}$$

The transfer function goes through a phase change of $-90°$ for a simple pole. The asymptotes are shown as the dotted blue lines in Figure 1-14, and will serve as our basic *uncorrected* sketch for Bode phase plots. The true phase (1.45) is shown as the solid line for frequencies within two orders of magnitude of the break point $\omega=1$; unlike amplitude plots the phase approaches the low- and high-frequency asymptotes more slowly, taking approximately two decades of frequency above or below the break point to closely approach the asymptote. Exactly *at* the breakpoint the function passes through $-\tan^{-1}(1)=-45°$, or halfway between the asymptotes. A decade *below* the break point the phase passes through $-\tan^{-1}(0.1)=5.7°$, or $\approx 6°$; a similar correction applies for a factor of ten *above* the break point ($-90°+6°=84°$).

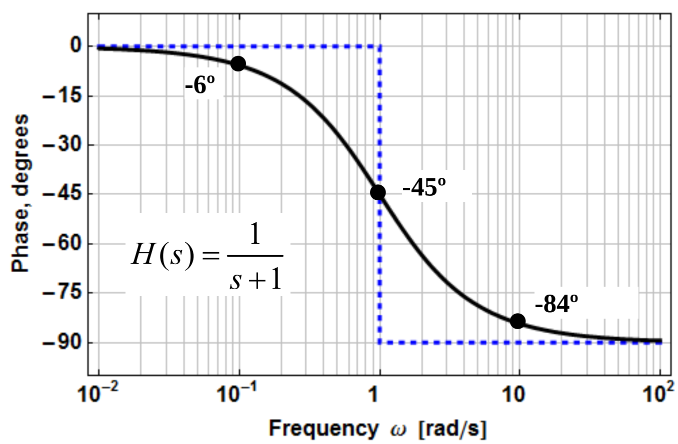


Figure 1-14 – Bode phase plot for a simple pole at $s=-1$

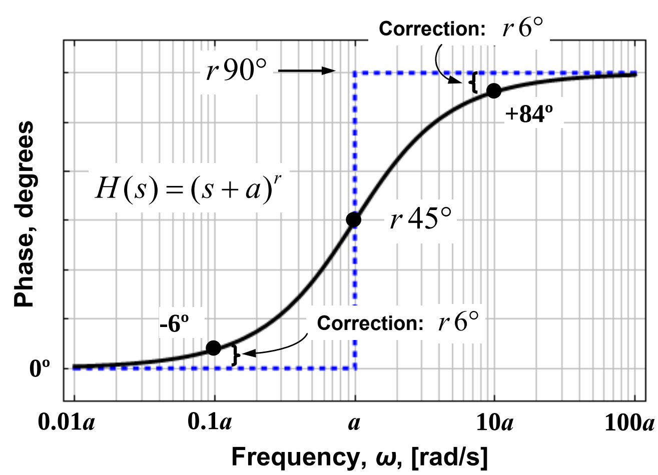


Figure 1-15 – Bode phase plot for a repeated zero at $s=-a$

The behavior for a *zero* is similar; the phase *increases* by $90°$ and passes though the midpoint of $+45°$ at the break point. We can extend the results for simple repeated poles and zeroes as before using the more general function

$$H(s)=(s+a)^{\pm r} \quad\Rightarrow\quad H(j\omega)=(j\omega+a)^{\pm r} \tag{1.47}$$

Here the +sign represents a zero and the –sign represents a pole. The phase is given by

$$\angle H(j\omega) = \pm r \tan^{-1}\left(\frac{\omega}{a}\right) \tag{1.48}$$

In this case the asymptotic behavior and value at the break point are

$$\angle H(j\omega) = \begin{cases} 0^\circ & \omega \ll a \\ \pm r\,90^\circ & \omega \gg a \end{cases} \qquad \angle H(ja) = \pm r45^\circ \tag{1.49}$$

The phase plot and asymptotes for (1.47) are shown in Figure 1-15. In general we find that the phase of the transfer function changes by 90° for each pole or zero, *increasing* 90° for a zero and *decreasing* 90° for each pole. The function passes through the midpoint of each phase jump, at *half* the net phase change or 45° for each pole or zero, and passes through a point that is 6° from the asymptote a decade above or below the break. For repeated roots all these values are multiplied by the number of times the root is repeated. The black dots in Figure 1-14 and Figure 1-15 serve as important references for drawing the corrected plot.

Phase Plots with Multiple Simple Poles and Zeroes

Our method for making the uncorrected Bode phase plots is simple: first determine the phase in the limit of $\omega \to 0$; that defines our starting point, then add dashed lines of constant phase, jumping up 90° for each zero and down 90° for each pole. We then add the reference points at each phase jump and a decade above and below to guide our sketch of the smoothed or corrected plot. As a first example let's revisit (1.20), given here again for convenience:

$$H(s) = \frac{10(s+100)}{(s+1)} \tag{1.50}$$

The Bode phase plot is shown in Figure 1-16. The plot starts at 0° because $H(j\omega) \to 10$ (a real number) as $\omega \to 0$. Then the uncorrected plot jumps down 90° at $\omega = 1$ for the pole, and then back up to 0° at $\omega = 100$ for the zero. Now we add the reference points. You can see that we have marked the mid-points of each phase jump at 45°, and the 6° corrections at $\omega = 0.1$ and $\omega = 10^3$. The only "new" issue here is the reference point at $\omega = 10$. This is a decade above the pole and a decade below the zero, so we get a *double* correction of 12°; that is, 6° for the pole and 6° for the zero. The two extra dashed lines in Figure 1-16 are included to help illustrate what is going on, representing the phase curves for the pole and zero terms separately. The key point to remember is that the total phase at any frequency is the sum of contributions from *all* the poles and zeroes. In Figure 1-16, the pole and zero are close enough together that their contributions overlap at $\omega = 10$. In general, whenever we mark a reference point for the corrected plot, we must always consider its proximity to other breaks.

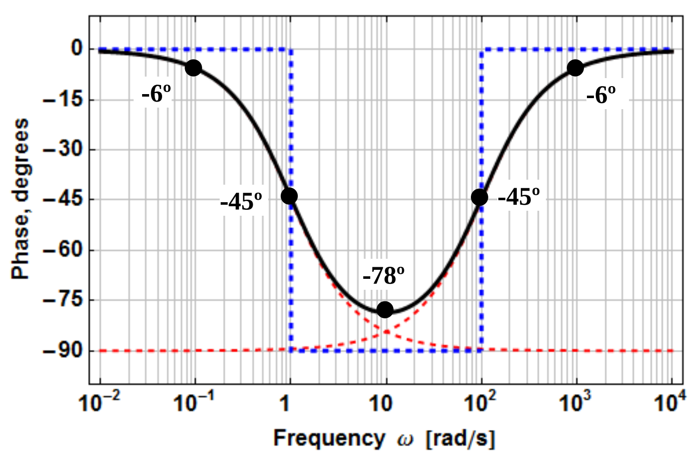


Figure 1-16 – Bode phase plot for (1.50).

A few more examples will help illustrate the issues involved. Consider:

$$H(s)=\frac{10^2 s}{(s+10)(s+100)} \tag{1.51}$$

The Bode phase plot for this function is shown in Figure 1-17. The plot in this case starts at $90°$ because $H(j\omega)\rightarrow 0.1j\omega$ (a positive imaginary number) as $\omega\rightarrow 0$. Then the uncorrected plot jumps down $90°$ at the first pole and down another $90°$ at the next pole. The reference points at $\omega=1$ and $\omega=10^3$ are easy because only one pole contributes to each. But the reference points at the phase jumps are slightly more challenging. For the first jump at $\omega=10$ you see we have marked the crossing point at $39°$, or $6°$ below $45°$. This is because we are a decade below the pole break at $\omega=10^2$. Similarly at $\omega=10^2$ we show the crossing point at $-39°$, or $6°$ above $-45°$ because we are a decade above the pole break at $\omega=10$.

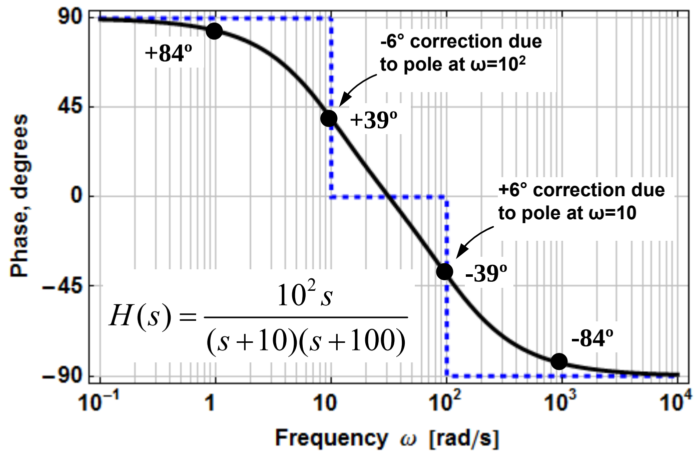


Figure 1-17 – Bode phase plot for (1.51).

So far the corrections have all been $\pm 6°$ because we have been dealing with simple poles. What if we have a repeated root? Consider a slight modification to the previous example:

$$H(s)=\frac{10^4 s}{(s+10)(s+100)^2} \tag{1.52}$$

The Bode phase plot for this function is shown in Figure 1-18. It begins much like Figure 1-17, but because of the repeated root the uncorrected plot jumps down by $180°$ at $\omega=10^2$. Similarly, all of the reference point corrections associated with this repeated root are doubled, as indicated for the points a decade above and below the repeated pole.

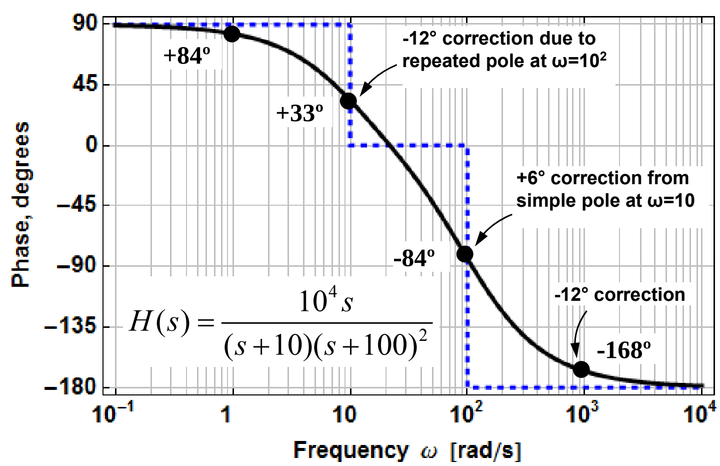


Figure 1-18 – Bode phase plot for (1.52).

The key takeaway from this discussion is that we should always consider a correction to the initial reference points when they involve contributions from multiple poles or zeroes. This is necessary when the poles or zeroes lie within two decades of each other in frequency. Thus we arrive at a simple set of rules for drawing Bode phase-plots with multiple simple poles & zeroes:

Rules for Drawing Bode Phase Plots with Simple Poles and Zeroes

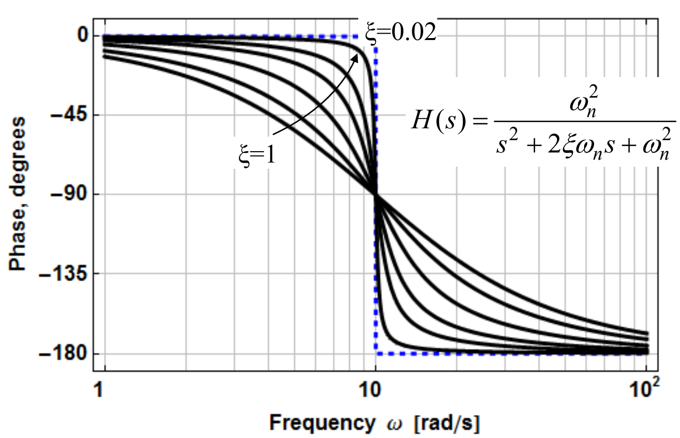
- First determine the phase of the transfer function in the limit of $\omega \to 0$. That will define the asymptotic starting point for the uncorrected plot. Remember that if the transfer function evaluates to a negative real number, the starting phase is $\pm 180°$.
- Sketch the uncorrected phase plot by drawing a sequence of lines of constant phase, beginning with the low-frequency asymptote above, and jumping discontinuously up or down at each break point, *increasing* $90°$ for a zero and *decreasing* $90°$ for each pole. For repeated roots the jump is $90°$ for each time the root is repeated.
- For each pole or zero, make an initial mark at the mid-point of each phase jump. Then consider whether those points are within a decade or less from a neighboring pole or zero. If so, make the appropriate correction to account for the influence of the neighboring pole or zero. Make similar corrections at points a decade above and below each pole location.
- Draw in the smoothed or corrected Bode phase plot, passing through the corrected reference points and meeting the asymptotes at points that are two decades away from the nearest pole or zero.

Second-Order Response with Complex Roots

A complex conjugate pair of roots presents an interesting challenge in connection with phase plots. For a complex pole pair as in (1.31) we find

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}\right) \tag{1.53}$$

This is plotted in Figure 1-19, along with the uncorrected plot that we'd expect for a repeated pole. As the damping factor decreases, the slope increases steadily to more closely approximate the uncorrected phase plot for small damping factors. Each curve passes through the mid-point of the phase jump.



ξ	$0.1\omega_n$	$0.2\omega_n$	$0.5\omega_n$
0.02	-0.2°	-0.5°	-1.5°
0.05	-0.6°	-1.2°	-3.8°
0.1	-1.2°	-2.4°	-7.6°
0.2	-2.3°	-4.8°	-15°
0.3	-3.5°	-7.1°	-22°
0.4	-4.6°	-9.5°	-28°
0.5	-5.8°	-12°	-34°
0.707	-8.1°	-16°	-43°
1	-11.4°	-23°	-53°

Figure 1-19 – Phase plot for a 2nd-order low-pass response as a function of the damping factor. Numerical data at three frequencies below the break are tabulated as an aid for sketching the response.

Superficially it seems relatively easy to deal with complex roots in a phase-plot, because the curves all have a qualitatively similar shape for damping factors in the range of $0 < \xi \le 1$. For small damping factors we simply draw the corrected plot a bit closer to the uncorrected plot. The challenge is getting the slope right, and how well we do that in a hand sketch is just a matter of how hard we want to work at it! For simple poles and zeroes our method consisted of drawing in reference points at the midpoint frequency and a decade above and below this point. For the second-order complex roots we can use the same method, but we must choose frequencies that are closer to the break in order to better approximate the slope; $\omega_n/2$ and $2\omega_n$ are practical choices. The table in Figure 1-19 includes the phase correction for three frequencies below the break, $\omega_n/10$, $\omega_n/5$, and $\omega_n/2$. Clearly the symmetry of the problem allows us to also use these corrections at $10\omega_n$, $5\omega_n$, and $2\omega_n$, respectively.

Let's finish with an illustrative example that involves complex-roots as well as simple poles and zeroes, such as (1.40), repeated here for convenience:

$$H(s) = \frac{10^5 s(s+100)}{(s+10)^2(s^2+400s+10^6)} \tag{1.54}$$

The Bode plot for this function is shown in Figure 1-20. For the uncorrected plot the phase starts at $+90°$ because of the zero at $s=0$, then drops $180°$ at $\omega=10$ because of the repeated pole. It jumps $90°$ at $\omega=10^2$ because of the simple zero, and drops again by $180°$ at $\omega=10^3$ for the complex pole pair. The damping factor is $\xi=0.2$, so from the table in Figure 1-19 we find a phase correction of $2.3°$ a decade away, and $15°$ an octave away. The annotations in Figure 1-20 explain how some of the reference points were calculated.

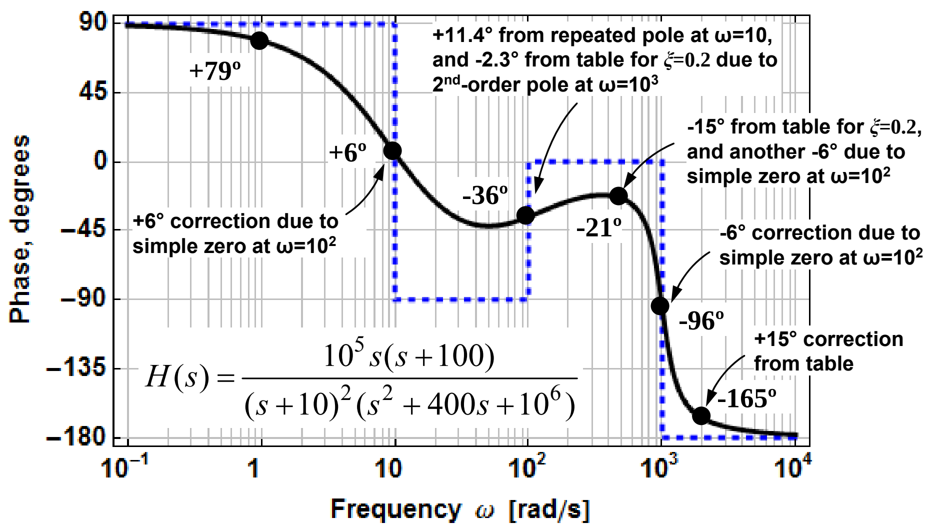


Figure 1-20 – Bode phase plot for (1.54)

Accuracy and Bode Plots

Looking back at the examples of phase plots in Figure 1-16 though Figure 1-20, you may notice that simply drawing the curves through the mid-points of each phase jump would give a reasonable good estimate of the actual curve. So it seems appropriate to ask, is all this business about calculating mid-point corrections even necessary?

In fact this is an important issue because it concerns the broader question of what we are trying to accomplish with our investigation of Bode plots. Nowadays we have the luxury of making computer-generated amplitude and phase plots in a fraction of the time it takes to draw a hand sketch. So in many respects, it simply does not make any sense to waste valuable time in trying to make a highly accurate hand sketch. If analytical accuracy is what we're after, then the computer is a better alternative. Furthermore, it turns out that in many practical applications it is rarely important to know the phase to a tenth of a degree. Often just knowing the phase to the nearest tens place is perfectly fine!

No, the real reason to persist in learning about Bode plots is the valuable *insight* it gives in connecting the shape of the frequency response to the transfer function. Knowing how poles and zeroes affect the amplitude and phase ultimately allows us to approach circuit analysis from a *design* perspective; that is, how do we design a circuit to give a desired frequency response? In this respect, computer-generated plots are not much help. They can tell you how a circuit will perform, but they can't tell you how to *improve* the circuit.

So if we keep in mind that our main goal in drawing Bode plots is usually to explore qualitative behavior of a circuit or transfer function, then the answer to the question is yes: we can usually take shortcuts like drawing the curve through the midpoint of the phase-jumps. If more accuracy is required, the simple first-order corrections that we have developed can be used to adjust the plot accordingly. If even greater accuracy is required, then a computer-generated plot is needed.